On the rank two geometries of the groups
PSL(2, q): part II*

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Received 30 November 2010, accepted 25 January 2013, published online 15 March 2013

Abstract

This document contains an appendix to the paper On the rank two geometries of the groups PSL(2, q): part II, Ars Math. Contemp. 6 (2013), 365–388.

Appendix

Proof of Lemma 9

Proof. In order to determine all subgroups $H$ of $PSL(2, q)$ such that $(H, D_{10})$ is a two-transitive pair we scan the list of maximal subgroups of $PSL(2, q)$. For each maximal subgroup we analyse its subgroup lattice. There are six cases to consider.

1. The group $E_q : \frac{q-1}{(2, q-1)}$ contains a subgroup $D_{10} \cong E_5 : 2$ if $5 | q$. In this situation and in view of (1) in Proposition 7, $H \cong E_5 : 4$ which is not a subgroup of $PSL(2, q)$, under the given conditions.

2. Take $D_{2d}$ with $d \mid \frac{q+1}{2}$. In view of (16)-(18) in Proposition 7, $D_{2d}$ acts two-transitively on the cosets of $D_{10}$ if and only if the index of $D_{10}$ in $D_{2d}$ equals 2 or 3 ($d = 10$ or 15). Therefore $(D_{20}, D_{10})$ and $(D_{30}, D_{10})$ are two-transitive pairs.

3. $A_4$ and $S_4$ do not contain any subgroup of order 10.

4. In view of (6) in Proposition 7 ($A_5, D_{10}$) is a two-transitive pair.

5. In view of (6), (7), (8) and (10) in Proposition 7, PSL(2, $q'$) acts two-transitively on the cosets of $D_{10} \cong E_{q'} : \frac{q'-1}{2}$ only if $q' = 5$, therefore $q = 5^r$ for $r$ an odd prime. $(PSL(2, 5), D_{10})$ is a two-transitive pair.

*This paper is a part of SIGMAP'10 special issue Ars Math. Contemp. vol. 5, no. 2.
†Supported by the “Communauté Française de Belgique - Actions de Recherche Concertées”

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6. In view of (12) in Proposition 7, $\text{PSL}(2, q')$ acts two-transitively on the cosets of $D_{10} \cong E_{q'}: \frac{q'-1}{2}$ only if $q' = 5$ and $q' - 1 = 2$, which leads to a contradiction.

\[ \square \]

**Proof of Lemma 10**

*Proof.* In order to determine all subgroups $H$ of $\text{PSL}(2, q)$ such that $(H, A_4)$ is a two-transitive pair we scan the list of maximal subgroups of $\text{PSL}(2, q)$. For each maximal subgroup we analyse its subgroup lattice. There are six cases to consider.

1. If $q = 5^r$, the group $E_q : \frac{q-1}{2}$ does not contain any subgroup isomorphic to $A_4 \cong E_4 : 3$ because $4 \mid q$ is in contradiction with the condition $q = 5^r$. If $q = p = \pm 1(5)$, the group $E_q : \frac{q-1}{2}$ does not contain any subgroup isomorphic to $E_4 : 3$ because $4 \mid p$ implies that $4 = p$, which is in contradiction with $p$ an odd prime, the same argument holds for $q = p^2 = -1(5)$. If $q = 4^r$ with $r$ prime, the $(2T)_1$ condition, the maximality and the conditions given on $q$ imply that the only candidate of the form $E_q : \frac{q-1}{2}$ is $E_{16} : 3$. Now $(E_{16} : 3, E_4 : 3)$ is a two-transitive pair.

2. Take $D_{2d}$ with $d \mid \frac{q+1}{(2, q-1)}$. We know that dihedral groups only contain cyclic groups and dihedral groups, they do not contain an $A_4$.

3. If $q = 4^r$ with $r$ prime, the group $\text{PSL}(2, q)$ does not contain a subgroup isomorphic to $S_4$, because this is in contradiction with $q = \pm 1(8)$. The same argument holds for $q = 5^r$ with $r$ an odd prime.

4. In view of (6) in Proposition 7 $(A_5, A_4)$ is a two-transitive pair.

5. If $q = p = \pm 1(5)$, the group $\text{PSL}(2, q)$ cannot contain any $\text{PSL}(2, q')$ with $q'^m = q$, $m$ an odd prime, the same argument holds for $q = p^2 = -1(5)$. If $q = 5^r$ with $r$ an odd prime; or if $q = 4^r$ with $r$ prime, the only candidates $q'$ for $\text{PSL}(2, q')$ are 4 and 5. In this situation we have $\text{PSL}(2, q') \cong \text{PSL}(2, 4) \cong \text{PSL}(2, 5) \cong A_5$. This situation has been treated in (4).

6. If $q = p = \pm 1(5)$; or $q = 5^r$ with $r$ an odd prime, the group $\text{PSL}(2, q)$ cannot contain any $\text{PGL}(2, q')$ with $q'^2 = q$.

*Part of proof of Proposition 13.*

*Proof.* Subcase 1: $G_{01} = G_0 \cap G_1 \cong D_{10}$.

By Lemma 9 the four possibilities for $G_1$ are $D_{20}$ provided $10 \mid \frac{q+1}{(2, q-1)}$, $D_{30}$ provided
15 \mid q^{\pm 1}_{(2, q-1)}$, PSL$(2, 5) \cong A_5$ provided $q = 5^r$ and $A_5$.

1.1 We consider the case where $G_1 \cong D_{20}$, provided $10 \mid q^{\pm 1}_{(2, q-1)}$.

The given conditions imply that either $q = p = \pm 1(20)$ or $q = p^2 = -1(20)$. In both situations there are two conjugacy classes of $A_5$ in PSL$(2, q)$. Since $\frac{q+1}{10}$ is even there are two conjugacy classes of $D_{10}$ in PSL$(2, q)$. The index of $D_{10}$ in $D_{20}$ equals two, therefore the $D_{10}$ in a $D_{20}$ are not all conjugate. The number of conjugacy classes of $D_{20}$ depends on whether $\frac{q+1}{20}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases where $\frac{q+1}{20}$ is even or odd.

* $\frac{q+1}{20}$ is even. This implies that $N_{PSL(2, q)}(D_{10}) = D_{20}$ and $N_{PSL(2, q)}(D_{20}) = D_{40}$, with two conjugacy classes of $D_{20}$. Therefore the number of $D_{20}$ containing a given $D_{10}$ is one.

There are two classes of $A_5$ and $D_{10}$ and the latter is contained in one $D_{20}$; therefore there exist exactly two RWPI and $(2T)_1$ geometries $\Gamma(PSL(2, q); A_5, D_{20}, D_{10})$ up to conjugacy, provided $\frac{q+1}{20}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of $D_{10}$, $D_{20}$ and $A_5$ are fused under the action of PGL$(2, q)$ and thus also under the action of PGL$(2, q)$. Therefore, there exists exactly one RWPI and $(2T)_1$ geometry $\Gamma(PSL(2, q); A_5, D_{20}, D_{10})$ up to isomorphism provided $\frac{q+1}{20}$ is even.

* $\frac{q+1}{20}$ is odd. In this situation there is only one conjugacy class of $D_{20}$ in PSL$(2, q)$. The condition on $q$ implies that $N_{PSL(2, q)}(D_{10}) = D_{20}$ and $N_{PSL(2, q)}(D_{20}) = D_{20}$. Therefore the number of $D_{20}$ containing a given $D_{10}$ is one.

Up to conjugacy, there exist exactly two RWPI and $(2T)_1$ geometries $\Gamma(PSL(2, q); A_5, D_{20}, D_{10})$ provided $\frac{q+1}{20}$ is odd.

Let us deal with the fusion of non-conjugate classes. Up to isomorphism there is exactly one such geometry, since following Lemma 8 the two classes of $D_{10}$ and $A_5$ are fused under the action of PGL$(2, q)$ and thus also under the action of PGL$(2, q)$.

To summarize, up to conjugacy there exist exactly two RWPI and $(2T)_1$ geometries $\Gamma_5 = \Gamma(PSL(2, q); A_5, D_{20}, D_{10})$ provided $q = p = \pm 1(20)$. Up to isomorphism there exists exactly one such geometry. Also, up to conjugacy there exist exactly two RWPI and $(2T)_1$ geometries $\Gamma_{12} = \Gamma(PSL(2, q); A_5, D_{20}, D_{10})$ provided $q = p^2 = -1(20)$. Up to isomorphism there exists exactly one such geometry.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 19, 41, 61$. For $q = 19$, it is also confirmed by [20].

1.2. We consider the case where $G_1 \cong D_{30}$, provided $15 \mid q^{\pm 1}_{(2, q-1)}$.

The condition $15 \mid q^{\pm 1}_{(2, q-1)}$ implies that either $q = 4^r$ with $r$ prime, $q = p = \pm 1(5)$ or $q = p^2 = -1(5)$. Hence, there are three cases namely $q = 4^r = \pm 1(15)$ with $r$ prime; $q = p = \pm 1(30)$; or $q = p^2 = -1(30)$. We distinguish the first case from the other two.

* Let us first assume that $q = 4^r = \pm 1(15)$ with $r$ prime. In this situation there is only one conjugacy class of $A_5$. The number of classes of $D_{30}$ and $D_{10}$ in PSL$(2, q)$ depends on whether $\frac{q+1}{15}$ is even or odd. The even case cannot occur because of the condition $q = 4^r$ given on $q$. If $\frac{q+1}{15}$ is odd there is only one conjugacy class of $D_{30}$ and also one of $D_{10}$ in PSL$(2, q)$. Then the index $\frac{|D_{30}|}{|D_{10}|} \neq 2$, and therefore all $D_{10}$ in $D_{30}$ are conjugate. And $A_5$ contains one $D_{10}$ up to conjugacy. The odd condition on $\frac{q+1}{15}$ implies
that $N_{\text{PSL}(2,q)}(D_{10}) = D_{10}$ and $N_{\text{PSL}(2,q)}(D_{30}) = D_{30}$. Therefore the number of $D_{30}$ containing a given $D_{10}$ is one.

To summarize, up to conjugacy there exists exactly one RW PRI and $(2T)_1$ geometry $\Gamma_1 = \Gamma(\text{PSL}(2, q); A_5, D_{30}, D_{10})$ and thus also exactly one up to isomorphism provided either $q = 4^r$ with $r$ prime; or $\frac{q+1}{15}$ odd. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by Magma for $q = 16$ and is also confirmed by [20].

- The cases $q = p = \pm 1(30)$ and $q = p^2 = -1(30)$ with $p$ an odd prime can be treated together. In this situation there are two conjugacy classes of $A_5$, but the number of conjugacy classes of $D_{30}$ and $D_{10}$ depends on whether $\frac{q+1}{30}$ is even or odd.

