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An infinite class of movable 5-configurations

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Abstract

A geometric 5-configuration is a collection of points and straight lines, typically in the Euclidean plane, in which every point has 5 lines passing through it and every line has 5 points lying on it; that is, it is an (n_5) configuration for some number n of points and lines. Using reduced Levi graphs and two elementary geometric lemmas, we develop a construction that produces infinitely many new 5-configurations which are movable; in particular, we produce infinitely many 5-configurations with one continuous degree of freedom, and we produce 5-configurations with $k-2$ continuous degrees of freedom for all odd $k>2$.

Keywords: Configurations, incidence geometry.

Math. Subj. Class.: 51A20, 51A45, 51E30, 05B30

A geometric k-configuration is a collection of points and straight lines, typically in the Euclidean plane, where every point lies on k lines and every line passes through k points. Geometric 3-configurations have been studied since the mid-1800s, and geometric 4-configurations since the late 1900s, with the first intelligible drawing of a 4-configuration appearing in a 1990 paper by Grünbaum and Rigby [15]. However, the situation for more *highly incident configurations*, that is, for (p_q, n_k) configurations with at least one of q, $k \geq 4$, is poorly understood in general.

Two constructions that produce infinite families of 5-configurations with a reasonably small number of points and lines are known [7, 9]. The (48_5) configuration shown in Figure 1a

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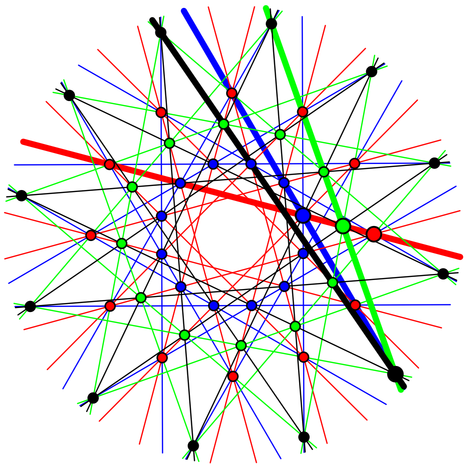
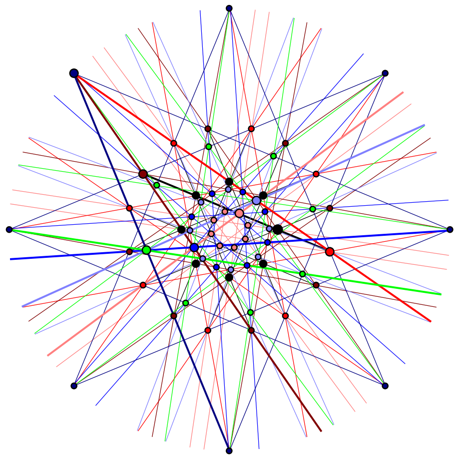
is the smallest known geometric 5-configuration and is the smallest example of the construction in [9]; a reasonably small example of the construction discussed in [7] is shown in Figure 1b (the smallest example is not intelligible at small scale). In his monograph on configurations [14, Section 4.1], Grünbaum spends only 5 pages (mostly pictures) discussing the little that is known about 5-configurations.



(a) A (48_5) configuration with 4 symmetry classes of points and lines



(b) A (64_5) configuration with 8 symmetry classes of points and lines

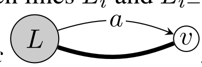
Figure 1: Examples of known small 5-configurations

In this paper, we present a new construction that produces infinitely many new geometric 5-configurations which are *movable*: that is, there is at least one continuous degree of freedom in the construction while fixing 4 points in general position. This construction significantly generalizes the construction presented in [9] and removes the need to complete the construction via a continuity argument, instead providing an entirely ruler-and-compass construction for those configurations, given an initial m-gon. The new construction technique uses two elementary geometric lemmas, the Circumcircle Construction Lemma and the Crossing Spans Lemma, which previously have been used separately in other configuration construction techniques.

1 Definitions; Levi and reduced Levi graphs

Given any (p_q, n_k) configuration, whether geometrically realizable or not, it is possible to construct a corresponding bipartite graph, called a *Levi graph*, which has one white vertex for each point of the configuration and one black vertex for each line of the configuration, with two vertices in the graph incident if and only if the corresponding point and line are incident in the configuration. More details on Levi graphs and configurations may be found in Grünbaum [14, Section 1.4] and Coxeter [12].

We say that a geometric k-configuration is *symmetric* if there exist non-trivial isometries of the Euclidean plane that map the configuration to itself. Note that in other places in the literature, the word ‘symmetric’ has been used to mean (p_q, n_k) configurations where $q = k$ (and thus $p = n$), i.e., k-configurations. Since we are interested in emphasizing the geometric nature of the configuration, we—following Grünbaum [14, p. 16]—refer to k-configurations as *balanced*, and reserve the word ‘symmetric’ to refer to the geometric structure. The *symmetry class* of an element (point or line) is the orbit of the element under the symmetry group of the configuration. If a geometric configuration has the property that every symmetry class under some fixed cyclic subgroup of the geometric symmetry group contains the same number of elements, then the configuration is called *polycyclic*; polycyclic configurations were first described by Boben and Pisanski [11].

