Improving upper bounds for the distinguishing index

Monika Piłśniak *

AGH University, Department of Discrete Mathematics,
al. Mickiewicza 30, 30-059 Krakow, Poland

Received 27 November 2015, accepted 13 February 2017, published online 6 March 2017

Abstract

The distinguishing index of a graph G , denoted by $D'(G)$, is the least number of colours in an edge colouring of G not preserved by any non-trivial automorphism. We characterize all connected graphs G with $D'(G) \geq \Delta(G)$. We show that $D'(G) \leq 2$ if G is a traceable graph of order at least seven, and $D'(G) \leq 3$ if G is either claw-free or 3-connected and planar. We also investigate the Nordhaus-Gaddum type relation: $2 \leq D'(G) + D'(\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$ and we confirm it for some classes of graphs.

Keywords: Edge colouring, symmetry breaking in graph, distinguishing index, claw-free graph, planar graph.

Math. Subj. Class.: 05C05, 05C10, 05C15, 05C45

1 Introduction

We follow standard terminology and notation of graph theory (cf. [12]). In this paper, we consider general, i.e. not necessarily proper, edge colourings of graphs. Such a colouring f of a graph G breaks an automorphism $\varphi \in \text{Aut}(G)$ if φ does not preserve colours of f . The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G admits an edge colouring with d colours that breaks all non-trivial automorphisms (such a colouring is called a *distinguishing edge d -colouring*). Clearly, $D'(K_2)$ is not defined, so in this paper, a graph G is called *admissible* if neither G nor $\overline{G}$ contains K_2 as a connected component.

The definition of $D'(G)$ introduced by Kalinowski and Piłśniak in [17] was inspired by the *distinguishing number* $D(G)$ which was defined for general vertex colourings by Albertson and Collins [1]. Another concept is the *distinguishing chromatic number* $\chi_D(G)$

*The research was partially supported by the Polish Ministry of Science and Higher Education. The author would like to thank an anonymous reviewer whose remarks and comments have been a great help in preparing the final version.

E-mail address: pilsniak@agh.edu.pl (Monika Piłśniak)

introduced by Collins and Trenk [7] for proper vertex colourings. Both numbers, $D(G)$ and $\chi_D(G)$, have been intensively investigated by many authors in recent years [4, 5, 6, 9, 16].

Our investigation was motivated by the renowned result of Nordhaus-Gaddum [18] who proved in 1956 the following lower and upper bounds for the sum of the chromatic numbers of a graph and its complement (actually, the upper bound was first proved by Zykov [22] in 1949).

Theorem 1.1 ([18]). *If G is a graph of order n with the chromatic number $\chi(G)$, then*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1.$$

Since then, Nordhaus-Gaddum type bounds were obtained for many graph invariants. An exhaustive survey is given in [2]. Here, we adduce only those closely related to the topic of our paper.

In 1964, Vizing [20] considered proper edge colourings and he proved Nordhaus-Gaddum type bounds for the chromatic index of a graph.

Theorem 1.2 ([20]). *If G is a graph of order n with the chromatic index $\chi'(G)$, then*

$$n - 1 \leq \chi'(G) + \chi'(\overline{G}) \leq 2(n - 1).$$

In 2013, Collins and Trenk [8] proved Nordhaus-Gaddum type inequalities for the distinguishing chromatic number.

Theorem 1.3 ([8]). *For every graph of order n and distinguishing number $D(G)$ the following inequalities are satisfied*

$$2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) \leq n + D(G).$$

Kalinowski and Piłśniak [17] also introduced a *distinguishing chromatic index* $\chi'_D(G)$ of a graph G as the least number of colours in a proper edge colouring that breaks all non-trivial automorphisms of G . They proved the following somewhat unexpected result.

Theorem 1.4 ([17]). *If G is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

unless $G \in \{C_4, K_4, C_6, K_{3,3}\}$ when $\chi'_D(G) \leq \Delta(G) + 2$.

The following Nordhaus-Gaddum type inequalities for the distinguishing chromatic index are the same as in Theorem 1.2 but we have to be more careful in the proof.

Theorem 1.5. *If G is an admissible graph of order $n \geq 3$, then*

$$n - 1 \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq 2(n - 1)$$

with the only exception $K_{1,4}$.

Proof. Without loss of generality we may assume that G is connected. It can be easily checked that the conclusion holds if $G \in \{K_4, C_6, \overline{C}_6, K_{3,3}\}$. Otherwise, $\chi'_D(G) \leq \Delta(G) + 1$. Suppose first that $\overline{G}$ is also connected. By Theorem 1.4,

$$\Delta(G) + \Delta(\overline{G}) \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq \Delta(G) + \Delta(\overline{G}) + 2.$$

Clearly, $n - 1 \leq \Delta(G) + \Delta(\overline{G}) \leq 2(n - 2)$ since both G and $\overline{G}$ are connected.

Now, let $\overline{G}$ be disconnected (but admissible). If there are two nonisomorphic components of $\overline{G}$ of orders k_1 and k_2 such that $3 \leq k_1 \leq k_2$, then $\Delta(\overline{G}) \leq n - k_1 - 1 \leq n - 4$, so $\chi'_D(\overline{G}) \leq n - 2$. If $\overline{G}$ has $t \geq 2$ components isomorphic to a graph H of order at least three, then $\chi'_D(H) \leq \frac{n}{t} + 1$ as $\Delta(H) \leq \frac{n}{t} - 1$. Even if we wastefully add an extra colour for each additional copy of H , we get $\chi'_D(tH) \leq \frac{n}{t} + 1 + t - 1 = \frac{n}{t} + t \leq n - 2$ unless $G = K_{3,3}$ but this we already checked.

To complete the proof it is enough to settle the case when $\overline{G}$ has only one component H of order at least three and some isolated vertices. Hence, $\Delta(H) \leq n - 2$. It is easy to check that $\chi'_D(G) + \chi'_D(\overline{G}) \leq 2(n - 1)$ for $H \in \{K_4, C_6, \overline{C}_6, K_{3,3}\}$ except for $H = K_4$ when $G = K_{1,4}$. Otherwise, $\chi'_D(\overline{G}) \leq n - 1$ and the conclusion holds unless $|G| = |H| + 1$ and $\Delta(H) = n - 2$. But then G has a unique vertex x of degree $n - 1$ (hence, x is fixed by every automorphism of G) with a pendant edge. The graph $G - x$ has a distinguishing colouring with $n - 1$ colours by Theorem 1.4 since $\Delta(G - x) \leq n - 2$. It suffices to colour the pendant edge with a colour missing at x to see that $\chi'_D(G) \leq n - 1$. $\square$

Collins and Trenk observed in [8] that the Nordhaus-Gaddum type relation is trivial for the distinguishing number, as $D(G) + D(\overline{G}) = 2D(G)$ since $\text{Aut}(\overline{G}) = \text{Aut}(G)$ and every colouring of $V(G)$ breaking all non-trivial automorphisms of G also breaks those of $\overline{G}$.