Assume $\frac{q+1}{30}$ is even. This implies that $N_{\text{PSL}(2,q)}(D_{10}) = D_{20}$ and $N_{\text{PSL}(2,q)}(D_{30}) = D_{60}$, with two conjugacy classes of $D_{10}$ and also two of $D_{30}$. The number of subgroups $D_{30}$ containing a given subgroup $D_{10}$ in $\text{PSL}(2,q)$ is equal to

$$\frac{|\text{PSL}(2,q)|}{|D_{60}|} \cdot \frac{|D_{30}|}{|D_{10}|} \cdot \frac{|D_{20}|}{|\text{PSL}(2,q)|} = 1.$$ 

Up to conjugacy, there exist exactly two RW PRI and $(2T)_1$ geometries $\Gamma(\text{PSL}(2, q); A_5, D_{30}, D_{10})$ provided $\frac{q+1}{30}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of $D_{10}$, $D_{30}$ and $A_5$ are fused under the action of $\text{PGL}(2,q)$ and thus also under the action of $\text{PGL}(2,q)$. Therefore there exists exactly one RW PRI and $(2T)_1$ geometry $\Gamma(\text{PSL}(2,q); A_5, D_{30}, D_{10})$ up to isomorphism provided $\frac{q+1}{30}$ is even.

Assume $\frac{q+1}{30}$ is odd. This implies that $N_{\text{PSL}(2,q)}(D_{10}) = D_{10}$ and $N_{\text{PSL}(2,q)}(D_{30}) = D_{30}$, with one conjugacy class of $D_{10}$ and also one of $D_{30}$. The number of subgroups $D_{30}$ containing a given subgroup $D_{10}$ in $\text{PSL}(2,q)$ is equal to

$$\frac{|\text{PSL}(2,q)|}{|D_{30}|} \cdot \frac{|D_{30}|}{|D_{10}|} \cdot \frac{|D_{10}|}{|\text{PSL}(2,q)|} = 1.$$ 

Up to conjugacy, there exist exactly two RW PRI and $(2T)_1$ geometries $\Gamma(\text{PSL}(2, q); A_5, D_{30}, D_{10})$ provided $\frac{q+1}{30}$ is odd.

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of $A_5$ are fused under the action of $\text{PGL}(2,q)$ and thus also under the action of $\text{PGL}(2,q)$. Therefore, there exists exactly one RW PRI and $(2T)_1$ geometry $\Gamma(\text{PSL}(2,q); A_5, D_{30}, D_{10})$ up to isomorphism provided $\frac{q+1}{30}$ is odd.

To summarize, there exist exactly two RW PRI and $(2T)_1$ geometries $\Gamma_0 = \Gamma(\text{PSL}(2,q); A_5, D_{30}, D_{10})$ up to conjugacy and exactly one up to isomorphism provided $q = p = \pm 1(30)$. Also, up to conjugacy there exist exactly two RW PRI and $(2T)_1$ geometries $\Gamma_{13} = \Gamma(\text{PSL}(2,q); A_5, D_{30}, D_{10})$ and exactly one up to isomorphism provided $q = p^2 = -1(30)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by Magma for $q = 29, 31, 61$.

1.3. Consider the case $G_0 \cong G_1 \cong A_5$.

There are four situations, which are $q = 5^r$ with $r$ odd prime, $q = p = \pm 1(5)$, $q = p^2 = -1(5)$ with $p$ an odd prime and $q = 4^r$ with $r$ prime. Cases 2 and 3 can be treated together. We distinguish them from the others in our discussion.
• Assume \( q = 5^r \) with \( r \) an odd prime. Using that \( \text{PSL}(2, 5) \cong A_5 \), there is only one conjugacy class of \( \text{PSL}(2, 5) \). We must check whether this geometry exists, that is, whether there are two subgroups isomorphic to \( A_5 \) in \( \text{PSL}(2, 5^r) \) that have a subgroup \( D_{10} \) in common. There is only one conjugacy class of \( E_5:2 \). Since \( \text{PSL}(2, 5^r) \) is simple and \( A_5 \) maximal, \( A_5 \) is self-normalized. Also, since \( \text{PSL}(2, 5) \) is simple and \( E_5:2 \) maximal, \( E_5:2 \) is self-normalized in \( \text{PSL}(2, 5) \) and also in \( \text{PSL}(2, 5^r) \). Therefore the number of subgroups \( \text{PSL}(2, 5) \) containing a given subgroup \( E_5:2 \) in \( \text{PSL}(2, 5^r) \) is equal to

\[
\frac{|\text{PSL}(2, 5^r)|}{|\text{PSL}(2, 5)|} \cdot \frac{|E_5:2|}{|E_5:2|} \cdot \frac{|\text{PSL}(2, 5^r)|}{|\text{PSL}(2, 5)|} = 1
\]

which implies that the geometry does not exist.

• Assume that either \( q = p = \pm 1(5) \) or \( q = p^2 = -1(5) \) with \( p \) an odd prime. There are two conjugacy classes of \( A_5 \). The number of conjugacy classes of \( D_{10} \) depends on whether \( \frac{q+1}{10} \) is even or odd.

If \( \frac{q+1}{10} \) is even there are two conjugacy classes of \( D_{10} \). Notice that all \( D_{10} \) in an \( A_5 \) are conjugate and \( N_{\text{PSL}(2, q)}(D_{10}) = D_{20} \). The number of subgroups \( A_5 \) containing a given subgroup \( D_{10} \) in \( \text{PSL}(2, q) \) is equal to

\[
\frac{|\text{PSL}(2, q)|}{|A_5|} \cdot \frac{|A_5|}{|D_{10}|} \cdot \frac{|D_{20}|}{|\text{PSL}(2, q)|} = 2.
\]

Therefore there exist exactly two RWPRI and \((2T)_1\) geometries

\[\Gamma_7 = \Gamma(\text{PSL}(2, q); A_5, A_5, D_{10}) \text{ up to conjugacy, provided } \frac{q+1}{10} \text{ is even with } q \text{ an odd prime and also exactly two RWPRI and } (2T)_1 \text{ geometries } \Gamma_{14} = \Gamma(\text{PSL}(2, q); A_5, A_5, D_{10}) \text{ up to conjugacy, provided } \frac{q+1}{10} \text{ is even with } q = p^2; \text{ one geometry for each class of } A_5.\]

Let us deal with the fusion of non-conjugate classes. Following Lemma 8 the two classes of \( A_5 \) and \( D_{10} \) are fused under the action of \( \text{PGL}(2, q) \) and thus also under the action of \( \text{PGL}(2, q) \). Therefore, there exists exactly one RWPRI and \((2T)_1\) geometry \(\Gamma_7 = \Gamma(\text{PSL}(2, q); A_5, A_5, D_{10}) \text{ up to isomorphism provided } \frac{q+1}{10} \text{ is even with } q \text{ an odd prime and also exactly one RWPRI and } (2T)_1 \text{ geometry } \Gamma_{14} = \Gamma(\text{PSL}(2, q); A_5, A_5, D_{10}) \text{ up to isomorphism provided } \frac{q+1}{10} \text{ is even with } q = p^2.\]

Assume that \( \frac{q+1}{10} \) is odd. There is only one conjugacy class of \( D_{10} \) and

\[N_{\text{PSL}(2, q)}(D_{10}) = D_{10}. \text{ The number of subgroups } A_5 \text{ containing a given subgroup } D_{10} \text{ in } \text{PSL}(2, q) \text{ is equal to}

\[
\frac{|\text{PSL}(2, q)|}{|A_5|} \cdot \frac{|A_5|}{|D_{10}|} \cdot \frac{|D_{10}|}{|\text{PSL}(2, q)|} = 1.
\]

Since there are two conjugacy classes of \( A_5 \) there exists exactly one RWPRI and \((2T)_1\) geometry \(\Gamma_8 = \Gamma(\text{PSL}(2, q); A_5, A_5, D_{10}) \text{ up to conjugacy and thus also exactly one up to isomorphism provided } \frac{q+1}{10} \text{ is odd with } q \text{ an odd prime. Also, there exists exactly one RWPRI and } (2T)_1 \text{ geometry } \Gamma_{15} = \Gamma(\text{PSL}(2, q); A_5, A_5, D_{10}) \text{ up to conjugacy and thus also exactly one up to isomorphism provided } \frac{q+1}{10} \text{ is odd with } q = p^2.\]

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by \textsc{Magma} for \( q = 9, 11, 19, 29, 31, 41, 49. \) For \( q = 9, \) it is also confirmed by
and for \( q = 11, 19 \) by [20].

- If \( q = 4^r \) with \( r \) prime. We know that there is only one conjugacy class of \( A_5 \). We must check whether this geometry exists, that is whether there are two subgroups isomorphic to \( A_5 \) in \( \text{PSL}(2, 4^r) \) that have a subgroup \( D_{10} \) in common. The condition given on \( q \) implies that \( \frac{q+1}{2} \) is odd, therefore there is only one class of \( D_{10} \) and \( N_{\text{PSL}(2, q)}(D_{10}) = D_{10} \). The number of subgroups \( A_5 \) containing a given subgroup \( D_{10} \) in \( \text{PSL}(2, q) \) is equal to

\[
\frac{|\text{PSL}(2, q)|}{|A_5|} \cdot \frac{|A_5|}{|D_{10}|} \cdot \frac{|D_{10}|}{|\text{PSL}(2, q)|} = 1.
\]

In this situation there is only one conjugacy class of \( A_5 \), therefore we may conclude that there exists no such geometry.

\[\square\]

**Proof of Proposition 14**

**Proof.** Let \( G_0 \cong A_4 \) with \( q \) prime, \( q > 3 \) and either \( q = 3, 13, 27, 37(40) \) or \( q = 5 \).

In view of (5) in Proposition 7 the only possibility for \( G_{01} \) is the cyclic subgroup of order 3. If \( H \) is a subgroup of \( G \) such that \((H, 3)\) is a two-transitive pair then one of the following holds: \( H \cong Z_6 \) provided \( 6 \mid \frac{q+1}{2} \), \( H \cong D_6 \) and \( H \cong A_4 \). They are the three only \( G_1 \)-candidates.

Notice that \( q \) prime, \( q > 3 \) and so 3 divides either \( \frac{q+1}{2} \) or \( \frac{q-1}{2} \).

We review all possibilities for \( G_1 \) as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism).

1. Consider the case where \( G_1 \cong Z_6 \), provided \( 6 \mid \frac{q+1}{2} \).

The conditions on \( q \) prime are that \( q = \pm 1(12) \) and \( q = 3, 13, 27, 37(40) \). This implies that \( q = 13, 37, 83, 107(120) \) with \( q \) prime. The group \( A_4 \) contains one cyclic group of order 3 up to conjugacy. The cyclic group of order 3 is contained in exactly one \( Z_6 \) and all \( Z_6 \) in \( \text{PSL}(2, q) \) are conjugate. Since \( \text{PSL}(2, q) \) is simple and \( A_4 \) maximal, \( A_4 \) is self-normalized. It is also the case for the cyclic subgroups of order 3 in \( A_4 \). Now \( N_{Z_6}(3) = Z_6 \) and \( N_{\text{PSL}(2, q)}(3) = N_{\text{PSL}(2, q)}(Z_6) = D_{q+1} \) provided \( 6 \mid \frac{q+1}{2} \) and \( D_{q-1} \) provided \( 6 \mid \frac{q-1}{2} \).