Given a polycyclic geometric configuration with cyclic symmetry group $\mathbb{Z}_m$, it is possible to construct an edge-labelled bipartite graph, called the *reduced Levi graph*, by associating one vertex of the graph to each symmetry class of points and of lines in the configuration, and connecting two vertices of the graph with an edge precisely when the corresponding elements of the configuration are incident. Suppose the elements of each symmetry class of elements are labelled cyclically counterclockwise, beginning from some chosen 0th element in each class; for example, line class L is labelled $(L)_0, \ldots, (L)_{m-1}$ and vertex class v is labelled $(v)_0, \ldots, (v)_{m-1}$. If for each i, line L_i and vertex v_{i+a} are incident (with indices computed modulo m), the corresponding directed edge in the reduced Levi graph from vertex L to vertex v is labelled a; in the case where L_i and vertex v_i are incident (that is, where $a = 0$), then we use an undirected thick edge. When vertices v_i and v_{i+a} both lie on line L_i, or from an alternate point of view, when lines L_i and L_{i-a} intersect at point v_i, then the reduced Levi graph contains a *double arc* 

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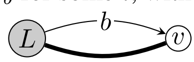
If p and q are any two points, we denote the line L passing through p and q as $p \vee q$. Similarly, if L and M are any two lines, we denote their point of intersection as $L \wedge M$ (possibly at infinity if $L \| M$). Given points $v_0, \ldots, v_{m-1}$ that form the vertices of a regular m-gon centered at $\mathcal{O}$, we say that a line is *span* b if it passes through v_i and v_{i+b} for some i, with all indices computed modulo m; span b lines correspond to double arcs 

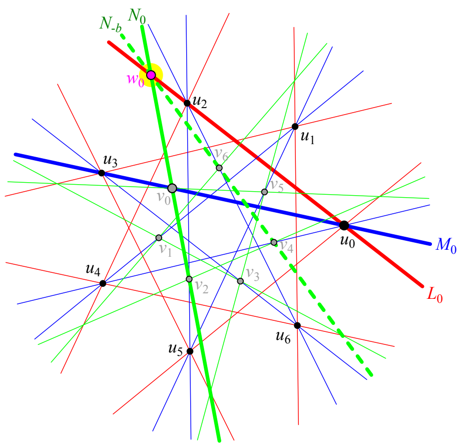
in the reduced Levi graph. A circle C is a *circumcircle of span* b if it passes through $v_i, \mathcal{O}$, and v_{i-b} for some i; to specify which i, we say that C is a *circumcircle of span* b *through* v_d. (Note that span b lines are constructed by moving counterclockwise from the initial point, and span b circumcircles by moving clockwise!)

2 Two construction lemmas

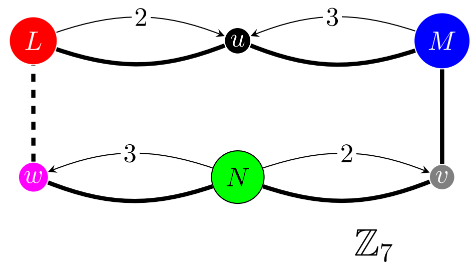
In 2006, one of the authors (LWB) discovered the Crossing Spans Lemma [3] (somewhat restated here):

Lemma 2.1 (Crossing Spans Lemma (CSL))**.** *Given a regular m-gon with vertices cyclically labelled as $u_0, u_1, \ldots, u_{m-1}$ and lines $L_i = u_i \vee u_{i+a}$ of span a and $M_i = u_i \vee u_{i+b}$ of span b, where $1 \leq a \neq b < \frac{m}{2}$, suppose that v_0 is an arbitrary point on M_0 (different from u_0, u_b to avoid degeneracies), and construct other points v_i to be the rotations of v_0 through $\frac{2\pi i}{m}$. Let $N_i = v_i \vee v_{i+a}$ and let $w_i = N_i \wedge N_{i-b}$. Then w_i also lies on L_i.*

Although easy to state and prove, the Crossing Spans Lemma has been used to produce a



(a) Illustrating the Crossing Spans Lemma; $m = 7, a = 2, b = 3$. Only point w_0 in class w has been shown, to better illustrate that the three lines L_i, N_{i-b}, and N_i really do intersect three at a time (that is, no almost-incidences are covered by points).



(b) The reduced Levi graph corresponding to Figure 2; the dashed edge corresponds to the forced incidence.

Figure 2: Illustrating the Crossing Spans Lemma

number of novel constructions for configurations [3, 5, 8, 9]. The Crossing Spans Lemma and its associated reduced Levi graph "gadget" are shown in Figure 2.

In fact, it is straightforward to show (by relabelling symmetry classes and applying duality arguments) that given either of the labelled subgraphs in a reduced Levi graph that are shown in Figure 3, the incidence given by the dashed line is induced, where white nodes correspond to point classes and gray nodes to line classes. These subgraphs, with various choices of labels, are used extensively in the proof of Theorem 4.1.

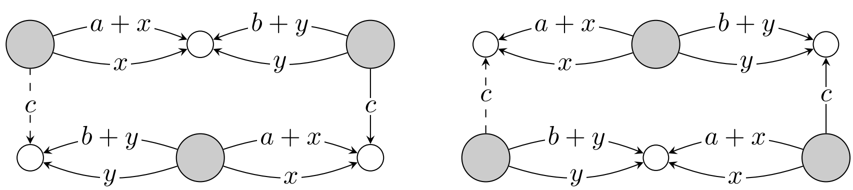
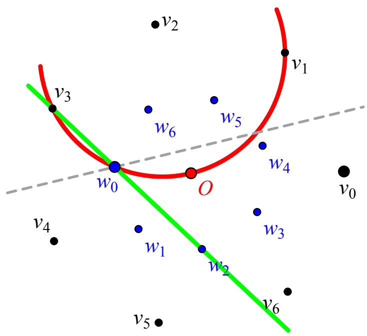


Figure 3: In either of these subgraphs in a reduced Levi graph (over $\mathbb{Z}_m$), the dashed line corresponds to a forced incidence via the CSL; , c, x, y are integers between 0 and $m-1$, and $1 \leq a \neq b < \frac{m}{2}$. Gray vertices correspond to line classes and white vertices to point classes. In the construction in Section 4, we typically take $c = 0$, $x = 0$, and $y = 0$ or δ.

