The parameters of Fibonacci and Lucas cubes

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Abstract
Motivated by the conjectures from Castro, et al. in 2011, in this paper we use integer programming formulations for computing the domination number, the 2-packing number and the independent domination number of Fibonacci cubes and Lucas cubes for $n \leq 13$.

Keywords: Fibonacci cubes, Lucas cubes, domination number, 2-packing number.
Math. Subj. Class.: 05C69, 05C25

1 Introduction

Hypercubes form one of the most applicable classes of graphs with many appealing properties. The $n$-cube $Q_n$ is the graph whose vertices are all binary strings of length $n$, and two vertices are adjacent if they differ in exactly one position. The Fibonacci cubes were introduced as a model for interconnection networks [4, 2]. They offer challenging mathematical and computational problems, and admit a recursive decomposition into smaller Fibonacci cubes (see [5], [6], [8] for their structural properties). The Fibonacci cubes can be recognized in $O(m \log n)$ time (where $n$ is the order and $m$ the size of a given graph) [10]. The Lucas cubes [7] form a class of graphs closely related to the Fibonacci cubes, obtained by removing some vertices from the Fibonacci cubes.

Let $Q_n$ be the $n$-dimensional hypercube. A Fibonacci string of length $n$ is a binary string $b_1 b_2 \ldots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words, Fibonacci strings are binary strings that contain no consecutive ones. The Fibonacci cube $\Gamma_n$, for $n \geq 1$ is the subgraph of $Q_n$ induced by the Fibonacci strings of length $n$. A Fibonacci string $b_1 b_2 \ldots b_n$ is a Lucas string if $b_1 \cdot b_n = 0$. In other words, Lucas strings are binary strings

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that contain no consecutive ones circularly. The Lucas cube $\Lambda_n$, for $n \geq 1$ is the subgraph of $Q_n$ induced by the Lucas strings of length $n$. It is well-known that $|V(\Gamma_n)| = F_{n+2}$, where $F_n$ are the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Similarly, $|V(\Lambda_n)| = L_n$ for $n \geq 1$, where $L_n$ are the Lucas numbers: $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.

Let $G$ be a graph. Set $D \subseteq V(G)$ is a dominating set if every vertex from $V(G)$ either belongs to $D$ or is adjacent to some vertex from $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any two different vertices $u$ and $v$ of $X$. The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing of $G$. It is well-known that for any graph $G$ holds $\gamma(G) \geq \rho(G)$.

An independent set or stable set is a set of vertices in a graph, no two of which are adjacent. The independent domination number $i(G)$ of a graph $G$ is the size of the smallest independent dominating set (or, equivalently, the size of the smallest maximal independent set). The minimum dominating set in a graph will not necessarily be independent, but the size of a minimum dominating set is always less than or equal to the size of a minimum maximal independent set, $\gamma(G) \leq i(G)$.

Pike and Zou in [9] obtained a lower bound for the domination number of Fibonacci cube of order $n$ and determined the exact value of the domination number of Fibonacci cubes of order at most 8. Castro et al. in [1] obtained upper and lower bounds for the domination and 2-packing number of Fibonacci and Lucas cubes. Furthermore, the authors obtained the exact values for $\gamma(\Gamma_n)$ and $\gamma(\Lambda_n)$ for $n \leq 9$ and for $\rho(\Gamma_n)$ and $\rho(\Lambda_n)$ for $n \leq 10$.

In this paper we use integer programming method to compute the exact values of the domination, 2-packing and independent domination number of Fibonacci and Lucas cubes for $n \leq 13$, which resolves the conjecture from [1].

2 Main results

For each subset of the vertex set $S \subseteq V(G)$ define

$$x_i = \begin{cases} 
1 & \text{if } i \in S \\
0 & \text{if } i \in V \setminus S.
\end{cases}$$

The neighborhood $N(v)$ of a vertex $v$ in a graph $G$ is the induced subgraph of $G$ consisting of all vertices adjacent to $v$ and all edges connecting two such vertices. Let $N[v] = N(v) \cup \{v\}$ denote the closed neighborhood of the vertex $v$.

The domination number of $G$ can be formulated as the following $0-1$ integer programming problem:

$$\gamma(G) = \min \sum_{i=1}^{n} x_i$$

subject to

$$\sum_{j \in N[i]} x_j \geq 1,$$

$$x_i \in \{0, 1\}, \quad \text{for all } 1 \leq i \leq n.$$

It is easy to see that the conditions (2.2) and (2.3) define dominating set $S$ and vice versa [3]. For Fibonacci cube $\Gamma_n$ this formulation has $F_{n+2}$ variables and $2F_{n+2}$ constraints,
while each condition from (2.2) contains at most \( n \) variables. For Lucas cube \( \Lambda_n \) this formulation has \( L_n \) variables and \( 2L_n \) constrains, while each condition from (2.2) contains at most \( n \) variables.