The number of subgroups \( Z_6 \) containing a given cyclic subgroup of order 3 in \( \text{PSL}(2, q) \) is equal to

\[
\frac{|\text{PSL}(2, q)|}{|q + 1|} \cdot \frac{|q + 1|}{|\text{PSL}(2, q)|} = 1.
\]

Therefore, there exists exactly one RWPI and \((2T)_1\) geometry \( \Gamma_1 = \Gamma(\text{PSL}(2, q); A_4, Z_6, 3) \) up to conjugacy, and also exactly one up to isomorphism, provided \( q = 13, 37, 83, 107(120) \). This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for \( q = 13, 37, 83 \).

2. Consider the case where \( G_1 \cong D_6 \).

All cyclic subgroups of order 3 are conjugate in \( \text{PSL}(2, q) \). The number of conjugacy classes of \( D_6 \) depends on whether \( \frac{q+1}{6} \) is odd or even. We distinguish the cases \( \frac{q+1}{6} \) odd or even.

The group \( A_4 \) contains one cyclic group of order 3 up to conjugacy. We know that the normalizer of \( D_6 \) in \( \text{PSL}(2, q) \) is \( D_6 \) provided \( \frac{q+1}{6} \) is odd, and that it is \( D_{12} \) provided...
is even. The normalizer of the cyclic group of order 3 in $D_6$ is $D_6$ and its normalizer in $\mathrm{PSL}(2, q)$ is a dihedral group of order $q \pm 1$. Therefore the number of subgroups $D_6$ containing a given cyclic subgroup of order 3 in $\mathrm{PSL}(2, q)$ is equal to

$$\left\{ \begin{array}{ll}
\frac{|\mathrm{PSL}(2, q)|}{|D_6|} \cdot 1 \cdot \frac{|q+1|}{|\mathrm{PSL}(2, q)|} = \frac{q \pm 1}{6} & \text{if } \frac{q \pm 1}{6} \text{ odd} \\
\frac{|\mathrm{PSL}(2, q)|}{|D_{12}|} \cdot 1 \cdot \frac{|q+1|}{|\mathrm{PSL}(2, q)|} = \frac{q \pm 1}{12} & \text{if } \frac{q \pm 1}{6} \text{ even.}
\end{array} \right.$$

To get the number of geometries up to conjugacy we need to know whether the subgroup $A_4$ normalizes each of the $D_6$, which is the case because

$$|N_{\mathrm{PSL}(2, q)}(3) \cap N_{\mathrm{PSL}(2, q)}(A_4)| = 3.$$

In order to determine the number of classes of geometries up to conjugacy we distinguish the cases $\frac{q \pm 1}{6}$ odd or even.

- Assume $\frac{q \pm 1}{6}$ is odd. There is only one class of $D_6$ and every given cyclic subgroup of order 3 in $\mathrm{PSL}(2, q)$ is contained in exactly $\frac{q \pm 1}{6}$ dihedral groups $D_6$. Up to conjugacy there exist exactly $\frac{q \pm 1}{6}$ geometries.

- Assume $\frac{q \pm 1}{6}$ is even. There are two classes of $D_6$ and every given cyclic subgroup of order 3 in $\mathrm{PSL}(2, q)$ is contained in exactly $\frac{q \pm 1}{12}$ dihedral groups $D_6$. Up to conjugacy there exist exactly $\frac{q \pm 1}{6}$ geometries.

To summarize, up to conjugacy there exist exactly $\frac{q - 1}{6}$ RWPRIs and $(2T)_1$ geometries $\Gamma_3 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ provided $\frac{q - 1}{6}$ is odd and exactly $\frac{q - 1}{6}$ RWPRIs and $(2T)_1$ geometries $\Gamma_5 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to conjugacy, provided $\frac{q - 1}{6}$ is even.

Also, there exist exactly $\frac{q + 1}{6}$ RWPRIs and $(2T)_1$ geometries $\Gamma_2 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to conjugacy, provided $\frac{q + 1}{6}$ is odd and exactly $\frac{q + 1}{6}$ RWPRIs and $(2T)_1$ geometries $\Gamma_4 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to conjugacy, provided $\frac{q + 1}{6}$ is even.

Let us deal with the fusion of non-conjugate classes. We remember that $q$ is prime and thus $\mathrm{PSL}(2, q) \cong \mathrm{PGL}(2, q)$. We also find that $N_{\mathrm{PGL}(2, q)}(A_4) = S_4$, $N_{\mathrm{PGL}(2, q)}(3) = D_{2(q \pm 1)}$, and $N_{\mathrm{PGL}(2, q)}(D_6) = D_{12}$. In order to determine the number of classes of geometries up to isomorphism we distinguish the cases $\frac{q \pm 1}{6}$ odd or even.

- Assume $\frac{q \pm 1}{6}$ odd. There is only one conjugacy class of $D_6$. If we fix $A_4$ and the cyclic group of order 3, there is one $D_6$ which is fixed and the others are exchanged two by two, because $D_6$ in $\mathrm{PSL}(2, q)$ is its own normalizer. They merge two by two under the action of $\mathrm{PSL}(2, q)$. Therefore, the number of RWPRIs and $(2T)_1$ geometries $\Gamma_2 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to isomorphism, provided $\frac{q + 1}{6}$ odd, is exactly $\frac{q + 1}{6} + 1$, and the number of RWPRIs and $(2T)_1$ geometries $\Gamma_3 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to isomorphism, provided $\frac{q - 1}{6}$ odd, is exactly $\frac{q - 1}{6} + 1$.

- Assume $\frac{q \pm 1}{6}$ is even. There are two conjugacy classes of $D_6$. They both merge under the action of $\mathrm{PGL}(2, q)$ and thus also in $\mathrm{PSL}(2, q)$ (see Lemma 11). If we fix $A_4$ and the cyclic group of order 3, we fix two $D_6$, one of each conjugacy class and all others are exchanged two by two. They merge two by two under the action of $\mathrm{PSL}(2, q)$. Therefore, the number of RWPRIs and $(2T)_1$ geometries $\Gamma_2 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to isomorphism, provided $\frac{q + 1}{6}$ even, is exactly $\left(\frac{q + 1}{6} - 2\right) + 1 = \frac{q + 1}{12}$, and the number of RWPRIs and $(2T)_1$ geometries $\Gamma_3 = \Gamma(\mathrm{PSL}(2, q); A_4, D_6, 3)$ up to isomorphism, provided $\frac{q - 1}{6}$ even,
This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for \( q = 5, 13, 37, 43, 53, 67 \). For \( q = 5 \), it is also confirmed by [3] and for \( q = 13 \) by [20].

3. Consider the case where \( G_0 \cong G_1 \cong A_4 \). We must check whether this geometry exists or not, that is whether there are two subgroups isomorphic to \( A_4 \) in \( \text{PSL}(2, q) \) that have a cyclic subgroup of order 3 in common. We know that \( N_{\text{PSL}(2, q)}(A_4) = A_4 \) and that \( N_{A_4}(3) = 3 \). Moreover, the group \( A_4 \) contains 4 maximal cyclic subgroups of order 3, all conjugate. The normalizer of 3 in \( \text{PSL}(2, q) \) is \( D_{q-1} \) if \( 3 \mid q - 1 \) and \( D_{q+1} \) if \( 3 \mid q + 1 \). Therefore the number of subgroups \( A_4 \) containing a given cyclic subgroup of order 3 in \( \text{PSL}(2, q) \) is equal to

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{|\text{PSL}(2, q)|}{|A_4|} \cdot 4 \cdot \frac{q-1}{|\text{PSL}(2, q)|} = \frac{q-1}{3} & \text{if } 3 \mid q - 1 \\
\frac{|\text{PSL}(2, q)|}{|A_4|} \cdot 4 \cdot \frac{q+1}{|\text{PSL}(2, q)|} = \frac{q+1}{3} & \text{if } 3 \mid q + 1.
\end{array} \right.
\]

Knowing that there exists only one conjugacy class of \( A_4 \) and using the conditions on \( q \) we know that this geometry exists. There exist exactly, up to conjugacy, \( \frac{q-1}{3} - 1 \) RWPI and \((2T)_1\) geometries \( \Gamma_7 = \Gamma (\text{PSL}(2, q); A_4, A_4, 3) \), provided \( 3 \mid q - 1 \) and exactly \( \frac{q+1}{3} - 1 \) RWPI and \((2T)_1\) geometries \( \Gamma_6 = \Gamma (\text{PSL}(2, q); A_4, A_4, 3) \) up to conjugacy, provided \( 3 \mid q + 1 \).

Let us deal with the fusion of non-conjugate classes. We remember that \( q \) is prime and thus \( \text{PGL}(2, q) \cong \text{PGL}(2, q) \). We find that \( N_{\text{PGL}(2, q)}(A_4) = S_4 \) and \( N_{\text{PGL}(2, q)}(3) = D_{2(q+1)} \). Therefore the number of subgroups \( A_4 \) containing a given cyclic subgroup of order 3 in \( \text{PGL}(2, q) \) is equal to \( \frac{q+1}{3} \). To count the geometries up to isomorphism we need to know the action of \( \text{PGL}(2, q) \) on subgroups \( A_4 \) containing a given cyclic subgroup of order 3. If we fix \( A_4 \cong G_0 \) and the cyclic subgroup of order 3 we know that \( |N_{\text{PGL}(2, q)}(3) \cap N_{\text{PGL}(2, q)}(A_4)| = |D_6| = 2|3| \). This \( D_6 \) is contained in two \( S_4 \) in \( \text{PGL}(2, q) \), which implies that there is one other \( A_4 \) fixed and all others are exchanged two by two. Thus they merge under the action of \( \text{PGL}(2, q) \). Hence, there exist exactly \( \frac{q+1}{3} - 1 \) RWPI and \((2T)_1\) geometries

\[
\Gamma_7 = \Gamma (\text{PSL}(2, q); A_4, A_4, 3) \text{ up to isomorphism, provided } 3 \mid q - 1 \text{ and exactly } \frac{q+1}{3} - 1 \text{ RWPI and } (2T)_1 \text{ geometries } \Gamma_6 = \Gamma (\text{PSL}(2, q); A_4, A_4, 3) \text{ up to isomorphism, provided } 3 \mid q + 1.
\]

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for \( q = 5, 13, 37, 43, 53, 67 \). For \( q = 13 \), it is also confirmed by [20].