In Section 4 we formulate and discuss the following conjecture.

Conjecture 1.6. *Let G be an admissible graph of order $n \geq 7$, and let $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$. Then*

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2.$$

In Section 2 we characterize graphs G which need exactly $\Delta(G)$ colours to break all non-trivial automorphisms. In Section 3 we give upper bounds for the distinguishing index of traceable graphs, claw-free graphs, planar graphs and 2-connected graphs.

2 Improved general upper bound

In the sequel, we make use of some facts proved in [17].

Proposition 2.1 ([17]). $D'(P_n) = 2$ for every $n \geq 3$.

Proposition 2.2 ([17]). $D'(C_n) = 3$ for $n \leq 5$, and $D'(C_n) = 2$ for $n \geq 6$.

Proposition 2.3 ([17]). $D'(K_n) = 3$ if $3 \leq n \leq 5$, and $D'(K_n) = 2$ if $n \geq 6$.

Proposition 2.4 ([17]). $D'(K_{3,3}) = 3$, and $D'(K_{n,n}) = 2$ if $n \geq 4$.

By the well-known theorem of Jordan (cf. [12]), every finite tree T has either a central vertex or a central edge, which is fixed by every automorphism of T . In the proof of Theorem 2.8, which is the main result of this section, we use Lemma 2.5, a simple generalization of the theorem of Jordan. Recall that the *eccentricity* of a vertex v in a connected graph G is the number

$$\varepsilon_G(v) = \max\{d(v, u) : u \in V(G)\}.$$

The *center* of a graph G is the set $Z(G)$ of vertices with minimum eccentricity. Clearly, the center of G is setwise fixed by every automorphism $\varphi \in \text{Aut}(G)$, i.e. $\varphi(v) \in Z(G)$ if $v \in Z(G)$. A proper subgraph H of G is called *pendant* if it has only one vertex adjacent to vertices outside H .

Lemma 2.5. *Let G be a connected graph such that every cycle is contained in a clique. Then the center of G is either a single vertex or a maximal clique.*

Proof. The claim is true if G is a clique K_k of order $k \geq 1$. Otherwise, $\kappa(G) = 1$, and each block of G is a clique of order at least two. We then modify the standard proof of the theorem of Jordan for trees. Let G^- be a graph obtained from G by deleting $k - 1$ vertices of degree $k - 1$ in every pendant clique K_k with $k \geq 2$. Clearly, $\varepsilon_{G^-}(v) = \varepsilon_G(v) - 1$ for each $v \in V(G^-)$. Consequently, $Z(G^-) = Z(G)$. We continue this process until only one clique K_k is left for some $k \geq 1$. This clique is maximal whenever $k \geq 2$. $\square$

A *symmetric tree*, denoted by $T_{h,d}$, is a tree with a central vertex v_0 , all leaves at the same distance h from v_0 and all vertices that are not leaves of equal degree d . A *bisymmetric tree*, denoted by $T''_{h,d}$, is a tree with a central edge e_0 , all leaves at the same distance h from the edge e_0 and all vertices which are not leaves of equal degree d .

Theorem 2.6 ([17]). *If T is a tree of order $n \geq 3$, then $D'(T) \leq \Delta(T)$. Moreover, equality is achieved if and only if T is either a symmetric or a bisymmetric tree.*

For connected graphs in general there is the following upper bound for $D'(G)$.

Theorem 2.7 ([17]). *If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

unless G is C_3 , C_4 or C_5 .

It follows for connected graphs that $D'(G) > \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and G is a cycle of length at most 5. The equality $D'(G) = \Delta(G)$ holds for cycles of length at least 6, for K_4 , $K_{3,3}$ and for all symmetric or bisymmetric trees. Now, we show that $D'(G) < \Delta(G)$ for all other connected graphs. A *palette* of a vertex is the multiset of colours of edges incident to it.

Theorem 2.8. *Let G be a connected graph that is neither a symmetric nor a bisymmetric tree. If the maximum degree of G is at least 3, then*

$$D'(G) \leq \Delta(G) - 1$$

unless G is K_4 or $K_{3,3}$.

Proof. Denote $\Delta = \Delta(G)$. The conclusion holds for trees due to Theorem 2.6. Then assume that G contains a cycle. The general idea of the proof is the following. If G does not contain a cycle of length greater than three, then we define G' as an empty graph. Otherwise, we consecutively delete pendant trees and pendant triangles until we obtain a subgraph G' . Then, we construct an edge colouring f with $\Delta - 1$ colours stabilizing all vertices of G' by every automorphism preserving f . Finally, we colour pendant subtrees and pendant triangles to complete a distinguishing colouring with $\Delta - 1$ colours of the whole graph G .

If $\Delta(G') = 2$, then G' is a cycle C_p having a distinguishing colouring with $\Delta - 1$ colours unless $p \in \{4, 5\}$ and $\Delta = 3$. In this case, it can be easily checked that the graph G'_+ induced by C_p and the independent edges of G incident to C_p can always be coloured with two colours such that the vertices of C_p are fixed by every colour preserving

automorphism. So we can assume that $\Delta(G') \geq 3$. If $G' \in \{K_4, K_{3,3}\}$, then $G' \neq G$ due to the assumption, hence $\Delta \geq 4$, so we can stabilize K_4 or $K_{3,3}$ with three colours.

Let $N_i(v)$ denote the i -th sphere in v , i.e. the set of vertices of distance i from the vertex v . Let x be a vertex with maximum degree in G' . We colour with 1 all edges incident with x . In our edge colouring f of the graph G' , the vertex x will be the unique vertex of maximum degree with the monochromatic palette $\{1, \dots, 1\}$. Hence, x will be fixed by every automorphism φ preserving f . Consequently, φ maps each sphere $N_i(x)$ onto itself.

The first sphere $N_1(x)$ can be partitioned into subsets M_k , for $k = 0, \dots, \Delta - 1$, defined as

$$M_k = \{v \in N_1(x) : |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \dots, v_{l_k}\}$. Thus, $l_0 + l_1 + \dots + l_{\Delta-1} = \Delta$.

We want to find a colouring f of the edges of $G'[N_1(x) \cup N_2(x)]$ and, if necessary, of some subsequent spheres, such that each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving this colouring. To do this, we proceed in a number of steps M_k , for $k = 0, \dots, \Delta - 1$. In each step M_k , we find a colouring that fixes the vertices of M_k and their neighbours in $N_2(x)$.

Step M_0 . First we consider the case when the subgraph $G'[M_0]$ induced by the vertices of M_0 is connected. Observe that $\Delta(G'[M_0]) \leq \Delta - 1$ and, by Theorem 2.7, we can colour distinguishingly the edges of $G'[M_0]$ with $\Delta - 1$ colours, even if $G'[M_0]$ is a short cycle C_p with $3 \leq p \leq 5$. Indeed, if $G'[M_0] = C_3$ and $\Delta = 3$, then we would have $G = K_4$, but K_4 is excluded. Otherwise, $\Delta \geq 4$ and we can use a third colour in a short cycle C_p . It may happen that there exists a vertex $v \in M_0$ of degree Δ in G' (so $|M_0| = \Delta$) with a monochromatic palette $\{1, \dots, 1\}$ in a colouring of $G'[M_0]$ given by Theorem 2.7. In this case, either G is a complete graph K_n with $n \geq 5$ so $D'(K_n) \leq \Delta - 1$ by Proposition 2.3, or it is not difficult to see that there exists a colour c such that there is no vertex with all incident edges coloured with c ; whence we can exchange c and 1 in this colouring of $G'[M_0]$.