The 2-packing number of \( G \) can be formulated as the following \( 0 - 1 \) integer programming problem:

\[
\rho(G) = \max \sum_{i=1}^{n} x_i
\]  

subject to

\[
\sum_{j \in N[i]} x_j \leq 1, \quad (2.5)
\]

\[
x_i \in \{0, 1\}, \quad \text{for all } 1 \leq i \leq n. \quad (2.6)
\]

We will prove that the conditions (2.5) and (2.6) define 2-packing set \( S \) and vice versa. Let \( S \) be a 2-packing set. Since \( S \) does not contain two vertices on distance 1 or 2, for each \( v \in V(G) \) there is at most one vertex from the closed neighborhood \( N[v] \) which belongs to \( S \). Assume now that the set \( S \) satisfies the condition (2.5) and let \( u \) and \( v \) be two vertices from \( S \) on distance 2. In that case for the shortest path \( vwu \), we have \( \sum_{j \in N[w]} x_j \geq 2 \), which is impossible. Therefore, \( S \) is a 2-packing set.

The independent domination number \( G \) can be formulated as the following \( 0 - 1 \) integer programming problem:

\[
i(G) = \min \sum_{i=1}^{n} x_i
\]  

subject to

\[
\sum_{j \in N[i]} x_j \geq 1, \quad (2.8)
\]

\[
(n - 1)x_i + \sum_{j \in N(i)} x_j \leq n - 1, \quad (2.9)
\]

\[
x_i \in \{0, 1\}, \quad \text{for all } 1 \leq i \leq n. \quad (2.10)
\]

The conditions (2.8) and (2.10) define domination set \( S \), while the condition (2.9) ensures the independence. For \( x_i = 0 \) we have always true \( \sum_{j \in N(i)} x_j \leq n - 1 \), while for \( x_i = 1 \) we have \( \sum_{j \in N(i)} x_j \leq 0 \) which is equivalent to \( \sum_{j \in N[i]} x_j = 1 \). This proves that the formulation is correct. For Fibonacci cube \( \Gamma_n \) this formulation has \( F_{n+2} \) variables and \( 3F_n \) constraints, while each conditions from (2.8) and (2.9) contain at most \( n \) variables. For Lucas cube \( \Lambda_n \) this formulation has \( L_n \) variables and \( 3L_n \) constrains, while each condition from (2.8) and (2.9) contain at most \( n \) variables.

The tests were performed on the Intel Core 2 Duo T5800 2.0 GHz with 2 GB RAM running the Linux operating system and using CPLEX 8.1. The results are summarized in Tables 1 and 2. In Tables 3 and 4 we give some examples of dominating sets and 2-packing sets that were obtained during the computation of these values.

These results resolve the conjecture from [1] and support Problem 5.1 for \( n \leq 12 \).
Table 1: Parameters of small Fibonacci cubes.

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<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
<th>10</th>
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<td>$</td>
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Table 2: Parameters of small Lucas cubes.

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<th>11</th>
<th>12</th>
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<td>24</td>
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Table 3: Examples of minimal dominating sets for $\Gamma(10)$ and $\Lambda(11)$

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<th>$\Lambda(11)$</th>
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<td>$(1,0,1,0,0,0,0,1,0,0,0,0,0)$, $(0,1,0,0,0,0,0,0,1,0,0,0,0)$, $(0,0,0,0,0,0,0,1,1,0,0,0,0)$</td>
<td>$(0,0,0,0,0,0,0,0,0,1,0,0,0,0)$, $(0,0,0,0,0,0,0,0,0,0,1,0,0,0)$, $(0,0,0,0,0,0,0,0,0,0,0,1,0,0)$</td>
</tr>
<tr>
<td>$(0,0,0,0,0,0,0,0,0,1,0,0,0,0)$, $(0,0,0,0,0,0,0,0,0,0,0,1,0,0)$, $(0,0,0,0,0,0,0,0,0,0,0,0,1,0)$</td>
<td>$(0,0,0,0,0,0,0,0,0,0,0,0,0,1)$, $(0,0,0,0,0,0,0,0,0,0,0,0,0,0)$, $(0,0,0,0,0,0,0,0,0,0,0,0,0,0)$</td>
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Table 4: Examples of 2-packing sets for $\Gamma(11)$ and $\Lambda(12)$

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<th>$\Lambda(12)$</th>
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Table 4: Examples of 2-packing sets for $\Gamma(11)$ and $\Lambda(12)$
References