**Proof of Proposition 18**

**Proof.** Let \( G_0 \cong S_4 \).

We subdivide our discussion in three cases, namely the three \( G_{01} \)-candidates given by (11), (12) and (13) in Proposition 7 which are: \( D_6, D_8 \) and \( A_4 \). In each of these three cases we review all possibilities for \( G_1 \) given in the previous Lemmas as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism).
Subcase 1: $G_0 = G_0 \cap G_1 \cong D_6$.

By Lemma 15 the three possibilities for $G_1$ are $D_{12}$ provided $6 \mid \frac{q+1}{2}$, $D_{18}$ provided $9 \mid \frac{q+1}{2}$ and $S_4$.

The number of conjugacy classes of $D_6$ depends on whether $\frac{q+1}{6}$ is odd or even. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q+1}{6}$ odd or even.

Recall that when $q > 2$ is a prime and $q = \pm 1(8)$ there are two conjugacy classes of $S_4$ in $\PSL(2, q)$.

1.1. Consider the case where $G_1 \cong D_{12}$, provided $6 \mid \frac{q+1}{2}$.

Since $\frac{q+1}{6}$ is even, following Lemma 4 there are two conjugacy classes of $D_6$ in $\PSL(2, q)$. The number of conjugacy classes of $D_{12}$ depends on whether $\frac{q+1}{12}$ is even or odd. The conditions on $q$ are that $q = \pm 1(8)$ and $q = \pm 1(12)$. Which implies that $\frac{q+1}{12}$ is even. In this situation there are two classes of $D_{12}$ in $\PSL(2, q)$. Now the index of $\frac{|D_{12}|}{|D_6|} = 2$, therefore the $D_6$ in a $D_{12}$ are not all conjugate. Also, every $D_{12}$ contains two $D_6$ which are not conjugate. And $S_4$ contains one $D_6$ up to conjugacy. Since $\frac{q+1}{12}$ is even we have $N_{\PSL(2, q)}(D_6) = D_{12} = N_{D_{12}}(D_6)$ and $N_{\PSL(2, q)}(D_{12}) = D_{24}$. Therefore the number of $D_{12}$ containing a given $D_6$ is one. Since there are two classes of $S_4$, $D_6$ and $D_{12}$, there exist exactly two RWpRI and $(2T)_1$ geometries $\Gamma_1 = \Gamma(\PSL(2, q); S_4, D_{12}, D_6)$ up to conjugacy when $\frac{q+1}{12}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $D_6$, $D_{12}$ and $S_4$ are fused under the action of $\PGL(2, q)$ and thus also under the action of $\PGammaL(2, q)$. Therefore, there exists exactly one RWpRI and $(2T)_1$ geometry $\Gamma_1 = \Gamma(\PSL(2, q); S_4, D_{12}, D_6)$ up to isomorphism, provided $\frac{q+1}{12}$ is even.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 23$.

1.2. Consider the case where $G_1 \cong D_{18}$, provided $9 \mid \frac{q+1}{2}$.

The number of conjugacy classes of $D_{18}$ and $D_6$ depends on whether $\frac{q+1}{18}$ is even or odd. The conditions on $q$ are that $q = \pm 1(8)$ and $q = \pm 1(18)$. Which implies that $\frac{q+1}{18}$.

Now the index $\frac{|D_{18}|}{|D_6|} \neq 2$, therefore all $D_6$ in a $D_{18}$ are conjugate. And $S_4$ contains one $D_6$ up to conjugacy.

- Assume $\frac{q+1}{18}$ is even. This implies that $N_{\PSL(2, q)}(D_6) = D_{12}$ and $N_{\PSL(2, q)}(D_{18}) = D_{36}$. In this situation there are two conjugacy classes of $D_6$ and also two of $D_{18}$. The number of subgroups $D_{18}$ containing a given subgroup $D_6$ in $\PSL(2, q)$ is equal to

$$\frac{|\PSL(2, q)|}{|D_{36}|} \cdot \frac{|D_{18}|}{|D_6|} \cdot \frac{|D_{12}|}{|\PSL(2, q)|} = 1.$$ 

Since there are two conjugacy classes of $S_4$ there exist exactly two RWpRI and $(2T)_1$ geometries $\Gamma(\PSL(2, q); S_4, D_{18}, D_6)$ up to conjugacy, provided $\frac{q+1}{18}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $D_6$, $D_{18}$ and $S_4$ are fused under the action of $\PGL(2, q)$ and thus also under the action of $\PGammaL(2, q)$. Therefore, there exists exactly one RWpRI and $(2T)_1$ geometry $\Gamma(\PSL(2, q); S_4, D_{18}, D_6)$ up to isomorphism, provided $\frac{q+1}{18}$ is even.

- Assume $\frac{q+1}{18}$ is odd. This implies that $N_{\PSL(2, q)}(D_6) = D_6$ and $N_{\PSL(2, q)}(D_{18}) = D_{18}$. In this situation there is one conjugacy class of $D_6$ and also one of $D_{18}$. The number
of subgroups $D_{18}$ containing a given subgroup $D_6$ in $\text{PSL}(2, q)$ is equal to
\[
\frac{|\text{PSL}(2, q)|}{|D_{18}|} \cdot \frac{|D_{18}|}{|D_6|} \cdot \frac{|D_6|}{|\text{PSL}(2, q)|} = 1.
\]
Since there are two conjugacy classes of $S_4$ there exist exactly two RWRI and $(2T)_1$ geometries $\Gamma(\text{PSL}(2, q); S_4, D_{18}, D_6)$ up to conjugacy, provided $\frac{q+1}{18}$ is odd.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $S_4$ are fused under the action of $\text{PGL}(2, q)$ and thus also under the action of $\text{PGL}(2, q)$. Therefore, there exists exactly one RWRI and $(2T)_1$ geometry $\Gamma(\text{PSL}(2, q); S_4, D_{18}, D_6)$ up to isomorphism, provided $\frac{q+1}{18}$ is odd.

To summarize, there exist exactly two RWRI and $(2T)_1$ geometries $\Gamma_1 = \Gamma(\text{PSL}(2, q); S_4, D_{18}, D_6)$ up to conjugacy and one up to isomorphism, provided $q = \pm 1(72)$ or $q = \pm 17(72)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by Magma for $q = 17$ and is also confirmed by [20].

1.3. Finally we consider the case where $G_0 \cong G_1 \cong S_4$.

- Assume $\frac{q+1}{6}$ is even. There are two conjugacy classes of $D_6$. Now all the $D_6$ are contained in a $S_4$ and all $D_6$ in a $S_4$ are conjugate. The normalizer of $D_6$ in $\text{PSL}(2, q)$ is $D_{12}$. The number of subgroups $S_4$ containing a given subgroup $D_6$ in $\text{PSL}(2, q)$ is equal to
\[
\frac{|\text{PSL}(2, q)|}{|S_4|} \cdot \frac{|S_4|}{|D_6|} \cdot \frac{|D_{12}|}{|\text{PSL}(2, q)|} = 2.
\]
Therefore, there exist exactly two RWRI and $(2T)_1$ geometries $\Gamma_3 = \Gamma(\text{PSL}(2, q); S_4, S_4, D_6)$ up to conjugacy, provided $\frac{q+1}{6}$ is even, one for each class of $S_4$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $S_4$ are fused under the action of $\text{PGL}(2, q)$ and thus also under the action of $\text{PGL}(2, q)$. Therefore, there exists exactly one RWRI and $(2T)_1$ geometry $\Gamma_3 = \Gamma(\text{PSL}(2, q); S_4, S_4, D_6)$ up to isomorphism, provided $\frac{q+1}{6}$ is even.

- Assume $\frac{q+1}{6}$ is odd. There is one conjugacy class of $D_6$. This implies that normalizer $N_{\text{PSL}(2, q)}(D_6) = D_6$. The number of subgroups $S_4$ containing a given subgroup $D_6$ in $\text{PSL}(2, q)$ is equal to
\[
\frac{|\text{PSL}(2, q)|}{|S_4|} \cdot \frac{|S_4|}{|D_6|} \cdot \frac{|D_6|}{|\text{PSL}(2, q)|} = 1.
\]
Since there are two conjugacy classes of $S_4$, there exists exactly one RWRI and $(2T)_1$ geometry $\Gamma_4 = \Gamma(\text{PSL}(2, q); S_4, S_4, D_6)$ up to conjugacy and thus also one up to isomorphism, provided $\frac{q+1}{6}$ is odd.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by Magma for $q = 7, 17, 23, 31, 41$. For $q = 17$, it is also confirmed by [20].

Subcase 2: $G_{01} = G_0 \cap G_1 \cong D_8$.

By Lemma 16 the three possibilities for $G_1$ are $D_{16}$ provided $8 \mid \frac{q+1}{2}$, $D_{24}$ provided $12 \mid \frac{q+1}{2}$ and $S_4$. Observe that under the hypothesis there are two conjugacy classes of $S_4$ in $\text{PSL}(2, q)$. 

2.1. Consider the case where $G_1 \cong D_{16}$, provided $8 \mid \frac{q^{\pm 1}}{2}$.

Since $\frac{q^{\pm 1}}{2}$ is even there are two conjugacy classes of $D_8$. The conditions on $q$ are that $q \pm 1(8)$ and $q \pm 1(16)$. Which implies that $q = \pm 1(16)$. The index of $D_8$ in $D_{16}$ equals two, therefore the $D_8$ in a $D_{16}$ are not all conjugate. And also, every $D_{16}$ contains two $D_8$ which are not conjugate. Moreover $S_4$ contains one $D_8$ up to conjugacy. The number of conjugacy classes of $D_{16}$ depends on whether $\frac{q^{\pm 1}}{16}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q^{\pm 1}}{16}$ odd or even.

- Assume $\frac{q^{\pm 1}}{16}$ is even. This implies that $N_{\text{PSL}(2,q)}(D_8) = D_{16} = N_{D_{16}}(D_8)$ and $N_{\text{PSL}(2,q)}(D_{16}) = D_{32}$, with two conjugacy classes of $D_{16}$. Therefore the number of $D_{16}$ containing a given $D_8$ is one.

Since there are two classes of $S_4$, $D_8$ and $D_{16}$, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\text{PSL}(2,q); S_4, D_{16}, D_8)$ up to conjugacy, provided $\frac{q^{\pm 1}}{16}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $D_8$, $D_{16}$ and $S_4$ are fused under the action of $\text{PGL}(2,q)$ and thus also under the action of $\text{PGF}(2,q)$. Therefore there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\text{PSL}(2,q); S_4, D_{16}, D_8)$ up to isomorphism provided $\frac{q^{\pm 1}}{16}$ is even.