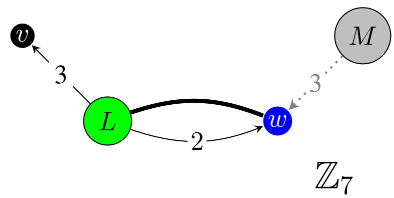
In [14, p. 116–118], Branko Grünbaum described a geometric technique to constructing a certain class of 3-configurations. This technique was extended in [7] to the Circumcircle Construction Lemma. Although the lemma can be stated as a more general incidence theorem [8], we state it as follows in order to facilitate the main construction in Section 4.

Lemma 2.2 (Circumcircle Construction Lemma (CCL))**.** *Let $v_0, v_1, \ldots, v_{m-1}$ and $w_0, w_1, \ldots, w_{m-1}$ form the vertices, labelled cyclically counterclockwise, of two regular convex m-gons centered at $\mathcal{O}$. The point w_0 lies on the circle passing through $v_d, v_{d-b}, \mathcal{O}$ if and only if the points w_0, w_b, v_d are collinear.*

That is, if w_0 lies on the circumcircle of span b through v_d, then the line L_0 of span b through w_0 passes through v_d, and conversely. By symmetry, the line L_{-d} will also pass through the point w_0, and in general, if w_0 is defined to also lie on some other line M_0, then each rotated image w_i will lie on the three lines L_i, L_{i-d} and M_i. The Circumcircle Construction Lemma, along with its reduced Levi graph structure, is illustrated in Figure 4.



(a) Illustrating the Circumcircle Construction Lemma; $m = 7, b = 2, d = 3$. The green line is L_0, and the dashed gray line is a possible other line M_0 passing through w_0 (i.e., w_0 could be defined as the intersection of M_0 and C); other elements of line classes L and M have been suppressed for clarity.



(b) The "gadget" in a reduced Levi graph corresponding to Figure 4a. (The connection between w and M is optional, depending on whether there happens to be a line M_0 passing through w_0; this is the typical situation in applications of the CCL.)

Figure 4: The Circumcircle Construction Lemma.

3 Celestial 4-configurations

The building blocks for the new construction of 5-configurations presented in Section 4 are the *celestial 4-configurations*, which are configurations that have the property that every point has two lines from each of two symmetry classes of lines passing through it, and every line has two points from each of two symmetry classes of points lying on it. An example of such a configuration is shown in Figure 5, along with a general reduced Levi graph. Celestial 4-configurations were first described in detail (aside from a handful of examples, e.g., [15, 16]) in Boben and Pisanski's article *Polycyclic Configurations* [11], as the main class of 4-configurations analyzed in that paper. Their description was expanded in Grünbaum's monograph *Configurations of Points and Lines* [14, Sections 3.5–3.8], although in that chapter, he unfortunately called them k-*astral* configurations (even though as he defined previously [14, p. 34], a k-astral configuration is simply a configuration with k symmetry classes of points and of lines, and there exist k-astral 4-configurations that are not k-celestial [13]).

Every k-celestial 4-configuration can be described by a celestial symbol

$$m\#(s_1, t_1; \ldots; s_k, t_k)$$

that satisfies four axioms:

Axiom 1: (order condition) $s_i \neq t_i \neq s_{i+1}$ (with indices taken modulo m)

Axiom 2: (even condition) $\sum_{i=1}^{k}(s_i - t_i) = 2\delta$ for some integer δ

Axiom 3: (cosine condition) $\prod_{i=1}^{k} \cos\left(\frac{s_i \pi}{m}\right) = \prod_{i=1}^{k} \cos\left(\frac{t_i \pi}{m}\right)$

Axiom 4: (substring condition) no substring $s_i, t_i; \ldots ; s_j, t_j$ or $t_i; s_{i+1}, \ldots, t_j; s_{j+1}$ satisfies the previous axioms.

A symbol satisfying the 4 axioms is said to be *valid*. Although celestial 4-configurations are probably the most well-understood class of 4-configuration, they are still poorly understood in general. The collection of 2-celestial configurations is completely classified ([2], with a clearer proof in [14, p. 210-211]), but general k-celestial configurations are not completely classified, and the problem appears to be non-tractable (since it depends on being able to solve certain trigonometric diophantine equations). However, some known families of valid k-celestial configurations, primarily for $k = 3, 4$, were presented in [1].

Given a valid symbol, there is a corresponding *cohort* $m\#S; T$, where $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ (as sets), which corresponds to a collection of valid symbols; in particular, the sets in a cohort must satisfy the even and cosine conditions, and it must be possible to find an ordering of the s_i and t_i that satisfies the order condition.

To construct a k-celestial 4-configuration $m\#(s_1, t_1; \ldots; s_k, t_k)$ with k point classes $v_1, \ldots, v_k$ and k line classes $L_1, \ldots, L_k$, do the following:

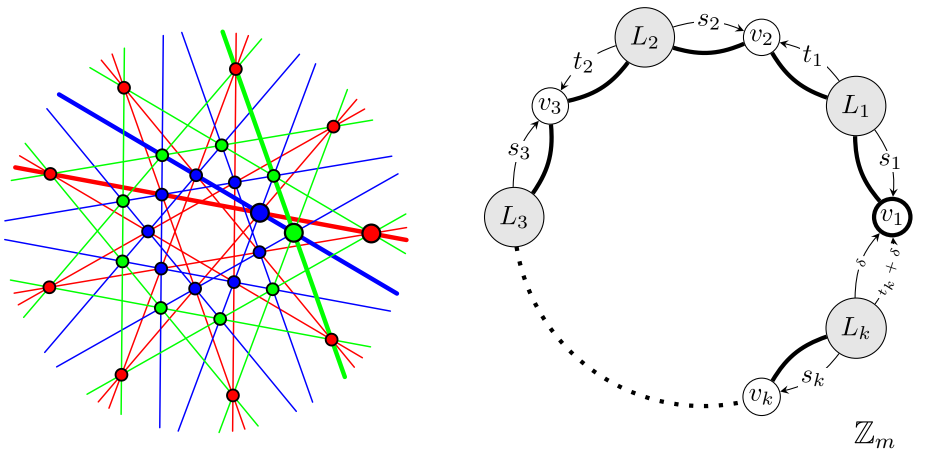
Algorithm 1 (Constructing a celestial 4-configuration)**.**

Input: A valid celestial symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$.