Now, let $G'[M_0]$ be disconnected. Let $z_1, \dots, z_s$ be isolated vertices or end-vertices of isolated edges in $G'[M_0]$. Clearly, $s \leq \Delta - 1$ by the definition of G' . If $s = \Delta - 1$, then we colour with i every edge $z_i u$, where $u \in N_1(x) \setminus M_0$. Otherwise, we colour $z_i u$ with $i + 1$ for $i = 1, \dots, s$. Thus, we avoid a monochromatic palette of $\{1, \dots, 1\}$ at another vertex of maximum degree in G' .

We also have to distinguish all isomorphic components of $G'[M_0]$ of order greater than 2. Denote such a component by H and suppose that $G'[M_0]$ contains t components isomorphic to H , for some $t \geq 2$. Hence $t \leq \frac{\Delta}{3}$ and $\Delta(H) \leq \frac{\Delta}{t} - 1$. Therefore, we can choose distinct sets of $\frac{\Delta}{t}$ colours for every component since

$$\left(\frac{\Delta - 1}{\frac{\Delta}{t}}\right) \geq \left(\frac{\Delta - 1}{3}\right) \geq \frac{\Delta}{3} \geq t.$$

Thus each vertex of M_0 is fixed.

Step M_1 . For every $i = 1, \dots, l_1$, we colour the edge $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \dots, \Delta - 1\}$. This is impossible only if $l_1 = \Delta$, when we have to have two vertices $a, b \in M_1$ with the same colour of edges aa' and bb' , where a' and b' are neighbours of a and b in $N_2(x)$, respectively. If $G'[M_1]$ contains an edge e , then we colour it with 1, and all other edges of $G'[M_1]$ with 2. Then we choose exactly one of the vertices a, b incident to e . We proceed analogously when $G'[N_2(x)]$ contains an edge. Then all

vertices of M_1 are fixed unless $l_1 = \Delta$ and neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge.

If $|N_2(x)| = 1$, then G' is isomorphic to $K_{2,\Delta}$. It is easy to see that $D'(K_{2,\Delta}) \leq \Delta - 1$ for $\Delta \geq 3$ (for $\Delta \geq 4$ this immediately follows from Lemma 3.1 and Corollary 3.8). If $2 \leq |N_2(x)| \leq \Delta - 1$, then choosing a and b such that a' has at least two neighbours in $N_1(x)$ and $b' \neq a'$ yields a colouring fixing $N_1(x) \cup N_2(x)$.

Suppose $|N_2(x)| = \Delta$. If there is a vertex $v \in N_2(x)$ with less than $\Delta - 1$ neighbours in $N_3(x)$, then we choose a such that $a' = v$, and it suffices to reserve a unique set of colours for the edges between a' and $N_3(x)$.

Hence, assume that every vertex of $N_2(x)$ has $\Delta - 1$ neighbours in $N_3(x)$. We select two vertices $a, b \in M_1$ and assume that the colours of the edges aa' and bb' are the same. Next, we implement the following Procedure SUBTREES (a, b) , which we also use in subsequent steps.

Procedure SUBTREES (a, b)

We are given two vertices $a, b \in N_1(x)$ such that each their neighbour in $N_2(x)$ is adjacent to $\Delta - 1$ vertices of $N_3(x)$.

Let T_a be a maximal subtree of the graph $G'[\{a\} \cup \bigcup_{i \geq 2} N_i(x)]$, rooted at a , such that all leaves of T_a belong to the same sphere $N_{l-1}(x)$ and each vertex of $V(T_a) \cap N_{i-1}(x)$ has $\Delta - 1$ neighbours in $N_i(x)$ for $i = 3, \dots, l$. Thus $l \geq 3$. Define a graph

$$\widetilde{T}_a = G'[\bigcup_{v \in V(T_a) \setminus \{a\}} N(v)],$$

i.e. $\widetilde{T}_a$ is a graph obtained from T_a by adding all edges incident with the leaves of T_a . Analogously, we define a tree T_b and a graph $\widetilde{T}_b$. Observe that the trees T_a and T_b are disjoint and non-empty.

The edges incident to the roots a and b are already coloured. For every other vertex of T_a and T_b , we colour its incident edges going to the next sphere with distinct colours from $\{1, \dots, \Delta - 1\}$. Thus we obtain an edge colouring f . The only automorphism of T_a (as well as of T_b) preserving f is the identity. The vertex x will be fixed by every colour preserving automorphism φ . Consequently, φ maps $\widetilde{T}_a$ onto $\widetilde{T}_b$ whenever $\varphi(a) = b$. Thus, if $\widetilde{T}_a$ and $\widetilde{T}_b$ are not isomorphic, then f distinguishes all vertices in $V(T_a) \cup V(T_b)$. Hence, assume that the rooted graphs $\widetilde{T}_a$ and $\widetilde{T}_b$ are isomorphic. Observe that there exists exactly one non-trivial isomorphism $\psi_0: V(T_a) \rightarrow V(T_b)$ preserving f since each vertex in T_a has a distinct coloured path from the root a .

Denote $W_l = (V(\widetilde{T}_a) \cup V(\widetilde{T}_b)) \cap N_l(x)$. By our choice of G' , all vertices in W_l are of degree at least two in G' . It follows that one of the following three cases has to hold.

Case 1. There exist vertices in W_l adjacent to more than one vertex of W_{l-1} . Then we modify f by colouring again all edges between such vertices and W_{l-1} in order to break any possible permutation of W_l . A permutation of a set $L \subseteq W_l$ can be extended to an automorphism of G' that fixes all leaves of $\widetilde{T}_a \cup \widetilde{T}_b$ only if every vertex from L have the same set of neighbours $U = \{u_1, \dots, u_d\}$ in W_{l-1} . Such a set L contains at most $\Delta - 1$ leaves since the number of edges joining U to W_l equals $d(\Delta - 1)$. Every permutation of L will be broken whenever for every vertex $w \in L$ the multiset of colours of the edges $wu_1, \dots, wu_d$ will be distinct. Clearly, $d \leq \Delta$. There are $\binom{\Delta+d-2}{d}$ such possible multisets of $\Delta - 1$ colours. Clearly, $\binom{\Delta+d-2}{d} - 1 \geq \Delta - 1$ for $\Delta \geq 3$ and $d \geq 2$. We can exclude a

Figure 1: An example of the subgraph $\tilde{T}_a$ for $\Delta = 4$ and $l = 4$. The edges of $\tilde{T}_a$ between W_3 and W_4 that do not belong to the tree T_a are dashed.

rainbow multiset $P = \{1, \dots, d\}$ (or an almost rainbow multiset $P = \{1, \dots, \Delta-1, \Delta-1\}$ if $d = \Delta$) and we still have enough multisets to colour the edges incident with vertices of L . Moreover, for $d = \Delta$ we can also exclude a monochromatic palette $\{1, \dots, 1\}$ since $\binom{2\Delta-2}{\Delta} - 2 \geq \Delta - 1$ for $\Delta \geq 3$.