- Assume $\frac{q^{\pm 1}}{16}$ is odd. This implies that $N_{\text{PSL}(2,q)}(D_8) = D_{16}$ and $N_{\text{PSL}(2,q)}(D_{16}) = D_{16}$, with one conjugacy class of $D_{16}$. Therefore the number of $D_{16}$ containing a given $D_8$ is one.

Hence, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\text{PSL}(2,q); S_4, D_{16}, D_8)$ up to conjugacy, provided $\frac{q^{\pm 1}}{16}$ is odd.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $D_8$ and $S_4$ are fused under the action of $\text{PGL}(2,q)$ and thus also under the action of $\text{PGF}(2,q)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma(\text{PSL}(2,q); S_4, D_{16}, D_8)$ up to isomorphism, provided $\frac{q^{\pm 1}}{16}$ is odd.

To summarize, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_5 = \Gamma(\text{PSL}(2,q); S_4, D_{16}, D_8)$ up to conjugacy and exactly one up to isomorphism, provided $q = \pm 1(16)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 17, 31$. For $q = 17$, it is also confirmed by [20].

2.2. We now consider the case $G_1 \cong D_{24}$, provided $12 \mid \frac{q^{\pm 1}}{2}$.

The index $\frac{|D_{24}|}{|D_8|} \neq 2$, therefore all $D_8$ in a $D_{24}$ are conjugate. And $S_4$ contains one $D_8$ up to conjugacy. The number of conjugacy classes of $D_8$ and $D_{24}$ depends on whether $\frac{q^{\pm 1}}{24}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q^{\pm 1}}{24}$ odd or even.

- Assume $\frac{q^{\pm 1}}{24}$ is even. This implies that $N_{\text{PSL}(2,q)}(D_8) = D_{16}$ and $N_{\text{PSL}(2,q)}(D_{24}) = D_{18}$. In this situation there are two conjugacy classes of $D_8$ and also two of $D_{24}$. The number of subgroups $D_{24}$ containing a given subgroup $D_8$ in $\text{PSL}(2,q)$ is equal to

$$\frac{|\text{PSL}(2,q)|}{|D_{48}|} \cdot \frac{|D_{24}|}{|D_8|} \cdot \frac{|D_{16}|}{|\text{PSL}(2,q)|} = 1.$$ 

Therefore, there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma(\text{PSL}(2,q); S_4, D_{24}, D_8)$ up to conjugacy, provided $\frac{q^{\pm 1}}{24}$ is even.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $D_8$, $D_{24}$ and $S_4$ are fused under the action of $\text{PGL}(2,q)$ and thus also under...
the action of $\mathrm{PGL}(2, q)$. Therefore, there exists exactly one RWPR and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2, q); S_4, D_{24}, D_8)$ up to isomorphism provided $\frac{q + 1}{24}$ is even.

- Assume $\frac{q + 1}{24}$ is odd. This implies that $N_{\mathrm{PSL}(2, q)}(D_8) = D_8$ and $N_{\mathrm{PSL}(2, q)}(D_{24}) = D_{24}$. In this situation there is one conjugacy class of $D_8$ and also one of $D_{24}$. The number of subgroups $D_{18}$ containing a given subgroup $D_8$ in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{|\mathrm{PSL}(2, q)|}{|D_{24}|} \cdot \frac{|D_{24}|}{|D_8|} \cdot \frac{|D_8|}{|\mathrm{PSL}(2, q)|} = 1.$$ 

To summarize, there exist exactly two RWPR and $(2T)_1$ geometries $\Gamma(\mathrm{PSL}(2, q); S_4, D_{24}, D_8)$ up to conjugacy.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $S_4$ are fused under the action of $\mathrm{PGL}(2, q)$ and thus also under the action of $\mathrm{PGL}(2, q)$. Therefore, there exists exactly one RWPR and $(2T)_1$ geometry $\Gamma(\mathrm{PSL}(2, q); S_4, D_{24}, D_8)$ up to isomorphism, provided $\frac{q + 1}{24}$ is odd.

To summarize, there exist exactly two RWPR and $(2T)_1$ geometries $\Gamma_6 = \Gamma(\mathrm{PSL}(2, q); S_4, D_{24}, D_8)$ up to conjugacy and exactly one up to isomorphism provided $q = \pm 1(24)$. This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by Magma for $q = 23$.

2.3. At last, consider the case $G_0 \cong G_1 \cong S_4$.

The number of conjugacy classes of $D_8$ depends on whether $\frac{q + 1}{8}$ is even or odd. In order to determine all geometries under the given conditions we distinguish the cases $\frac{q + 1}{8}$ odd or even.

- Assume $\frac{q + 1}{8}$ is even. There are two conjugacy classes of $D_8$. In $S_4$ all $D_8$ are conjugate and the normalizer of $D_8$ in $\mathrm{PSL}(2, q)$ is $D_{16}$. The number of subgroups $S_4$ containing a given subgroup $D_8$ in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{|\mathrm{PSL}(2, q)|}{|S_4|} \cdot \frac{|S_4|}{|D_8|} \cdot \frac{|D_8|}{|\mathrm{PSL}(2, q)|} = 2.$$ 

Therefore, up to conjugacy there exist exactly two RWPR and $(2T)_1$ geometries $\Gamma_7 = \Gamma(\mathrm{PSL}(2, q); S_4, S_4, D_8)$ provided $\frac{q + 1}{8}$ is even, one for each class of $S_4$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $S_4$ are fused under the action of $\mathrm{PGL}(2, q)$ and thus also under the action of $\mathrm{PGL}(2, q)$. Therefore, there exists exactly one RWPR and $(2T)_1$ geometry $\Gamma_7 = \Gamma(\mathrm{PSL}(2, q); S_4, S_4, D_8)$ up to isomorphism, provided $\frac{q + 1}{8}$ is even.

- Assume $\frac{q + 1}{8}$ is odd. There is one conjugacy class of $D_8$. This implies that normalizer $N_{\mathrm{PSL}(2, q)}(D_8) = D_8$. The number of subgroups $S_4$ containing a given subgroup $D_8$ in $\mathrm{PSL}(2, q)$ is equal to

$$\frac{|\mathrm{PSL}(2, q)|}{|S_4|} \cdot \frac{|S_4|}{|D_8|} \cdot \frac{|D_8|}{|\mathrm{PSL}(2, q)|} = 1.$$ 

Since there are two conjugacy classes of $S_4$, there exists exactly one RWPR and $(2T)_1$ geometry $\Gamma_8 = \Gamma(\mathrm{PSL}(2, q); S_4, S_4, D_8)$ up to conjugacy and thus also exactly one up to isomorphism provided $\frac{q + 1}{8}$ is odd.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by Magma for $q = 7, 17, 23, 31, 41$. For $q = 7$, it is also confirmed by [3]
and for \( q = 17 \) by [20].

**Subcase 3:** \( G_{01} = G_0 \cap G_1 \cong A_4 \).

By Lemma 17 the possibilities for \( G_1 \) are \( S_4 \) and \( A_5 \) provided \( q = \pm 1(5) \). In the latter situation there are two conjugacy classes of \( A_5 \).

3.1. Consider the case where \( G_0 \cong G_1 \cong S_4 \).

We have \( q = \pm 1(8) \) which implies that there are two conjugacy classes of \( S_4 \) and also two of \( A_4 \). Now all \( A_4 \) in a \( S_4 \) are conjugate and every given \( A_4 \) is contained in exactly one \( S_4 \), which implies that there exists no geometry in this situation.

3.2. Consider the case where \( G_1 \cong A_5 \).

If \( q = p = \pm 1(5) = \pm 1(8) \) with \( p \) prime, this case has already been dealt with in Proposition 13. Therefore, there exist exactly two RWPRI and \( (2T)_1 \) geometries \( \Gamma_9 = \Gamma (PSL(2, q); S_4, A_5, A_4) \) up to conjugacy and exactly one up to isomorphism for \( q = p = \pm 1(40) \) and for \( q = p = \pm 9(40) \) with \( p \) an odd prime. \( \square \)

**Proof of Proposition 20**

**Proof.** Let \( G_0 \cong PSL(2, 2^n) \).

We subdivide our discussion in three cases according to the three \( G_{01} \)-candidates given by (3), (4), (6) and (10) in Proposition 7 namely: the case of the cyclic subgroup of order 3 provided \( q' = 2 \); the case of \( D_{10} \) provided \( q' = 4 \) and the case of \( E_{2n} : (2^n - 1) \).

In each of these three cases we review all possibilities for \( G_1 \) given in the previous Lemmas as well as the number of classes of geometries with respect to conjugacy (resp. isomorphism). In order to determine all geometries under the given conditions we subdivide our discussion in a particular case and a general one depending on whether \( n = 1 \) or not.

**Particular case:** \( n = 1 \) and \( m = 2 \).

In this situation \( q' = 2 \) and \( q = 4 \). In view of (3) and (4) in Proposition 7 there are two cases to consider: the cyclic group of order 3 and the cyclic group of order 2.

**Subcase 1:** \( G_{01} = G_0 \cap G_1 \cong 2 \).

Since \( G \cong PSL(2, 4), (PSL(2, 2), 2) \) and \( (2^2, 2) \) are the only two-transitive pairs. We obtain the following geometries

\[
\Gamma_2 = \Gamma (PSL(2, 4); PSL(2, 2), PSL(2, 2), 2) \quad \text{and} \quad \Gamma_3 = \Gamma (PSL(2, 4); PSL(2, 2), 2^2, 2).
\]

They are indeed RWPRI and \( (2T)_1 \) geometries as we need because we already met them in [5], Proposition 15. Since \( PSL(2, 4) \cong PSL(2, 5) \) and \( PSL(2, 2) \cong S_3 \), these are the RWPRI and \( (2T)_1 \) geometries corresponding to the Petersen graph and the Desargues’ configuration.

**Subcase 2:** \( G_{01} = G_0 \cap G_1 \cong 3 \).

Since \( G \cong PSL(2, 4) \cong A_5, (PSL(2, 2), 3) \) and \( (A_4, 3) \) are the only two-transitive pairs.

The geometry \( \Gamma (PSL(2, 4); PSL(2, 2), PSL(2, 2), 3) \) has been treated in [5] Proposition 15 since \( PSL(2, 2) \cong D_6 \) and it does not exist. We obtain the following geometry

\[
\Gamma_4 = \Gamma (PSL(2, 4); PSL(2, 2), A_4, 3),
\]

which is indeed a RWPRI and \( (2T)_1 \) geometry as we need because we already met it in Proposition 14 since \( PSL(2, 4) \cong PSL(2, 5) \).
**General case:** $n \neq 1$ and $m$ is a prime.

In view of (10) in Proposition 7 there are two cases to consider: $E_{2n} : (2^n - 1)$ and $D_{10}$ provided $q' = 4$.