1. Construct the vertices of a regular m-gon centered at $\mathcal{O}$, labelled $(v_1)_0, \ldots (v_1)_{m-1}$.
2. Let L_1 be the collection of lines of span s_1 with respect to point class v_1: that is, let $(L_1)_i = (v_1)_i \vee (v_1)_{i+s_1}$.
3. Construct point class v_2 to be the set of t_1-st intersection points of the lines L_1: that is, $(v_2)_i = (L_1)_i \wedge (L_1)_{i-t_1}$.
4. Continue in this fashion; line class L_2 is the set of lines of span s_2 with respect to point class v_2, point class v_3 is the set of t_2-nd intersection points of the lines L_2, etc., stopping after the construction of line class L_k.

Because the symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$ is valid, the point class v_{k+1} corresponds, as a set, to point class v_1, and in particular, $(v_{k+1})_0 = (v_1)_\delta$, where $2\delta = \sum_{i=1}^{k}(s_i - t_i)$.

The general reduced Levi graph for the configuration $m\#(s_1, t_1; \ldots; s_k, t_k)$ is shown in Figure 5b; δ, the "twist" [11], is guaranteed to be an integer by the even condition. In general, the underlying graph for every reduced Levi graph of a celestial 4-configuration is a *double cycle* of even length; that is, an even cycle in which every edge is replaced by a pair of parallel edges.



(a) The celestial 4-configuration $9\#(4,3;2,3;1,3)$. The 0th element of each symmetry class is shown larger (points) or thicker (lines), and elements in different symmetry classes are distinguished by color (class 1 is red, class 2 is blue, and class 3 is green).

(b) The reduced Levi graph, a *double cycle*, for a general celestial 4-configuration, where $\delta = \frac{1}{2}\sum_{i=1}^{k}(s_i - t_i)$.

Figure 5: Celestial 4-configurations

4 Constructing movable 5-configurations

The general idea of the construction is to produce a 5-configuration whose reduced Levi graph consists of concentric double cycles, each of which corresponds to a particular celestial 4-configuration, where the double cycles are successively linked by single edges by applying the CSL, and finally, the innermost cycle is linked to the outermost cycle using the CCL; if $k > 2$ the construction will produce a movable 5-configuration. The reduced Levi graph is shown in Figure 6.

More specifically, the reduced Levi graph contains k concentric double cycles, each of which corresponds to a k-celestial 4-configuration with cohort $m\#S;T$ where $S \cap T = \varnothing$. If the outermost cycle corresponds to the configuration with symbol

$$m\#(s_1, t_1; s_2, t_2; \ldots; s_{k-1}, t_{k-1}; s_k, t_k),$$

then each successive cycle has the s_i's permuted cyclically one step while the t_i's remain fixed: that is, the second cycle has symbol

$$m\#(s_2, t_1; s_3, t_2; \ldots; s_k, t_{k-1}; s_1, t_k),$$

the third has symbol

$$m\#(s_3, t_1; s_4, t_2; \ldots; s_1, t_{k-1}; s_2, t_k),$$

and so on, so that the innermost cycle has symbol

$$m\#(s_k, t_1; s_1, t_2; \ldots; s_{k-2}, t_{k-1}; s_{k-1}, t_k).$$

The point classes of the celestial configuration corresponding to cycle j are labelled $v_1^j, \ldots v_k^j$ and the line classes $L_1^j, \ldots L_k^j$; that is, the superscript indicates the cycle, and the subscript the symmetry class in the celestial configuration. In Figure 6, the first point class of each celestial configuration is highlighted.

Given a valid configuration symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$ with cohort $m\#S;T$ with the property that $S \cap T = \varnothing$, the geometric construction algorithm to produce a 5-configuration with $k-2$ continuous degrees of freedom is given in Algorithm 2. If $k = 2$ the configuration is static and has been described previously in [9]; however, the construction algorithm given here, which uses the CCL to complete the construction, eliminates the need for completing the configuration via a continuity argument as described in that paper.

Algorithm 2 (Constructing a 5-configuration)**.**

Input: A valid celestial symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$ with the property that $S \cap T = \varnothing$.