We partition the set W_l into maximal subsets L with the same set of neighbours and assign suitable multisets of colours to each set L . We thus obtain a colouring fixing all vertices from W_l unless ψ_0 can be extended to an isomorphism $\tilde{\psi}_0$ of $\tilde{T}_a$ onto $\tilde{T}_b$ preserving this colouring. To break every such possible extension $\tilde{\psi}_0$, it suffices to assign the excluded multiset P to one vertex of one set L .

Case 2. Every vertex in W_l has only one neighbour in W_{l-1} and the set of edges $F = E(G'[W_l])$ is non-empty. Then we colour one edge of F with 1, and all other edges in F with 2. This colouring fixes all vertices of $\tilde{T}_a$ and $\tilde{T}_b$ unless all edges in F are of the form $w\tilde{\psi}_0(w)$, where $w\tilde{\psi}_0(w)$ is one of possible extensions of ψ_0 to an isomorphism of $\tilde{T}_a$ onto $\tilde{T}_b$. In such a case, we choose one edge $ww' \in F$ and exchange colours of the edge wu , where $u \in W_{l-1}$, with another edge between u and W_l .

Case 3. Every vertex in W_l has only one neighbour in W_{l-1} and no neighbours in W_l . By the maximality of the trees T_a and T_b and the definition of G' , each vertex in W_l has at least one neighbour in $N_{l+1}(x)$ and there exists a vertex $w_0 \in W_l$ with $s < \Delta - 1$ neighbours $y_1, \dots, y_s \in N_{l+1}(x)$. We colour each edge w_0y_j with colour $j + 1$ for $j = 1, \dots, s$. Next, for every vertex $w \in W_l$, we colour the set of edges between w and $N_{l+1}(x)$ with a set of $\Delta - 1$ colours excluding the set $\{2, \dots, s + 1\}$.

We thus obtained a colouring f of the edges of $G'[V(\tilde{T}_a) \cup V(\tilde{T}_b)]$, and the edges incident to W_l in Case 3, fixing all vertices of $\tilde{T}_a$ and $\tilde{T}_b$.

End of Procedure SUBTREES (a, b)

Step M₂. For every $i = 1, \dots, l_2$, we colour the edges $v_iu_i^1, v_iu_i^2$ where $\{u_i^1, u_i^2\} \subseteq N_2(x)$, with distinct sets of colours from among $\binom{\Delta-1}{2}$ sets. This is impossible only in the following three cases (in each case, we can assume that neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge, otherwise we could construct a distinguishing colouring f of $G'[N_1(x) \cup N_2(x)]$ analogously as in step M_1):

- a) $l_2 = \Delta = 4$. If there exist two vertices a and b in M_2 such that $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour with 2 both edges incident with b , and for the remaining vertices in M_2 we have distinct sets of colours from among $\binom{3}{2}$ sets. If for every two vertices $a, b \in M_2$, the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then two vertices a and b are assign the same pair of distinct colours, and we can distinguish them in next spheres using the procedure SUBTREES (a, b) .
- b) $l_2 = \Delta - 1$ and $\Delta = 3$. Let $M_2 = \{a, b\}$. If $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour edges incident with a with colours 1 and 2, and both edges incident with b with 2. If the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then a and b get the same pair of distinct colours and we can distinguish them in next spheres by the procedure SUBTREES (a, b) .
- c) $l_2 = \Delta = 3$. Let $M_2 = \{a, b, c\}$. If for two vertices of M_2 , say a and b , the set $N(a) \cap N(b) \cap N_2(x)$ is non-empty, then we can colour with 2 both edges incident with b and we colour edges incident with the remaining vertices of M_2 with a couple $\{1, 2\}$. It is not difficult to verify that this way, for every configuration of neighbours of M_2 , we can obtain colouring fixing the vertices of $N_1(x) \cup N_2(x)$ unless $|N(a) \cap N(b) \cap N(c) \cap N_2(x)| = 2$. But then $G' = G = K_{3,3}$, contrary to the assumption. If every vertex of $N_2(x)$ is adjacent only to one vertex of M_2 , then the pairs of edges incident to a and b are assign the same pair of colours $\{1, 2\}$, and we distinguish them using the procedure SUBTREES (a, b) . Both edges cu^1, cu^2 incident with c are coloured with 2, and to distinguish them, we split c into two vertices c^1 and c^2 , each joined by an edge coloured with 2 to u^1 and u^2 , respectively, and apply the procedure SUBTREES (c^1, c^2) .

Step M_k , for $k \geq 3$. For every $i = 1, \dots, l_k$, we colour the edges between v_i and $N_2(x)$ with distinct sets of k colours from among $\binom{\Delta-1}{k}$ sets. It is always possible whenever $\binom{\Delta-1}{k} \geq l_k$. This inequality does not hold only in two cases:

- a) $k = \Delta - 2$ and $l_k = \Delta$. In this case we define a colouring with $\Delta - 1$ colours like in step M_2 a). Namely, if either a vertex of M_k or its neighbour in $N_2(x)$ is adjacent to a vertex in the same sphere, then we can define a colouring fixing all these vertices analogously as in step M_1 and step M_2 . Also, if there are two vertices $a, b \in M_{\Delta-2}$ with a common neighbour in $N_2(x)$, we can assign the same palette to a and b as in the previous steps. Otherwise, two vertices $a, b \in M_{\Delta-2}$ are assign the same palette of $\Delta - 2$ colours and we distinguish them using Procedure SUBTREES (a, b) .
- b) $k = \Delta - 1$ and $l_k \geq 2$. Hence, $\Delta \geq 4$. For every $i = 1, \dots, l_k$, the set of edges between $v_i \in M_{\Delta-1}$ and $N_2(x)$ will be assign a distinct multiset P^i of colours from the set $\{1, \dots, \Delta - 1\}$, where only colour i appears twice. Moreover, one vertex can assign a rainbow palette $\{1, \dots, \Delta - 1\}$. Thus every vertex of $M_{\Delta-1}$ will have a distinct palette, and hence will be stabilized. To stabilize the two vertices of $N_2(x)$ joined to v_i by edges of colour i , we examine the vertices $v_1, \dots, v_{\Delta-1}$ of $M_{\Delta-1}$ in the following order.