**Subcase 1:** $G_0 = G_0 \cap G_1 \cong D_{10}$, provided $q' = 4$.

This situation has been treated in Proposition 13, Subcase 1. We obtained the following RWPI and $(2T)_1$ geometry $\Gamma_5 = \Gamma (\text{PSL}(2, 4^m) ; \text{PSL}(2, 4), D_{30}, D_{10})$, provided $\frac{q + 1}{15}$ is odd.

**Subcase 2:** $G_0 = G_0 \cap G_1 \cong E_{2n} : (2^n - 1)$.

By Lemma 19 the possibilities for $G_1$ are $E_{22n} : (2^n - 1)$ provided $m = 2$, and $\text{PSL}(2, 2^n)$.

Notice that if $n = 2$, $\text{PSL}(2, 2^n) \cong A_5$.

2.1. Consider the case where $G_1 \cong E_{22n} : (2^n - 1)$ provided $m = 2$.

In this situation there is only one conjugacy class of $\text{PSL}(2, 2^n)$ and also one of $E_{22n} : (2^n - 1)$ in $\text{PSL}(2, 2^{2n})$. There is one conjugacy class of $E_{2n} : (2^n - 1)$ in $\text{PSL}(2, 2^n)$ and also one in $\text{PSL}(2, 2^{2n})$. Notice that there are $2^n + 1$ conjugacy classes of $E_{2n} : (2^n - 1)$ in $E_{22n} : (2^n - 1)$. Since $\text{PSL}(2, 2^{2n})$ is simple and both $\text{PSL}(2, 2^n)$ and $E_{2n} : (2^n - 1)$ are maximal, $\text{PSL}(2, 2^n)$ and $E_{2n} : (2^n - 1)$ are self-normalized. Moreover the normalizer of $E_{2n} : (2^n - 1)$ in $\text{PSL}(2, 2^{2n})$ is itself. We also find that $N_{\text{PSL}(2, 2^{2n})} (E_{22n} : (2^n - 1)) = E_{22n} : (2^n - 1)$. Therefore the number of subgroups $E_{22n} : (2^n - 1)$ containing a given subgroup $E_{2n} : (2^n - 1)$ in $\text{PSL}(2, 2^{2n})$ is equal to

$$\frac{|\text{PSL}(2, 2^{2n})|}{|E_{22n} : (2^n - 1)|} \cdot \frac{|E_{22n} : (2^n - 1)|}{|E_{2n} : (2^n - 1)|} \cdot (2^n + 1) \cdot \frac{|E_{2n} : (2^n - 1)|}{|\text{PSL}(2, 2^{2n})|} = 1.$$  

Hence, the RWPI and $(2T)_1$ geometry $\Gamma_1 = \Gamma (\text{PSL}(2, 2^{2n}) ; \text{PSL}(2, 2^n), E_{22n} : (2^n - 1), E_{2n} : (2^n - 1))$ provided $n \neq 1$ exists and is unique up to conjugacy and also up to isomorphism.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 16, 64$. For $q = 16$, it is also confirmed by [20].

The particular situation where $n = 2$, has also been dealt with in Proposition 13, which showed that $\Gamma (\text{PSL}(2, 4^2) ; A_5, E_{16} : 3, A_4)$ exists and is unique up to conjugacy, and also up to isomorphism.

2.2. Consider the case where $G_0 \cong G_1 \cong \text{PSL}(2, 2^n)$.

In this situation there is only one conjugacy class of $\text{PSL}(2, 2^n)$ in $\text{PSL}(2, 2^{nm})$. We must check whether this geometry exists, that is whether there exist two subgroups isomorphic to $\text{PSL}(2, 2^n)$ in $\text{PSL}(2, 2^{nm})$ that have the subgroup $E_{2n} : (2^n - 1)$ in common. Since $\text{PSL}(2, 2^{nm})$ is simple and $\text{PSL}(2, 2^n)$ maximal, $\text{PSL}(2, 2^n)$ is self-normalized. Moreover, the group $\text{PSL}(2, 2^n)$ contains $2^n + 1$ maximal subgroups $E_{2n} : (2^n - 1)$ all conjugate. The normalizer of $E_{2n} : (2^n - 1)$ in $\text{PSL}(2, q)$ is the group itself. Therefore the number of subgroups $\text{PSL}(2, 2^n)$ containing a given subgroup $E_{2n} : (2^n - 1)$ in $\text{PSL}(2, q)$ is equal to

$$\frac{|\text{PSL}(2, 2^{nm})|}{|\text{PSL}(2, 2^n)|} \cdot \frac{|\text{PSL}(2, 2^n)|}{|E_{2n} : (2^n - 1)|} \cdot \frac{|E_{2n} : (2^n - 1)|}{|\text{PSL}(2, 2^{nm})|} = 1$$

which implies that the geometry does not exist.
The particular situation where \( n = 2 \), has also been treated in Proposition 13 since \( \text{PSL}(2, 4) \cong A_5 \), which showed that \( \Gamma(\text{PSL}(2, 4^n); \text{PSL}(2, 4), \text{PSL}(2, 4), A_4) \) does not exist.

**Proof of Proposition 24**

**Proof.** Let \( G_0 \cong \text{PSL}(2, p^n) \).

We subdivide our discussion in three cases according to the four \( G_{01} \)-candidates given by (5)-(10) in Proposition 7 namely: \( A_4 \) provided \( q' = 5 \), \( S_4 \) provided \( q' = 7 \), \( A_5 \) provided \( q' = 9, 11 \) and \( E_{q'} : \frac{q' - 1}{2} \).

In each of these four cases we review all possibilities for \( G_1 \) given in the previous Lemmas as well as the number of classes of such geometries with respect to conjugacy (resp. isomorphism). In order to determine all geometries under the given conditions we subdivide our discussion in a particular case and a general one depending on whether \( n = 1 \) or not.

**Particular case:** \( n = 1 \).

In this situation \( q' = p \). The candidates for \( G_{01} \) are \( E_p : \frac{p - 1}{2} \), \( A_4 \) provided \( q' = 5 \), \( S_4 \) provided \( q' = 7 \), \( A_5 \) provided \( q' = 11 \).

**Subcase 1:** \( G_{01} = G_0 \cap G_1 \cong E_p : \frac{p - 1}{2} \).

By Lemma 21 the only possibility for \( G_1 \) is \( \text{PSL}(2, p) \). We distinguish two particular situations, namely \( \text{PSL}(2, 3) \cong A_4 \) (provided \( p = 3 \)) and \( \text{PSL}(2, 5) \cong A_5 \) (provided \( p = 5 \)). All other situations will be treated in the general case, where \( n \) can take any value.

1.1 Consider the case where \( G_0 \cong \text{PSL}(2, 3) \cong A_4 \cong G_1 \).

In this situation \( G_{01} \) is the cyclic group of order 3. There is only one conjugacy class of \( A_4 \) in \( \text{PSL}(2, 3^m) \). We must check whether this geometry exists, that is whether there exist two subgroups isomorphic to \( A_4 \) in \( \text{PSL}(2, 3^m) \) that have the cyclic subgroup of order 3 in common. Since \( \text{PSL}(2, 3^m) \) is simple and \( A_4 \) maximal, \( A_4 \) is self-normalized. The cyclic subgroup of order 3 is self-normalized in \( A_4 \). Moreover \( A_4 \) contains four cyclic subgroups of order 3 which are all conjugate. The normalizer of 3 in \( \text{PSL}(2, 3^m) \) is an elementary abelian subgroup of order \( 3^m \). Therefore the number of subgroups \( A_4 \) containing a given subgroup 3 in \( \text{PSL}(2, 3^m) \) is equal to

\[
\frac{|\text{PSL}(2, 3^m)|}{|A_4|} \cdot \frac{|3^m|}{|\text{PSL}(2, 3^m)|} = 3^{m-1}
\]

and thus the geometry exists. There exist exactly \( 3^{m-1} - 1 \) RWPRI and \((2T)_1\) geometries \( \Gamma_1 = \Gamma(\text{PSL}(2, 3^m); A_4, A_4, 3) \) up to conjugacy when \( m \neq 3 \). There exist exactly 8 RWPRI and \((2T)_1\) geometries \( \Gamma_2 = \Gamma(\text{PSL}(2, 3^3); A_4, A_4, 3) \) up to conjugacy when \( m = 3 \).

Let us deal with the fusion of non-conjugate classes. We find that \( N_{\text{PGL}(2, q)}(A_4) = (S_4 : C_m) \) and \( N_{\text{PGL}(2, q)}(3) = (3^m : C_m) \). Therefore the number of subgroups \( A_4 \) containing a given cyclic subgroup of order 3 in \( \text{PGL}(2, 3^m) \) is equal to

\[
\frac{|\text{PGL}(2, 3^m)|}{|S_m|} \cdot \frac{|A_4|}{3} \cdot \frac{|3^m |}{|\text{PGL}(2, 3^m)|} = 3^{m-1}.
\]
To count the geometries up to isomorphism we need to know the action of $P\Gamma L(2, 3^m)$ on the subgroups $A_4$ containing a given cyclic subgroup of order 3. If we fix $A_4 \cong G_0$ and the cyclic subgroup of order 3 we know that $|N_{P\Gamma L(2, 3^m)}(A_4) \cap N_{P\Gamma L(2, 3^m)}(3)| = |D_6| \cdot |C_m|$. 

We distinguish the cases $m = 3$ and $m \neq 3$:

- Let us first assume that $m = 3$. In this situation there are three subgroups $A_4$ fixed and the others are exchanged 6 by 6. Thus they merge under the action of $P\Gamma L(2, 3^m)$. Therefore, there exist exactly $\frac{3^3 - 1}{6} + 1 = 2$ RWPRI and $(2T)_1$ geometries $\Gamma_2 = \Gamma(PGL(2, 3^3); A_4, A_4, 3)$ up to isomorphism for $m = 3$.

- Now we assume $m \neq 3$. Using Fermat’s Last Theorem for $m$ an odd prime we know that $m \mid 3^m - 1 - 1$. In this situation there is only one $A_4 \cong G_0$ fixed. All others are exchanged $2m$ by $2m$. Therefore, there exist exactly $\frac{3^m - 1}{2m}$ RWPRI and $(2T)_1$ geometries $\Gamma_1 = \Gamma(PGL(2, 3^m); A_4, A_4, 3)$ up to isomorphism, provided $m \neq 3$ is an odd prime.

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 27$.

1.2 Consider the case where $G_0 \cong PSL(2, 5) \cong A_5 \cong G_1$. This RWPRI and $(2T)_1$ geometry $\Gamma(PSL(2, 5^m), PSL(2, 5), A_5, E_5 : 2)$ has already been dealt with in Proposition 13 and it does not exist.

**Subcase 2:** $G_0 = G_0 \cap G_1 \cong A_4$, provided $q = 5^m$ with $m$ an odd prime.
This RWPRI and $(2T)_1$ geometry $\Gamma(PSL(2, 5^m), PSL(2, 5), A_5, A_4)$ has already been dealt with in Proposition 13, Subcase 2.3 and it does not exist.