1. Construct the first k-celestial 4-configuration with symbol $m\#(s_1, t_1; \ldots; s_k, t_k)$, with point classes $v_1^1, \ldots, v_k^1$ and line classes $L_1^1, \ldots, L_k^1$.
2. If $k > 2$, for $j = 2, \ldots, k-1$:
   (a) Place a new point $(v_1^j)_0$ arbitrarily on line $(L_1^{j-1})_0$, and construct the rest of the points $(v_1^j)_i$ in point class v_1^2 by rotating $(v_1^1)_0$ by $\frac{2\pi i}{m}$ for $i = 0, \ldots m-1$.
   (b) Using the point class v_1^j as the starting m-gon, construct the configuration
   $$m\#(s_j, t_1; s_{j+1}, t_2; \ldots; s_{j-2}, t_{k-1}; s_{j-1}, t_k)$$
   (where the sequence $s_1, s_2, \ldots, s_{k-1}, s_k$ has been cyclically permuted j steps but the sequence $t_1, \ldots, t_k$ remains fixed).
3. To construct the k-th celestial configuration:
   (a) Construct a circumcircle $\mathcal{C}$ of span s_k through $(v_1^1)_c$, choosing c (and varying continuous parameters if possible/necessary) so that $\mathcal{C}$ intersects line $(L_1^{k-1})_0$.
   (b) Let $(v_1^k)_0$ be the intersection of $\mathcal{C}$ with line $(L_1^{k-1})_0$, and let $(v_1^k)_i$ be the rotation of $(v_1^k)_0$ through $\frac{2\pi i}{m}$ about $\mathcal{O}$.
   (c) Construct configuration
   $$m\#(s_k, t_1; s_1, t_2; \ldots; s_{k-2}, t_{k-1}; s_{k-1}, t_k)$$
   using the points $(v_1^k)_i$ as the initial set of points.

Theorem 4.1. *Algorithm 2, beginning with $m\#(s_1, t_1; \ldots; s_k, t_k)$, creates a valid 5-configuration with mk^2 points, mk^2 lines and $k-2$ continuous degrees of freedom.*

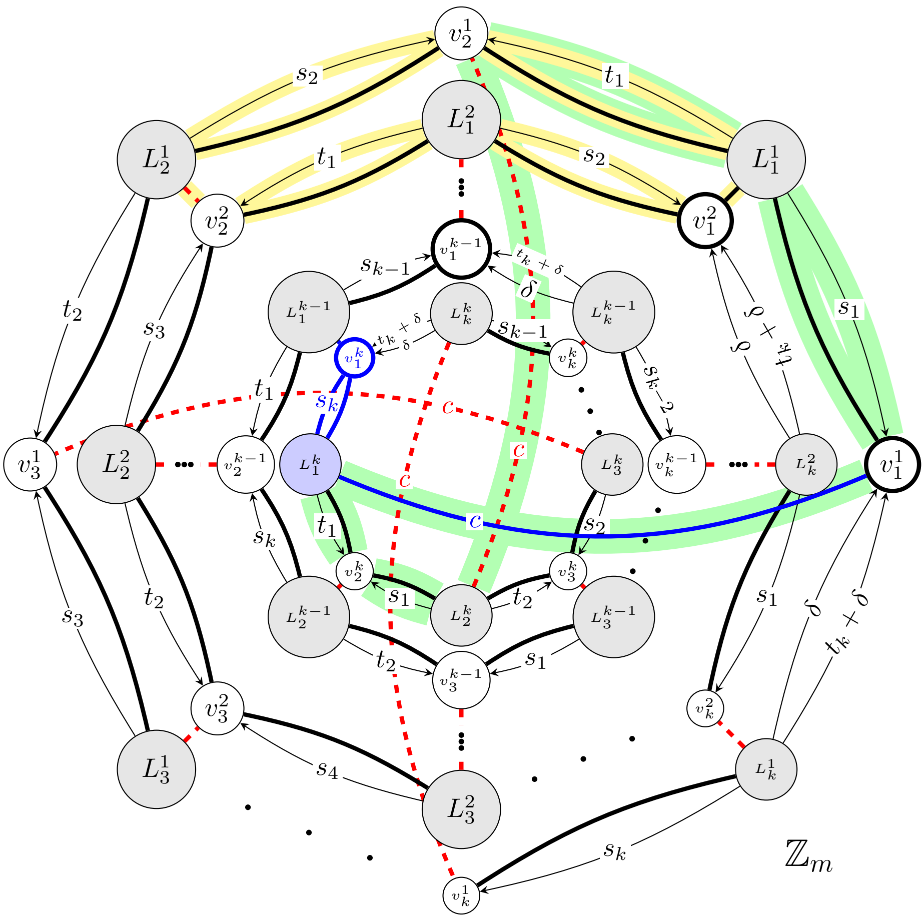


Figure 6: The reduced Levi graph, over $\mathbb{Z}_m$, for a movable 5-configuration with k^2 point classes and k^2 line classes. It consists of k concentric double cycles, each corresponding to a particular celestial 4-configuration, with the double cycles linked by arcs. The arcs shown red and dashed are induced by the Crossing Spans Lemma, with example CSL gadgets inducing the dashed edges highlighted in yellow and green, while the structure shown in blue is constructed via the Circumcircle Construction Lemma.

Proof. First, note that Algorithm 2 constructs k celestial configurations; each celestial configuration contains k symmetry classes of points and of lines, and each symmetry class contains m elements, for a total of mk^2 points and mk^2 lines.

Second, for $j = 2, \ldots, k-1$, the point $(v_1^j)_0$ is placed arbitrarily on line $(L_1^{j-1})_0$, for $(k-1) - 2 + 1 = k - 2$ continuous degrees of freedom.

Thus, the nontrivial part of the proof is to show that every point lies on 5 lines, and every line passes through 5 points.

Recall that the symbol for celestial configuration j is

$$m\#(s_j, t_1; s_{j+1}, t_2; \ldots; s_{j+\ell}, t_\ell; \ldots; s_{j-1}, t_k).$$

By construction, for each $j = 1, \ldots, k-1$, each line $(L_1^j)_i$ passes through the point $(v_1^{j+1})_i$ (that is, the first symmetry class of points in celestial configuration $j+1$ lies on the first symmetry class of lines in celestial configuration j), as well as through points $(v_1^j)_i$, $(v_1^j)_{i+s_j}$, $(v_2^j)_i$, and $(v_2^j)_{i+t_1}$ from celestial configuration j.