First, we consider each vertex v_i that have a neighbour $w_i \in N_2(x)$ with at least one but at most $\Delta - 2$ neighbours in $N_3(x)$. We choose another neighbour $w'_i \in N_2(x)$ of v_i and assign two distinct sets of colours for the edges going to $N_3(x)$ from w_i and w'_i , respectively. We colour the edges $v_i w_i$ and $v_i w'_i$ with the same colour i . Thus all neighbours of v_i are stabilized.

In the next stage, we consider every vertex v_i with every neighbour in $N_2(x)$ adjacent to $\Delta - 1$ vertices of $N_3(x)$. We colour the set of edges between v_i and $N_2(x)$ with the palette P^i , where two edges $v_i u_1, v_i u_2$ are coloured with i . Then we delete v_i and introduce two vertices v_i^1, v_i^2 and edges $v_i^1 u_1$ and $v_i^2 u_2$ coloured with i . Then we use the procedure SUBTREES (v_i^1, v_i^2) to stabilize u_1 and u_2 .

Further, we consider each vertex v_i with a neighbour $w_i \in N_2(x)$ incident to an edge $w_i u$, where $u \in N_2(x)$. First, we look for such an edge $w_i u$, which is already coloured. If there is no such edge, we take an uncoloured $w_i u$ and colour it with colour 3. In both cases, we put colour i on the edge $v_i w_i$ and another edge $v_i w$ with $w \neq u$. After we examine each such vertex v_i , we colour with 2 all remaining edges contained in $N_2(x)$.

Finally, we are left with at most Δ vertices v_i such that every neighbour of v_i is adjacent only to (at least two) vertices of $N_1(x)$. We take a first such vertex v_i and assign colour i to two its incident edges $v_i w_i$ and $v_i w'_i$. Thus all neighbours of v_i are stabilized unless common neighbours of w_i and w'_i were not considered yet. Then we take such a neighbour v_j and colour its incident edges with the palette P^j such that the edges $v_j w_i$ and $v_j w'_i$ have distinct colours. We repeat this procedure until only one vertex of $M_{\Delta-1}$ is left. We put a rainbow palette $\{1, \dots, \Delta - 1\}$ on its incident edges.

After we accomplish steps $M_0, \dots, M_{\Delta-1}$, we colour all uncoloured edges in subgraphs $G'[N_1(x)]$ and $G'[N_2(x)]$ with 2. Each vertex of $N_1(x) \cup N_2(x)$ is now fixed by every automorphism preserving our colouring f of edges of $G'[\{x\} \cup N_1(x) \cup N_2(x)]$, and of some edges between next spheres, if the procedure SUBTREES was used.

Then we recursively colour all yet uncoloured edges incident to consecutive spheres $N_i(x)$ as follows: for $v \in N_i(x)$, $i \geq 2$, we colour all edges vu , where $u \in N_{i+1}(x)$, with distinct colours from $\{1, \dots, \Delta - 1\}$. This is always possible since every vertex of $N_i(x)$ has at most $\Delta - 1$ neighbours in $N_{i+1}(x)$. Finally, we colour all uncoloured edges with end-vertices in the same sphere with 2. Hence, all vertices of G' are fixed by any automorphism preserving our colouring f . It is also easily seen that the already coloured edges can save their colours. Moreover, it is not difficult to observe that x is the unique vertex of maximum degree with a monochromatic palette $\{1, \dots, 1\}$. Thus, the whole subgraph G' (or G'_+) is fixed.

To end the proof, we colour pendant trees and triangles deleted from G at the beginning. First assume that G' is not empty. Let $N_i(G')$, for $i \geq 0$, be the set of vertices of distance i from G' . Then we recursively colour the edges incident to consecutive spheres $N_i(G')$ in the following way: for $v \in N_i(G')$, $i \geq 0$, we colour all edges vu , where $u \in N_{i+1}(G')$, with distinct colours from $\{1, \dots, \Delta - 1\}$ and the remaining edges incident to v , contained in $N_i(x)$, with 2. Hence, all vertices of G will be fixed by any automorphism preserving our colouring f .

If G' is empty, then we start with the centre $Z(G)$ that is setwise fixed by every automorphism. It follows from Lemma 2.5 that $Z(G)$ either induces K_3 , or K_2 (not contained in K_3), or K_1 . Let first $Z(G)$ induce a triangle K_3 . If $\Delta = 3$, then we stabilize $Z(G)$ by colouring with two colours all edges incident with vertices of $Z(G)$. When $\Delta \geq 4$, we can colour the edges of the triangle $Z(G)$ with three colours. Next, we recursively colour edges incident to subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours.

If $Z(G)$ is an edge e , then $G - e$ has two components. We distinguish each of them

by colouring subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours. If the components are isomorphic, then by assumption, each of them has a triangle. We colour two edges of these triangles contained in a sphere $N_i(Z(G))$, for some $i \geq 2$, with two distinct colours.

Finally, let $Z(G)$ be a single vertex z . Hence, $G - z$ has $q \geq 2$ components, each joined to z by one or two edges. If $q < \Delta$, then we can easily colour distinguishingly the edges incident with subsequent spheres $N_i(z)$, $i \geq 0$, with $\Delta - 1$ colours. If $q = \Delta$, then we choose two components of $G - z$, at least one of them with a triangle, and colour their two edges incident with z with the same colour. Then we distinguish these two components by an edge of the triangle. $\square$

3 Some classes of graphs

A graph G is called *asymmetric* if its automorphism group is trivial. Then obviously $D'(G) = 1$.

We say that a graph G is *almost spanned* by a subgraph H (not necessarily connected) if $G - v$ is spanned by H for some $v \in V(G)$. The following observation will play a crucial role in this section.

Lemma 3.1. *If a graph G is spanned or almost spanned by a subgraph H , then*

$$D'(G) \leq D'(H) + 1.$$

Proof. We colour the edges of H with colours $1, \dots, D'(H)$, and all other edges of G with an additional colour 0. If φ is an automorphism of G preserving this colouring, then $\varphi(x) = x$, for each $x \in V(H)$. Moreover, if H is a spanning subgraph of $G - v$, then also $\varphi(v) = v$. Therefore, φ is the identity. $\square$

3.1 Traceable graphs

Recall that a graph is *traceable* if it contains a Hamiltonian path.

Theorem 3.2. *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

Proof. Let $P_n = v_1 v_2 \dots v_n$ be a Hamiltonian path of G . If $G = P_n$, then the conclusion follows from Proposition 2.1. If G is isomorphic to $P_n + v_1 v_3$, then we colour the edge $v_1 v_3$ with 1, and all other edges with 2 breaking all non-trivial automorphisms of G . So suppose that G contains an edge $v_i v_j$ distinct from $v_1 v_3$ and $v_{n-2} v_n$ with $i < j - 1$. Without loss of generality we may assume that $i - 1 \leq n - j$ (otherwise we reverse the labeling). It is easy to see that at least one of the graphs $P_n + v_i v_j - v_{j-1} v_j$, $P_n + v_i v_j - v_{j-1}$ or $P_n + v_i v_j - v_n$ is an asymmetric spanning or almost spanning subgraph of G for any $n \geq 7$. The conclusion follows from Lemma 3.1. $\square$

The assumption $n \geq 7$ is substantial in Theorem 3.2 as $D'(K_{3,3}) = 3$.