**Subcase 3:** $G_0 = G_0 \cap G_1 \cong S_4$, when $q = 7^m$ with $m$ an odd prime.
By Lemma 23 the possibility for $G_1 \cong PSL(2, 7) \cong G_0$. In this situation there is only one conjugacy class of $PSL(2, 7)$ in $PSL(2, 11^m)$ and two conjugacy classes of $S_4$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $PSL(2, 7)$ in $PSL(2, 7^m)$ which have the subgroup $S_4$ in common. Since $PSL(2, 7^m)$ is simple and $PSL(2, 7)$ maximal, $PSL(2, 7)$ is self-normalized. The normalizer of $S_4$ in $PSL(2, 7^m)$ and in $PSL(2, 7)$ is the group $S_4$ itself. Therefore the number of subgroups $PSL(2, 7)$ containing a given subgroup $S_4$ in $PSL(2, 7^m)$ is equal to

$$\frac{|PSL(2, 7^m)|}{|PSL(2, 7)|} \cdot \frac{|PSL(2, 7)|}{|S_4|} \cdot \frac{|S_4|}{|PSL(2, 7^m)|} = 1$$

which implies that the geometry does not exist.

**Subcase 4:** $G_0 = G_0 \cap G_1 \cong A_5$, when $q = 11^m$ with $m$ an odd prime.
By Lemma 25 the possibility for $G_1 \cong PSL(2, 11) \cong G_0$. In this situation there is only one conjugacy class of $PSL(2, 11)$ in $PSL(2, 11^m)$ and two conjugacy classes of $A_5$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $PSL(2, 11)$ in $PSL(2, 11^m)$ which have the subgroup $A_5$ in common. Since $PSL(2, 11^m)$ is simple and $PSL(2, 11)$ maximal, $PSL(2, 11)$ is self-normalized. The normalizer of $A_5$ in $PSL(2, 11^m)$ and in $PSL(2, 11)$ is the group $A_5$ itself. Therefore the number of subgroups $PSL(2, 11)$ containing a given subgroup $A_5$ in $PSL(2, 11^m)$ is equal to

$$\frac{|PSL(2, 11^m)|}{|PSL(2, 11)|} \cdot \frac{|PSL(2, 11)|}{|A_5|} \cdot \frac{|A_5|}{|PSL(2, 11^m)|} = 1$$
which implies that the geometry does not exist.

**General case:**
Let us now discuss the general case, where \( n \) can take any value and \( p^n \) is different from 3 and 5 because these two cases have been discussed in the particular case. The two candidates for \( G_{01} \) are \( E_{q'} : q' - \frac{1}{2} \) and \( A_5 \) provided \( q' = 3^2 \).

**Subcase 1:** \( G_{01} = G_0 \cap G_1 \cong E_{p^n} : \frac{p^n - 1}{2} \).
By Lemma 21 the only possibility for \( G_1 \) is \( PSL(2, p^n) \cong G_0 \). In this situation there is only one conjugacy class of \( PSL(2, p^n) \) in \( PSL(2, p^{nm}) \). We must check whether this geometry exists, that is whether there are two subgroups isomorphic to \( PSL(2, p^n) \) in \( PSL(2, p^{nm}) \) that have the subgroup \( E_{p^n} : \frac{p^n - 1}{2} \) in common. Since \( PSL(2, q) \) is simple and \( PSL(2, p^n) \) maximal, \( PSL(2, p^n) \) is self-normalized. Moreover, the group \( PSL(2, p^n) \) contains \( 2^n + 1 \) maximal subgroups \( E_{p^n} : \left( \frac{p^n - 1}{2} \right) \) all conjugate. There is only one conjugacy class of \( E_{p^n} : \left( \frac{p^n - 1}{2} \right) \) in \( PSL(2, p^{nm}) \). The normalizer of \( E_{p^n} : \left( \frac{p^n - 1}{2} \right) \) in \( PSL(2, q) \) is the group itself. Therefore the number of subgroups \( PSL(2, p^n) \) containing a given subgroup \( E_{p^n} : \left( \frac{p^n - 1}{2} \right) \) in \( PSL(2, q) \) is equal to

\[
\frac{|PSL(2, p^{mn})|}{|PSL(2, p^n)|} \cdot \frac{|PSL(2, p^n)|}{|E_{p^n} : \frac{p^n - 1}{2}|} \cdot \frac{|E_{p^n} : \frac{p^n - 1}{2}|}{|PSL(2, p^{mn})|} = 1
\]

which implies that the geometry does not exist.

**Subcase 2:** \( G_{01} = G_0 \cap G_1 \cong A_5 \), when \( q = 9^m \) with \( m \) an odd prime.
By Lemma 22 the possibility for \( G_1 \cong PSL(2, 9) \cong G_0 \). In this situation there is only one conjugacy class of \( PSL(2, 9) \) in \( PSL(2, 9^m) \) and two conjugacy classes of \( A_5 \). We must check whether this geometry exists, that is whether there are two subgroups isomorphic to \( PSL(2, 9) \) in \( PSL(2, 9^m) \) which have the subgroup \( A_5 \) in common. Since \( PSL(2, 9^m) \) is simple and \( PSL(2, 9) \) maximal, \( PSL(2, 9) \) is self-normalized. The normalizer of \( A_5 \) in \( PSL(2, 9^m) \) and in \( PSL(2, 9) \) is the group \( A_5 \) itself. Therefore the number of subgroups \( PSL(2, 9) \) containing a given subgroup \( A_5 \) in \( PSL(2, 9^m) \) is equal to

\[
\frac{|PSL(2, 9^m)|}{|PSL(2, 9)|} \cdot \frac{|PSL(2, 9)|}{|A_5|} \cdot \frac{|A_5|}{|PSL(2, 9^m)|} = 1
\]

which implies that the geometry does not exist.

\[ \square \]

**Proof of Proposition 29**

*Proof.* Let \( G_0 \cong PGL(2, p^n) \).
We subdivide our discussion in four cases, namely the four \( G_{01} \)-candidates given by (11), (12), (13) and (20) in Proposition 7 namely: \( E_{p^n} : (p^n - 1) \), \( PSL(2, p^n) \), \( D_8 \) for \( p^n = 3 \) and the case of \( S_4 \) provided \( q = 5^2 \). In each of these four cases we review all possibilities for \( G_1 \) given in the previous Lemmas as well as the number of classes of such geometries with respect to conjugacy (resp. isomorphism).

**Subcase 1:** \( G_{01} = G_0 \cap G_1 \cong D_8 \), provided \( q = 9 \).
By Lemma 25 the only case to consider is $G_0 \cong G_1 \cong \text{PGL}(2, 3)$.

Since $q = 9$, there is only one conjugacy class of $D_8$ and $D_8$ is self-normalized in $\text{PSL}(2, 9)$. Therefore the number of subgroups $\text{PGL}(2, 3)$ containing a given subgroup $D_8$ in $\text{PSL}(2, 9)$ is equal to

$$|\text{PSL}(2, 9)| \cdot |\text{PGL}(2, 3)| \cdot |D_8| = |\text{PSL}(2, 9)| = 1.$$  

There are 2 conjugacy classes of $\text{PGL}(2, 3)$ in $\text{PSL}(2, 9)$. Hence, up to conjugacy and also up to isomorphism there exists exactly one RWPRI and $(2T)_1$ geometry

$$\Gamma_3 = \Gamma (\text{PSL}(2, 9); \text{PGL}(2, 3); \text{PGL}(2, 3); D_8).$$  

This is confirmed by [3].

**Subcase 2:** $G_{01} = G_0 \cap G_1 \cong E_{p^n} : (p^n - 1)$.

By Lemma 26 the possibilities for $G_1$ are $E_{p^{2n}} : (p^n - 1)$ and $\text{PGL}(2, p^n)$. Notice that $S_4$ is a particular case of $\text{PGL}(2, p^n)$ provided $p^n = 3$.

2.1. Consider the case where $G_1 \cong E_{p^{2n}} : (p^n - 1)$.

In this situation there is only one conjugacy class of $E_{p^{2n}} : (p^n - 1)$ and two conjugacy classes of $\text{PGL}(2, p^n)$ in $\text{PSL}(2, p^{2n})$. Each $\text{PGL}(2, p^n)$ contains one conjugacy class of $E_{p^n} : (p^n - 1)$ and there are two conjugacy classes of $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^{2n})$. Notice that there are $p^n + 1$ conjugacy classes of $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^{2n})$.

Since $\text{PSL}(2, p^{2n})$ is simple and both $\text{PGL}(2, p^n)$ and $E_{p^n} : (p^n - 1)$ maximal, $E_{p^n} : (p^n - 1)$ and $E_{p^n} : (p^n - 1)$ are self-normalized. Moreover the normalizer of $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^{2n})$ is itself. We also find that $N_{\text{PSL}(2, p^{2n})} (E_{p^{2n}} : (p^n - 1)) = E_{p^{2n}} : (p^n - 1)$. Therefore the number of subgroups $\text{PGL}(2, p^n)$ containing a given subgroup $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^{2n})$ is equal to

$$|\text{PSL}(2, p^{2n})| \cdot |\text{PGL}(2, p^n)| \cdot |E_{p^n} : (p^n - 1)| = 1.$$  

Therefore, up to conjugacy, there exist exactly two RWPRI and $(2T)_1$ geometries

$$\Gamma_1 = \Gamma (\text{PSL}(2, p^{2n}); \text{PGL}(2, p^n); E_{p^{2n}} : (p^n - 1); E_{p^n} : (p^n - 1)),$$

corresponding to the two conjugacy classes of subgroups isomorphic to $E_{p^n} : (p^n - 1)$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $\text{PGL}(2, p^n)$ are fused under the action of $\text{PGL}(2, p^{2n})$ and thus also under the action of $\text{PGL}(2, p^{2n})$. This is also the case for the two classes of $E_{p^n} : (p^n - 1)$. Therefore, up to isomorphism there exists exactly one RWPRI and $(2T)_1$ geometry

$$\Gamma_1 = \Gamma (\text{PSL}(2, p^{2n}); \text{PGL}(2, p^n); E_{p^{2n}} : (p^n - 1); E_{p^n} : (p^n - 1)).$$

This geometry is new and the number of classes up to conjugacy (resp. isomorphism) is confirmed by MAGMA for $q = 9, 25, 49$.