By careful choice of labels and the Crossing Spans Lemma, it follows that for all $\ell = 2, \ldots, k-1$ (with ℓ indexing the symmetry classes in the celestial configuration j), each line $(L_\ell^j)_i$ passes through point $(v_\ell^{j+1})_i$, as well as through points $(v_\ell^j)_i$, $(v_\ell^j)_{i+s_{j+\ell}}$, $(v_{\ell+1}^j)_i$ and $(v_{\ell+1}^j)_{i+t_\ell}$ from celestial configuration j.

A CSL gadget showing that points v_2^2 are incident with lines L_2^1 (dashed red line) beginning with the input that points v_1^2 are constructed incident with lines L_2^1 (solid black line) is highlighted in Figure 6 in yellow.

Finally, again by the CSL, line $(L_k^j)_i$ passes through point $(v_k^{j+1})_i$, as well as through points $(v_k^j)_i$, $(v_k^j)_{i+s_{j-1}}$, $(v_1^j)_{i+\delta}$ and $(v_1^j)_{i+\delta+t_k}$ from the completion of the celestial configuration j.

Thus, for $j = 1, \ldots, k-1$ (indexing the celestial configuration), $\ell = 1, \ldots k$ (indexing the symmetry class in the celestial configuration) and $i = 0, \ldots m-1$ (indexing the elements of the symmetry class) each line $(L_\ell^j)_i$ has 5 points lying on it. By inspection of the previous incidences, for $j = 2, \ldots, k-1$, each point $(v_\ell^j)_i$ has 5 lines passing through it; however, points $(v_\ell^1)_i$ only have 4 lines passing through them so far.

However, in step 3, we constructed $(v_1^k)_0$ be the intersection of $\mathcal{C}$ with line $(L_1^{k-1})_0$, where $\mathcal{C}$ is a circle of span s_k through $(v_1^1)_c$. By the Circumcircle Construction Lemma it follows that points $(v_1^k)_0$, $(v_1^k)_{s_k}$ and $(v_1^1)_c$ are collinear; that is line $(L_1^k)_0$, which is span s_k with respect to the points v_1^k by construction, passes through point $(v_1^1)_c$. By symmetry, it follows that line $(L_1^k)_i$ passes through $(v_1^1)_{i+c}$ for $i = 0, \ldots, m-1$. (This is represented by the thick blue line connecting the inner and outer rings in Figure 6.) By construction of the kth celestial configuration, it follows that line $(L_1^k)_i$ also passes through points $(v_1^k)_i$, $(v_1^k)_{i+s_k}$, $(v_2^k)_i$ and $(v_2^k)_{i+t_k}$.

A final application of the Crossing Spans Lemma on gadgets connecting the inner and outer ring shows that symmetry class L_ℓ^k in the k-th celestial configuration is incident with symmetry class v_ℓ^1 in the first celestial configuration. The CSL gadget showing that L_2^k is incident with v_2^1 (dashed red curve), beginning with the fact that L_1^k is incident with v_1^1

(thick blue curve) is highlighted in green in Figure 6. Specifically, for $\ell = 2, \ldots k-1$, $(L_\ell^k)_i$ passes through $(v_\ell^1)_i$, $(v_\ell^k)_i$, $(v_\ell^k)_{i+s_{\ell-1}}$, $(v_\ell^k)_i$, $(v_\ell^k)_{i+t_\ell}$. Finally, $(L_k^k)_i$ passes through $(v_k^1)_i$, $(v_k^k)_i$, $(v_k^k)_{i+s_{k-1}}$, $(v_1^k)_{i+\delta}$, and $(v_1^k)_{i+\delta+t_k}$. Thus, every point lies on 5 lines, and every line passes through 5 points.

□

5 Some valid inputs for Algorithm 2

Proposition 5.1. *The smallest movable 5-configuration produced by Algorithm 2 uses* $9\#(4,3;2,3;1,3)$ *(or another configuration with the same cohort) as its input and has 81 points and lines.*

Proof. If $k = 2$, Algorithm 2 produces static configurations. Inspection of a list of all valid symbols for small 3-celestial configurations (e.g., from [14, Table 3.7.1] or from the personal list of one of the the authors (LWB)) shows that the cohort $9\#\{4,2,1\};\{3,3,3\}$ is the smallest cohort with disjoint sets. □

This configuration is shown in Figure 7.

Theorem 5.2. *There exist infinitely many 5-configurations with one continuous degree of freedom.*

Proof. From [1] we know that

$$2q\#\{q-p, p, q-2r\};\{q-r, r, q-2p\}, \text{ for } q \geq 4 \text{ and } 0 < p, r < q$$

is a valid family of celestial 4-configuration cohorts.

Suppose that $r \neq p$, $r \neq \frac{q}{3}$, $p \neq \frac{q}{3}$ and $p + r \neq q$. Under these conditions, the sets S and T will always be disjoint. To see this, first note that $q - p \neq q - r$, because $p \neq r$; $q - p \neq r$, because $p + r \neq q$; and $q - p \neq q - 2p$ because $p \neq 0$. Next, $p \neq q - r$ because $p + r \neq q$; $p \neq r$ by hypothesis; and $p \neq q - 2p$ since $p \neq q/3$. Finally, $q - 2r \neq q - r$ because $r \neq 0$; $q - 2r \neq r$ since $r \neq q/3$; and $q - 2r \neq q - 2p$ because $r \neq p$. Thus, the sets are disjoint. Hence the cohort is valid as input for Algorithm 2.