3.2 Claw-free graphs

A $K_{1,3}$ -free graph, called also a *claw-free graph*, is a graph containing no copy of $K_{1,3}$ as an induced subgraph. Claw-free graphs have numerous applications, e.g., in operations research and scheduling theory. For a survey of claw-free graphs and their applications consult [10].

A k -tree of a connected graph is its spanning tree with maximum degree at most k . Win [21] investigated spanning trees in 1-tough graphs and proved the following result.

Theorem 3.3 ([21]). *A 2-connected claw-free graph has a 3-tree.*

We use this result to give an upper bound for the distinguishing number of claw-free graphs.

Theorem 3.4. *If G is a connected claw-free graph, then $D'(G) \leq 3$.*

Proof. Assume first that G is 2-connected. By Theorem 3.3, G contains a 3-tree T . By Theorem 2.6, we have $D'(T) \leq 2$ if T is neither symmetric nor bisymmetric tree. In such a case, $D'(G) \leq 3$ by Lemma 3.1.

Let T be a symmetric tree $T_{h,3}$. Denote a central vertex of T by x and its neighbours by a, b, c . Since G is a claw-free graph, there exists in G at least one edge, say bc , in the neighbourhood of x in T . Define a subgraph $\tilde{T} = T + bc$. We colour bc, xa and xb with 1, and xc with 2. Thus all vertices a, b, c, x are fixed by every non-trivial automorphism of $\tilde{T}$. We now colour the remaining edges in $\tilde{T}$ starting from the edges incident to a, b, c in such a way that two uncoloured adjacent edges obtain two different colours 1 and 2. This 2-colouring breaks all non-trivial automorphisms of $\tilde{T}$. Hence, $D'(G) \leq 3$ by Lemma 3.1.

Let T be a bisymmetric tree $T''_{h,3}$. Denote a central edge by xy and its neighbours by a, b adjacent to x , and c, d adjacent to y . We colour xy, xa and yc with 1, and xb and yd with 2. Since G is claw-free, there exists in G either at least one of the edges by, cx (or symmetrically dx or ay) or both ab and cd . We define a subgraph $\tilde{T}$ obtained from the tree T by adding either one of the edges by, cx (or symmetrically, dx or ay) or both ab and cd . In the first case we colour by or cx (or symmetrically, dx or ay) with 1, in the second case we colour ab with 1 and cd with 2. Now all vertices a, b, c, d, x, y are fixed by every non-trivial automorphism of $\tilde{T}$. We then colour the remaining edges of $\tilde{T}$ as above, and we obtain the claim.

If a graph G is not 2-connected, then its graph of blocks and cut-vertices is a path, since G is claw-free. We colour every block according to the rules described above. Then to break all non-trivial automorphisms of G , it is enough to break a possible automorphism $\psi \in \text{Aut}(G)$ that exchanges two terminal blocks. Let z be a cut-vertex that belongs to a terminal block B_0 . It follows that z and its neighbours in B_0 induce a clique K of order $k \geq 2$. We have three colours in our disposal, so it is easily seen that we can permute the colours to obtain a nonisomorphic colouring of K , thus breaking ψ . $\square$

The theorem is sharp for graphs of order at most 5. We conjecture that the distinguishing index of claw-free graphs of order big enough is 2.

3.3 Planar graphs

First, recall that by the famous Theorem of Tutte [19], every 4-connected planar graph G is Hamiltonian. Hence, its distinguishing index is at most 2, by Theorem 3.2, whenever $|G| \geq 7$. A similar result as for claw-free graphs we obtain for 3-connected planar graphs. In the proof, we use the following result of Barnette about spanning trees of such graphs.

Theorem 3.5 ([3]). *Every 3-connected planar graph has a 3-tree.*

Using a similar method as in the proof of Theorem 3.4, we obtain the following.

Theorem 3.6. *If G is 3-connected planar graph, then $D'(G) \leq 3$.*

Proof. Let T be a 3-tree of G . It follows from Theorem 2.6 that $D'(T) \leq 2$ and hence, $D'(G) \leq 3$ by Lemma 3.1, if T is neither a symmetric nor a bisymmetric tree.

Let then T be a symmetric tree $T_{h,3}$. Denote the central vertex by x , and by T_a, T_b and T_c the connected components of $T - x$ which are trees rooted at the neighbours a, b, c of a vertex x , respectively. Since G is 3-connected, there exist an edge e between T_a and T_b in G . Consider a spanning subgraph $\tilde{T} = T + e$. Then we colour xa and xc with 1, and xb with 2, and extend this colouring as in the proof of Theorem 3.4 to a colouring of $\tilde{T}$ breaking all non-trivial automorphisms of $\tilde{T}$ (the colour of e is irrelevant). Consequently, $D'(G) \leq 3$ by Lemma 3.1.

If T is a bisymmetric tree $T''_{h,3}$ with the central edge xy , then we can add to T one edge in a subtree of $T - xy$ rooted at x , and such a graph can be easily distinguished by two colours. Again, our claim follows from Lemma 3.1. $\square$

3.4 2-connected graphs

For a 2-connected planar graph G , the distinguishing index may attain $1 + \lceil \sqrt{\Delta(G)} \rceil$ as it is shown by the complete bipartite graph $K_{2,q}$ with $q = r^2$ for a positive integer r . In this case, $D'(K_{2,q}) = r + 1$ as it follows from the result obtained independently by Fisher and Isaak [11] and by Imrich, Jerebic and Klavžar [14]. They proved the following theorem. Actually, they formulated it for the distinguishing number $D(K_p \square K_q)$ of the Cartesian product of complete graphs, but $D'(K_{p,q}) = D(K_p \square K_q)$.

Theorem 3.7 ([11, 14]). *Let p, q, d be integers such that $d \geq 2$ and $(d-1)^p < q \leq d^p$. Then*

$$D'(K_{p,q}) = \begin{cases} d, & \text{if } q \leq d^p - \lceil \log_d p \rceil - 1, \\ d+1, & \text{if } q \geq d^p - \lceil \log_d p \rceil + 1. \end{cases}$$

If $q = d^p - \lceil \log_d p \rceil$ then the distinguishing index $D'(K_{p,q})$ is either d or $d+1$ and can be computed recursively in $O(\log^(q))$ time.*

In the next section, we make use of the following immediate corollary.

Corollary 3.8. *If $p \leq q$, then $D'(K_{p,q}) \leq \lceil \sqrt[3]{q} \rceil + 1$.*

In the proof of Proposition 3.10 we also make use of an earlier result of Imrich and Klavžar [15] which is a slightly weaker version of Theorem 3.7 for $d = 2$.