2.2 Let us now consider the case where $G_1 \cong G_0 \cong \text{PGL}(2, p^n)$.

In this situation there are two conjugacy classes of $\text{PGL}(2, p^n)$ and also two conjugacy classes of $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^{2n})$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\text{PGL}(2, p^n)$ in $\text{PSL}(2, p^{2n})$ that have the subgroup $E_{p^n} : (p^n - 1)$ in common. Since $\text{PSL}(2, p^{2n})$ is simple and $\text{PGL}(2, p^n)$ is maximal, $\text{PGL}(2, p^n)$ is self-normalized. The subgroup $E_{p^n} : (p^n - 1)$ is also its own normalizer.
in $\text{PGL}(2, p^n)$ and in $\text{PSL}(2, p^{2n})$. Therefore the number of subgroups $\text{PGL}(2, p^n)$ containing a given subgroup $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^{2n})$ is equal to

$$\frac{|\text{PSL}(2, p^{2n})|}{|\text{PGL}(2, p^n)|} \cdot \frac{|\text{PGL}(2, p^n)|}{|E_{p^n} : (p^n - 1)|} \cdot \frac{|E_{p^n} : (p^n - 1)|}{|\text{PSL}(2, 2^{2n})|} = 1.$$ 

Now all $E_{p^n} : (p^n - 1)$ in $\text{PSL}(2, p^n)$ are conjugate. This implies that the RWPRI and $(2T)_1$ geometry $\Gamma\left(\text{PSL}(2, p^{2n}); \text{PSL}(2, p^n); \text{PSL}(2, p^n); E_{p^n} : (p^n - 1)\right)$ does not exist.

Notice that in the particular case where $p^n = 3$ and thus $G_1 \cong S_4 \cong \text{PGL}(2, 3)$ the geometry does not exist.

**Subcase 3:** $G_{01} = G_0 \cap G_1 \cong \text{PGL}(2, p^n)$.

By Lemma 27 the possibilities for $G_1$ are $A_5$ provided $p^n = 3$, $\text{PGL}(2, p^n)$. Notice that $S_4$ is a particular case of $\text{PGL}(2, p^n)$ provided $p^n = 3$.

3.1. Consider the case where $G_1 \cong A_5$ when $p^n = 3$.

There are two conjugacy classes of $\text{PGL}(2, 3) \cong S_4$, of $A_4$ and of $A_5$ in $\text{PGL}(2, 9)$. All $A_4$ in $A_5$ are conjugate, it is also the case for all $A_4$ in $S_4$. Since $\text{PGL}(2, 9)$ is simple and both $S_4$ and $A_5$ are maximal, $S_4$ and $A_5$ are self-normalized. The normalizer of $A_4$ in $\text{PGL}(2, 9)$ and in $S_4$ is $A_5$. $A_4$ is self-normalized in $A_5$. The number of subgroups $A_5$ containing a given subgroup $A_4$ in $\text{PGL}(2, 9)$ is equal to

$$\frac{|\text{PSL}(2, q)|}{|A_5|} \cdot \frac{|A_5|}{|A_4|} \cdot \frac{|S_4|}{|\text{PSL}(2, q)|} = 2.$$ 

To count the geometries up to conjugacy we need to know if the $S_4$ normalizes each of the $A_5$ which is not the case because $|N_{\text{PGL}(2, q)}(A_4) \cap N_{\text{PGL}(2, q)}(S_4)| = |S_4| = 2|A_4|$. Therefore, up to conjugacy there exist exactly two RWPRI and $(2T)_1$ geometries $\Gamma_2 = \Gamma\left(\text{PGL}(2, 9); S_4, A_5, A_4\right)$.

Let us deal with the fusion of non-conjugate classes. Following Lemma 11 the two classes of $A_4$, $S_4$ and $A_5$ are fused under the action of $\text{PGL}(2, 9)$ and thus also under the action of $\text{PIL}(2, 9)$. Therefore, there exists exactly one RWPRI and $(2T)_1$ geometry $\Gamma_2 = \Gamma\left(\text{PGL}(2, 9); S_4, A_5, A_4\right)$ up to isomorphism. This is confirmed by [3].

3.2 Consider the case where $G_1 \cong G_0 \cong \text{PGL}(2, p^n)$.

In this situation there are two conjugacy classes of $\text{PGL}(2, p^n)$ and also two conjugacy classes of $\text{PSL}(2, p^n)$ in $\text{PSL}(2, p^{2n})$. We must check whether this geometry exists, that is whether there are two subgroups isomorphic to $\text{PGL}(2, p^n)$ in $\text{PSL}(2, p^{2n})$ that have the subgroup $\text{PSL}(2, p^n)$ in common. Since $\text{PSL}(2, p^{2n})$ is simple and $\text{PGL}(2, p^n)$ maximal, $\text{PGL}(2, p^n)$ is self-normalized. The normalizer of the subgroup $\text{PSL}(2, p^n)$ in $\text{PGL}(2, p^n)$ and in $\text{PSL}(2, p^{2n})$ is $\text{PGL}(2, p^n)$. Therefore the number of $\text{PGL}(2, p^n)$ containing a given $\text{PSL}(2, p^n)$ is one.

Now all $\text{PSL}(2, p^n)$ in $\text{PGL}(2, p^n)$ are conjugate, which implies that the RWPRI and $(2T)_1$ geometry $\Gamma\left(\text{PSL}(2, p^{2n}); \text{PGL}(2, p^n); \text{PGL}(2, p^n); \text{PGL}(2, p^n)\right)$ does not exist.

Notice that in the particular case where $p^n = 3$ we get $G_1 \cong S_4 \cong \text{PGL}(2, 3)$.

**Subcase 4:** $G_{01} = G_0 \cap G_1 \cong S_4$, provided $q = 5^2$.

By Lemma 28 the only case to consider is $G_0 \cong G_1 \cong \text{PGL}(2, 5)$. 

In this situation where \( q = 25 \), there are two conjugacy classes of \( \text{PGL}(2, 5) \) and also two conjugacy classes of \( S_4 \) in \( \text{PSL}(2, 5^2) \). Since \( \text{PSL}(2, 5^2) \) is simple and \( \text{PGL}(2, 5) \) is maximal, \( \text{PGL}(2, 5) \) is self-normalized and \( S_4 \) is self-normalized in \( \text{PGL}(2, 5) \) and also in \( \text{PSL}(2, 5^2) \). Therefore the number of \( \text{PGL}(2, 5) \) containing a given \( S_4 \) is one. Now all \( S_4 \) in \( \text{PGL}(2, 5) \) are conjugate, which implies that the RWPRI and \((2T)_1\) geometry \( \Gamma (\text{PSL}(2, 5^2); \text{PGL}(2, 5); \text{PGL}(2, 5); S_4) \) does not exist. \( \square \)

**Case of Table 3, geometry \( \Gamma_2 \)**

We know that \( s \geq 2 \). Consider a path \((a, b, c)\) as in the preceding case. Here, \( G_{abc} = Z_3 \). This acts on the three 1-elements \( d_1, d_2, d_3 \) other than \( b \) in \( c^\perp \). The action is transitive since otherwise \( Z_3 \) would be in the kernel of the action of \( G_c \) on \( c^\perp \). This kernel for the action of \( S_4 \) on the cosets of \( D_6 \) is reduced to the identity, a contradiction. This provides \( s \geq 3 \) for paths starting at a 0-arc.

Next consider a path \((h, i, j)\) as in the preceding case. Here, \( G_{hij} = Z_2 \). This acts on the two 0-elements \( k_1, k_2 \) other than \( i \) in \( j^\perp \). The action is transitive since otherwise \( Z_2 \) would be in the kernel of the action of \( G_j \) on \( j^\perp \). This kernel for the action \( D_{18} \) on the cosets of \( D_6 \) is a group \( Z_3 \), a contradiction. Hence \( s \geq 3 \).

Applying Leemans’ method we get \( s = 2 \) or 3. Thus \( s = 3 \).

**Case of Table 3, geometry \( \Gamma_5 \)**

This geometry \( \Gamma (\text{PSL}(2, q); D_{16}, S_4, D_8) \) is known as a locally 7-arc-transitive graph due to Wong [22], hence \( s = 7 \).

**Case of Table 3, geometry \( \Gamma_7 \) and \( \Gamma_8 \).**

This geometry \( \Gamma (\text{PSL}(2, q); S_4, S_4, D_8) \) is known as a locally 4-arc-transitive graph due to Biggs-Hoare [1], hence \( s = 4 \) in this case.

**Case of Table 4, geometry \( \Gamma_1 \)**

We know that \( s \geq 2 \). Consider a path \((a, b, c)\) as in the preceding case. Here, \( G_{abc} = 2^n \). This acts on the \( 2^n \) elements of type 1, \( d_1, ..., d_{2^n} \) other than \( b \) in \( c^\perp \). The action is transitive since otherwise a subgroup of order 2 would be in the kernel of the action of \( G_c \) on \( c^\perp \). This kernel for the action of \( \text{PSL}(2, 2^n) \) on the cosets of \( 2^n : (2^n - 1) \) is reduced to the identity, a contradiction. This provides \( s \geq 3 \) for paths starting at a 0-arc.

Next consider a path \((h, i, j)\) as in the preceding case. Here, \( G_{hij} = Z_{2^{n-1}} \). This acts on the \( 2^n - 1 \) elements of type 0, \( k_1, k_{2^n-1} \) other than \( i \) in \( j^\perp \). The action is transitive since otherwise \( \text{PSL}(2, 2^n) \) with \( t \) prime and dividing \( 2^n - 1 \) would be in the kernel of the action of \( G_j \) on \( j^\perp \). This kernel for the action of \( 2^{2^n} : (2^n - 1) \) on the cosets of \( 2^n : (2^n - 1) \) is not determined but its order divides \( 2^n \), a contradiction. Hence \( s \geq 3 \).

Applying Leemans’ method we get \( s = 2 \) or 3. Thus \( s = 3 \).

**Case of Table 6, geometry \( \Gamma_1 \)**

We know that \( s \geq 2 \). Consider a path \((a, b, c)\) as in the preceding case. Here, \( G_{abc} = p^n \). This acts on the \( p^n \) elements of type 1, \( d_1, ..., d_{p^n} \) other than \( b \) in \( c^\perp \). The action is
transitive since otherwise a subgroup of order $p$ would be in the kernel of the action of $G_c$ on $c^\perp$. This kernel for the action of $PGL(2, 2^n)$ on the cosets of $p^n : (p^n - 1)$ is reduced to the identity, a contradiction. This provides $s \geq 3$ for paths starting at a $0$ element.

Next consider a path $(h, i, j)$ as in the preceding case. Here, $G_{hij} = Z_{p^n-1}$. This acts on the $p^n - 1$ elements of type $0, k_1, k_{p^n-1}$ other than $i$ in $j^\perp$. The action is transitive since otherwise $Z_t$ with $t$ prime and dividing $p^n - 1$ would be in the kernel of the action of $G_j$ on $j^\perp$. This kernel for the action of $p^{2n} : (p^n - 1)$ on the cosets of $p^n : (p^n - 1)$ is not determined but its order divides $p^n$, a contradiction. Hence $s \geq 3$.

Applying Leemans’ method we get $s = 2$ or $3$. Thus $s = 3$. 