In particular, $p = 1$ and $r = 2$ produces the valid input cohort $2q\#\{q-1, 1, q-4\};\{q-2, 2, q-2\}$ for any $q \geq 4$. □

Lemma 5.3. *The cohort* $3q\#\{1, 2, \ldots, 2^{k-1}\};\{\underbrace{q, q, \ldots, q}_{k}\}$ *for* $q = \dfrac{2^k+1}{3}$, *k odd and $k > 2$ is a valid celestial cohort.*

Proof. Note that the cohort $9\#\{1,2,4\};\{3,3,3\}$ can be viewed as the case $k = 3$ of this cohort.

To show the cohort is valid, we need to show that $q = \dfrac{2^k+1}{3}$ is an integer and that the cohort satisfies the cosine and even conditions.

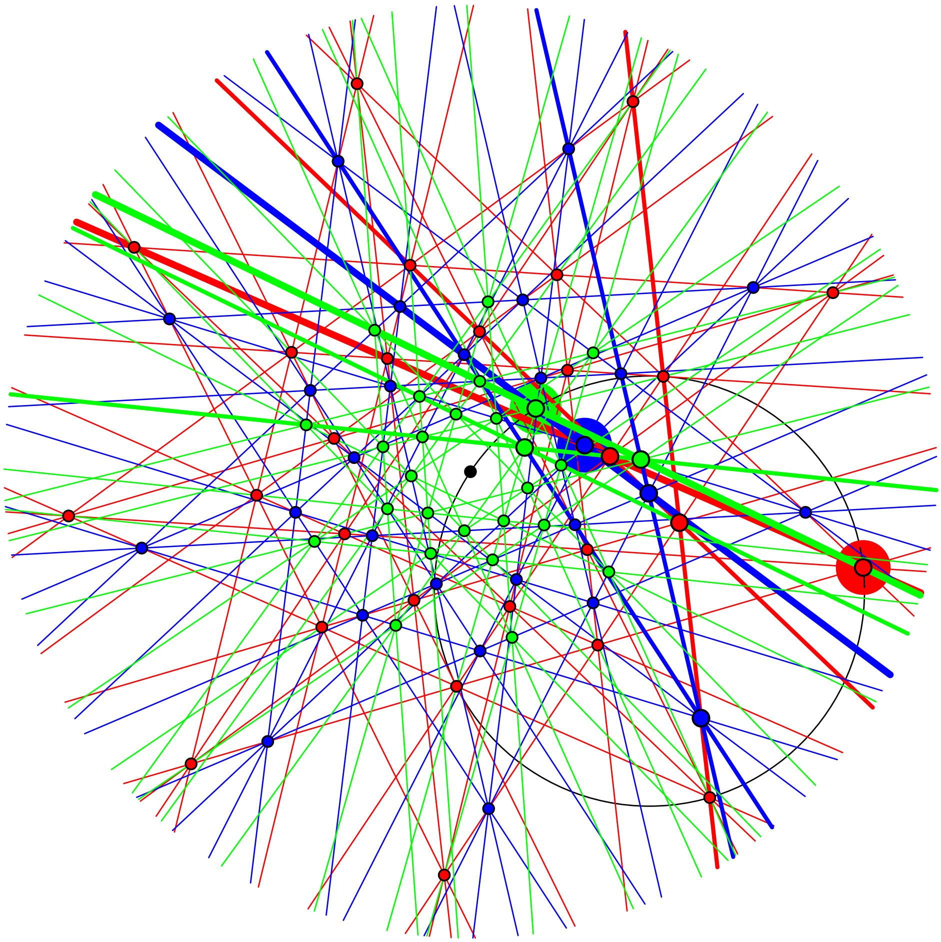


Figure 7: The smallest movable 5-configuration produced by Algorithm 2, an (81_5) configuration, with initial celestial configuration $9\#(4,3;2,3;1,3)$ shown in red, second celestial configuration $9\#(2,3;1,3;4,3)$ shown in blue, and final celestial configuration $9\#(1,3;4,3;2,3)$ shown in green. The point $(v_1^1)_0$ is highlighted in red, the line $(L_1^1)_0$ is the thickest red line, the point $(v_1^2)_0$ is highlighted in blue, and the line $(L_1^2)_0$ is the thickest blue line. The point $(v_1^3)_0$, which was constructed via the intersection of $(L_1^2)_0$ with the black circumcircle of span 1 through $(v_1^1)_0$, is highlighted in green, and $(L_1^3)_0$ is the thickest green line. Other 0th elements of symmetry classes are shown at medium weights. Already we have reached the limits of intelligibility of a small-scale diagram.

If $k = 2j+1$ for some integer j, it is straightforward to show that

$$2^k + 1 = 2^{2j+1} + 1 = (2+1)\sum_{i=0}^{2j}(-1)^i 2^i,$$

so $2^{2j}+1$ is clearly divisible by 3, and $q = \sum_{i=0}^{2j}(-1)^i 2^i$, which is odd.

Moreover, if $s_i = 2^{i-1}$, then $\sum_{i=1}^{k} 2^{i-1} = 2^k + 1$. Thus, if $t_i = q$ for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k}(s_i - t_i) = \left(2^k + 1\right) - (2j+1)q$$

is even, since both terms are odd.