Theorem 3.9 ([15]). *If $2 \leq p \leq q \leq 2^p - p + 1$, then $D'(K_{p,q}) = 2$.*

Proposition 3.10. *If $p \leq q \leq 2^p - p + 1$ and $p + q \geq 7$, then there exists a distinguishing edge 2-colouring of $K_{p,q}$ such that the edges in one of colours induce a connected spanning or almost spanning, asymmetric subgraph of $K_{p,q}$.*

Proof. The assumptions imply that $p \geq 3$, and $D'(K_{p,q}) = 2$ by Theorem 3.9. Let P and Q be the two sets of bipartition of $K_{p,q}$ with $|P| = p$ and $|Q| = q$. If $p = q$, then $p \geq 4$, and there exists a spanning asymmetric tree of $K_{p,p}$ (see [17]). If $p < q \leq 2^p - p + 1$, then for the proof of Theorem 3.9, Imrich and Klavžar in [15] constructed a distinguishing vertex 2-colouring of $K_p \square K_q$ that corresponds to a distinguishing edge 2-colouring f of $K_{p,q}$, where a colouring of vertices in a K_q -layer can be represented by a sequence from $\{1, 2\}^q$ and it corresponds to a colouring of edges incident to a vertex in P (the same is true

for K_p -layers and vertices in Q). We wish to show that this colouring yields a connected asymmetric subgraph of $K_{p,q}$ which is spanning or almost spanning.

First assume that $q = 2^p - p + 1$. In the coloring f , every vertex in P has distinct positive number of edges coloured with 1, and there exists a vertex v_1 with all incident edges coloured with 1. Moreover, distinct vertices from Q have distinct sets of neighbours joined by edges coloured with 1, and there exists a vertex, say v_2 , with all incident edges coloured with 2. Let S be a subgraph induced by edges coloured with 1. Then S is an almost spanning subgraph since v_2 is the only vertex outside S . The graph S is connected because v_1 is adjacent to every vertex in Q , and every vertex in P is joined to a vertex in Q by an edge coloured with 1. Moreover, S is also asymmetric since f breaks all non-trivial automorphisms of $K_{p,q}$ and any automorphism interchanging some parts of the sets P and Q does not preserve distances in S .

Following [15] for $p < q < 2^p - p + 1$, we exclude a relevant number of such pairs of sequences of colours that the sum of them is a sequence $(3, \dots, 3)$. Additionally, if both q and p are odd, we exclude the sequence $(0, \dots, 0)$. Again, we obtain a connected spanning (or almost spanning) asymmetric subgraph S of $K_{p,q}$ induced by the edges coloured with 1. $\square$

Proposition 3.10 and Lemma 3.1 immediately imply the following.

Corollary 3.11. *If a graph G of order at least 7 is spanned by $K_{p,q}$ and $p \leq q \leq 2^p - p + 1$, then $D'(G) \leq 2$.*

In general, for 2-connected graphs we conjecture that the complete bipartite graph K_{2,r^2} is the worst case, i.e. attains the highest value of the distinguishing index.

Conjecture 3.12. *If G is a 2-connected graph, then*

$$D'(G) \leq 1 + \left\lceil \sqrt{\Delta(G)} \right\rceil.$$

4 Nordhaus-Gaddum inequalities for D'

In this section, we discuss Conjecture 1.6, formulated at the end of Introduction, stating that

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2$$

for every admissible graph G of order $n \geq 7$, where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

The left-hand inequality is obvious. Indeed, if a graph G is asymmetric, then so is $\overline{G}$. Thus we are only interested in the right-hand inequality $D'(G) + D'(\overline{G}) \leq \Delta + 2$. Note also that at least one of the graphs G and $\overline{G}$ is connected.

The bound $\Delta + 2$ cannot be improved. To see this, consider a star $K_{1,n-1}$ of any order $n \geq 7$. As $\overline{K_{1,n-1}}$ is a disjoint union of a complete graph K_{n-1} and an isolated vertex, it follows from Proposition 2.3 that $D'(\overline{K_{1,n-1}}) = 2$. Therefore, $D'(K_{1,n-1}) + D'(\overline{K_{1,n-1}}) = n - 1 + 2 = \Delta + 2$.

If T is a tree, then $\Delta(T)$ can be much smaller than $\Delta = \Delta(\overline{T}) = n - 1$. However, the following holds.

Proposition 4.1. *If T is a tree of order $n \geq 7$, then*

$$D'(T) + D'(\overline{T}) \leq \Delta(T) + 2.$$

Proof. As it was shown above, the conclusion holds for stars. If T is not a star, then $D'(\overline{T}) \leq 2$ by Lemma 3.1. Indeed, as it was proved by Hedetniemi et al. in [13], a complete graph K_n contains edge disjoint copies of any two trees of order n distinct from a star $K_{1,n-1}$. Thus, the complement $\overline{T}$ contains a spanning asymmetric tree. By Theorem 2.6, we have the inequality $D'(T) + D'(\overline{T}) \leq \Delta(T) + 2$. $\square$

This fact emboldened us to formulate the following stronger conjecture.

Conjecture 4.2. *Every connected admissible graph G of order $n \geq 7$ satisfies the inequality*

$$D'(G) + D'(\overline{G}) \leq \Delta(G) + 2.$$

Now we show that Conjecture 1.6 holds not only for trees, but also for some other classes of graphs. To do this we use the following fact.

Theorem 4.3. *Let G be a connected admissible graph of order $n \geq 7$. If either G or every connected component of $\overline{G}$ has the distinguishing index at most 3, then*

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. Our claim is true for trees by Proposition 4.1. Observe also, that it is true if G is a path or a cycle of order at least 7 since its complement $\overline{G}$ is Hamiltonian, and $D'(G) + D'(\overline{G}) \leq 4$. So, now we can assume that $\Delta(G) \geq 3$ and neither G nor $\overline{G}$ is a tree. We consider two cases.

Case A. Every component H of $\overline{G}$ satisfies $D'(H) \leq 3$.

Then $D'(G) \leq \Delta(G) - 1$ by Theorem 2.8, and if $\overline{G}$ is connected, then our claim holds. Assume now that $\overline{G}$ is disconnected. Then G is spanned by $K_{p,q}$ with $p \leq q$ and $\Delta \geq q$, where $p + q = |V(G)|$. Suppose that the graph $\overline{G}$ has t isomorphic components. If we had a distinct set of three colours for every component, then $D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil$. We then consider two cases:

- a) If $q \leq 2^p - p + 1$, then $D'(G) = 2$ by Corollary 3.11. Moreover, we then have at most $\frac{n}{3}$ components of $\overline{G}$, so $D'(\overline{G}) \leq \lceil \sqrt[3]{2n} \rceil$. And we can easily see that

$$\lceil \sqrt[3]{2n} \rceil + 2 \leq \frac{n}{2} + 2$$

for every $n \geq 4$.