It remains to show the cosine condition is fulfilled: that is, we need to show that for $q = \frac{2^k+1}{3}$,

$$\prod_{i=1}^{k} \cos\left(\frac{2^{i-1}\pi}{3q}\right) = \prod_{i=1}^{k} \cos\left(\frac{q\pi}{3q}\right). \tag{5.1}$$

The right-hand side of equation (5.1) clearly evaluates to $\frac{1}{2^k}$. To see the left-hand side also evaluates to $\frac{1}{2^k}$, we use the following trigonometric identity, which can be proved using the identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and induction (see [10]):

$$2^k \prod_{j=0}^{k-1} \cos\left(2^j a\right) = \frac{\sin\left(2^j a\right)}{\sin(a)}. \tag{5.2}$$

Applying this identity to the left-hand side of (5.1), we see that

$$\begin{aligned}
\prod_{i=1}^{k} \cos\left(\frac{2^{i-1}\pi}{3q}\right) = \prod_{i=1}^{k} \cos\left(\frac{2^{i-1}\pi}{2^k+1}\right) &= \frac{1}{2^k}\left(\frac{\sin\left(\frac{2^k\pi}{2^k+1}\right)}{\sin\left(\frac{\pi}{2^k+1}\right)}\right) \\
&= \frac{1}{2^k}\sin\left(\pi - \frac{\pi}{2^k+1}\right)\csc\left(\frac{\pi}{2^k+1}\right) \\
&= \frac{1}{2^k}\left(\sin(\pi)\cos\left(\frac{\pi}{2^k+1}\right) - \cos(\pi)\sin\left(\frac{\pi}{2^k+1}\right)\right)\csc\left(\frac{\pi}{2^k+1}\right) \\
&= \frac{1}{2^k}\left(0 - (-1)\sin\left(\frac{\pi}{2^k+1}\right)\right)\csc\left(\frac{\pi}{2^k+1}\right) \\
&= \frac{1}{2^k},
\end{aligned}$$

so the cosine condition is satisfied. □

Theorem 5.4. *There exists at least one 5-configuration with s continuous degrees of freedom, for infinitely many values of s.*

Proof. Use the cohort $3q\#\{1, 2, \ldots, 2^{k-1}\}; \{\underbrace{q, q, \ldots, q}_{k}\}$ for $q = \frac{2^k+1}{3}$, k odd and $k > 2$ from Lemma 5.3; clearly, the sets S and T are disjoint. This produces a movable 5-configuration with $k-2$ degrees of freedom for all odd $k \geq 3$. □

6 Open Questions

Question 1. In [8], the Crossing Spans Lemma is generalized to allow larger and differently labelled subgraphs, as the Extended Crossing Spans Lemma. Are there interesting movable configurations that can be constructed from this generalization?
Question 2. This construction depends on two very simple geometric lemmas, which are straightforward to prove using basic Euclidean geometry. Are there other such useful lemmas? What techniques can be used, and which incidence theorems, to construct new configurations from known configurations while retaining useful symmetry properties?
Question 3. Finding movable 3-configurations is easy [6], and there are a number of known classes of movable 4-configurations [3, 4, 8, 14]. This paper presents a class of movable 5-configurations. Are there movable k-configurations for any $k > 5$? For all $k > 5$? In particular, are there movable 6-configurations?

References

[1] A. Berardinelli and L. W. Berman, Systematic celestial 4-configurations, *Ars Math. Contemp.* **7** (2014), 361–377.

[2] L. W. Berman, A characterization of astral (n_4) configurations, *Discrete Comput. Geom.* **26** (2001), 603–612.

[3] L. W. Berman, Movable (n_4) configurations, *Electron. J. Combin.* **13** (2006), Research Paper 104, 30, http://www.combinatorics.org/Volume_13/Abstracts/v13i1r104.html.

[4] L. W. Berman, A new class of movable (n_4) configurations, *Ars Math. Contemp.* **1** (2008), 44–50.

[5] L. W. Berman, Constructing highly incident configurations, *Discrete Comput. Geom.* **46** (2011), 447–470, doi:10.1007/s00454-010-9279-7, http://dx.doi.org/10.1007/s00454-010-9279-7.

[6] L. W. Berman, Geometric constructions for 3-configurations with non-trivial geometric symmetry, *Electron. J. Combin.* **20** (2013), Paper 9, 29.

[7] L. W. Berman and J. R. Faudree, Highly incident configurations with chiral symmetry, *Discrete Comput. Geom.* **49** (2013), 671–694, doi:10.1007/s00454-013-9494-0, http://dx.doi.org/10.1007/s00454-013-9494-0.

[8] L. W. Berman, J. R. Faudree and T. Pisanski, Polycyclic movable 4-configurations are plentiful, *Discrete Comput. Geom.* (Published Online 16 November 2015), doi:10.1007/s00454-015-9749-z.

[9] L. W. Berman and L. Ng, Constructing 5-configurations with chiral symmetry, *Electron. J. Combin.* **17** (2010), Research Paper 2, 14, http://www.combinatorics.org/Volume_17/Abstracts/v17i1r2.html.

[10] W. A. Beyer, J. D. Louck and D. Zeilberger, Math Bite: A Generalization of a Curiosity that Feynman Remembered All His Life, *Math. Mag.* **69** (1996), 43–44, http://www.jstor.org/stable/2691393?origin=pubexport.

[11] M. Boben and T. Pisanski, Polycyclic configurations, *European J. Combin.* **24** (2003), 431–457, doi:10.1016/S0195-6698(03)00031-3, http://dx.doi.org/10.1016/S0195-6698(03)00031-3.

[12] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455.

[13] B. Grünbaum, Musings on an example of Danzer's, *European J. Combin.* **29** (2008), 1910–1918, doi:10.1016/j.ejc.2008.01.004, http://dx.doi.org/10.1016/j.ejc.2008.01.004.

[14] B. Grünbaum, *Configurations of points and lines*, volume 103 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2009.

[15] B. Grünbaum and J. F. Rigby, The real configuration (21_4), *J. London Math. Soc. (2)* **41** (1990), 336–346, doi:10.1112/jlms/s2-41.2.336, http://dx.doi.org/10.1112/jlms/s2-41.2.336.

[16] D. Marušič and T. Pisanski, Weakly flag-transitive configurations and half-arc-transitive graphs, *European J. Combin.* **20** (1999), 559–570, doi:10.1006/eujc.1999.0302, http://dx.doi.org/10.1006/eujc.1999.0302.