- b) If $q \geq 2^p - p + 1$, then there exists a big component (of order q) in $\overline{G}$ and we can assume that $t \leq \frac{p}{3}$ remaining components are isomorphic. In this case, by assumptions we have $p \leq \lceil \log_2(q + p - 1) \rceil$, therefore

$$D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil \leq \sqrt[3]{2 \lceil \log_2(q + p - 1) \rceil}.$$

On the other hand, $D'(G) \leq \lceil \sqrt[p]{q} \rceil + 2$ by Corollary 3.8 and Theorem 3.1. Then it is not difficult to check that for $q \geq 2^p - p + 1$

$$\sqrt[3]{2 \lceil \log_2(q + p - 1) \rceil} + \lceil \sqrt[p]{q} \rceil + 2 \leq q + 2$$

what finishes the proof in Case A.

Case B. $D'(G) \leq 3$.

If graph $\overline{G}$ is connected, then the claim follows immediately from Theorem 2.7 whenever $D'(G) = 2$ or $D'(\overline{G}) = 2$, and it follows from Theorem 2.8 if $D'(G) = 3$. Assume now that $\overline{G}$ has $t \geq 2$ components. Then $\Delta \geq \frac{n}{2}$ and, in the worst case, all components of $\overline{G}$ are isomorphic. Observe that maximal degree of every component is at most $\frac{n}{t} - 1$. If we assign one extra colour to every component, then we need at most $\frac{n}{t} - 1 + (t - 1)$ colours to distinguish $\overline{G}$. Hence, if

$$\frac{n}{t} + t \leq \frac{n}{2} - 1,$$

then $D'(\overline{G}) \leq \Delta - 1$, and our claim is true. The above inequality holds unless $t = 2$.

If there exist two isomorphic components in $\overline{G}$, then $D'(G) \leq 2$ due to Corollary 3.11 since G is spanned by $K_{\frac{n}{2}, \frac{n}{2}}$. Then $D'(\overline{G}) \leq \frac{n}{2}$, and finally $D'(G) + D'(\overline{G}) \leq \frac{n}{2} + 2$. $\square$

Now we can formulate some consequences of Theorem 4.3 and suitable results proved in Section 3.

Corollary 4.4. *Let G be an admissible graph of order $n \geq 7$. If G satisfies at least one of the following conditions:*

- i) G is a traceable graph, or
- ii) G is a claw-free graph, or
- iii) G is a triangle-free graph, or
- iv) G is a 3-connected planar graph,

then

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. It suffices to apply Theorem 4.3 together with Theorem 3.2, Theorem 3.4 and Theorem 3.6, respectively. Observe also that if the girth of a graph G is at least 4, i.e., G is triangle-free, then its complement $\overline{G}$ is claw-free. $\square$

Finally, it has to be noted that there exist graphs of order less than 7 such that the right-hand inequality in Conjecture 1.6 is not satisfied. For example, for the graph $K_{3,3}$ we have $D'(K_{3,3}) = 3$, $D'(\overline{K_{3,3}}) = D'(2K_3) = 4$ and $\Delta = 3$, hence $D'(K_{3,3}) + D'(\overline{K_{3,3}}) = \Delta + 4$. Also, $D'(C_5) + D'(\overline{C_5}) = 3 + 3 = \Delta + 4$, and $D'(K_{1,i}) + D'(\overline{K_{1,i}}) = \Delta + 3$ for $i = 3, 4, 5$.

References

- [1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, *Electron. J. Combin.* **3** (1996), #R18, <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v3i1r18>.
- [2] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, *Discrete Appl. Math.* **161** (2013), 466–546, doi:10.1016/j.dam.2011.12.018.
- [3] D. W. Barnette, Trees in polyhedral graphs, *Canad. J. Math.* **18** (1966), 731–736, doi:10.4153/cjm-1966-073-4.

- [4] D. L. Boutin, The cost of 2-distinguishing Cartesian powers, *Electron. J. Combin.* **20** (2013), #P74, <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i1p74>.
- [5] J. O. Choi, S. G. Hartke and H. Kaul, Distinguishing chromatic number of Cartesian products of graphs, *SIAM J. Discrete Math.* **24** (2010), 82–100, doi:10.1137/060651392.
- [6] K. L. Collins, M. Hovey and A. N. Trenk, Bounds on the distinguishing chromatic number, *Electron. J. Combin.* **16** (2009), #R88, <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v16i1r88>.
- [7] K. L. Collins and A. N. Trenk, The distinguishing chromatic number, *Electron. J. Combin.* **13** (2006), #R16, <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v13i1r16>.
- [8] K. L. Collins and A. N. Trenk, Nordhaus-Gaddum theorem for the distinguishing chromatic number, *Electron. J. Combin.* **20** (2013), #P46, <http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i3p46>.
- [9] E. Estaji, W. Imrich, R. Kalinowski, M. Piłśniak and T. W. Tucker, Distinguishing Cartesian products of countable graphs, *Discuss. Math. Graph Theory* **37** (2017), 155–164, doi:10.7151/dmgt.1902.
- [10] R. Faudree, E. Flandrin and Z. Ryjáček, Claw-free graphs — a survey, *Discrete Math.* **164** (1997), 87–147, doi:10.1016/s0012-365x(96)00045-3.
- [11] M. J. Fisher and G. Isaak, Distinguishing colorings of Cartesian products of complete graphs, *Discrete Math.* **308** (2008), 2240–2246, doi:10.1016/j.disc.2007.04.070.
- [12] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Discrete Mathematics and Its Applications, CRC Press, Boca Raton, Florida, 2nd edition, 2011, doi:10.1201/b10959.
- [13] S. M. Hedetniemi, S. T. Hedetniemi and P. J. Slater, A note on packing two trees into K_n , *Ars Combin.* **11** (1981), 149–153.
- [14] W. Imrich, J. Jerebic and S. Klavžar, The distinguishing number of Cartesian products of complete graphs, *European J. Combin.* **29** (2008), 922–929, doi:10.1016/j.ejc.2007.11.018.
- [15] W. Imrich and S. Klavžar, Distinguishing Cartesian powers of graphs, *J. Graph Theory* **53** (2006), 250–260, doi:10.1002/jgt.20190.
- [16] W. Imrich, S. M. Smith, T. W. Tucker and M. E. Watkins, Infinite motion and 2-distinguishability of graphs and groups, *J. Algebraic Combin.* **41** (2015), 109–122, doi:10.1007/s10801-014-0529-2.
- [17] R. Kalinowski and M. Piłśniak, Distinguishing graphs by edge-colourings, *European J. Combin.* **45** (2015), 124–131, doi:10.1016/j.ejc.2014.11.003.
- [18] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, *Amer. Math. Monthly* **63** (1956), 175–177, doi:10.2307/2306658.
- [19] W. T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82** (1956), 99–116, doi:10.2307/1992980.
- [20] V. G. Vizing, The chromatic class of a multigraph, *Kibernetika* **1** (1965), 29–39, doi:10.1007/bf01885700.
- [21] S. Win, On a connection between the existence of k -trees and the toughness of a graph, *Graphs Combin.* **5** (1989), 201–205, doi:10.1007/bf01788671.
- [22] A. A. Zykov, On some properties of linear complexes, *Mat. Sbornik (N. S.)* **24** (1949), 163–188, <http://mi.mathnet.ru/eng/msb5974>.