The parameters of Fibonacci and Lucas cubes*

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Abstract

Motivated by the conjectures from Castro, et al. in 2011, in this paper we use integer programming formulations for computing the domination number, the 2-packing number and the independent domination number of Fibonacci cubes and Lucas cubes for $n \leq 13$.

Keywords: Fibonacci cubes, Lucas cubes, domination number, 2-packing number.

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1 Introduction

Hypercubes form one of the most applicable classes of graphs with many appealing properties. The n -cube Q_n is the graph whose vertices are all binary strings of length n , and two vertices are adjacent if they differ in exactly one position. The Fibonacci cubes were introduced as a model for interconnection networks [4, 2]. They offer challenging mathematical and computational problems, and admit a recursive decomposition into smaller Fibonacci cubes (see [5], [6], [8] for their structural properties). The Fibonacci cubes can be recognized in $O(m \log n)$ time (where n is the order and m the size of a given graph) [10]. The Lucas cubes [7] form a class of graphs closely related to the Fibonacci cubes, obtained by removing some vertices from the Fibonacci cubes.

Let Q_n be the n -dimensional hypercube. A Fibonacci string of length n is a binary string $b_1 b_2 \dots b_n$ with $b_i \cdot b_{i+1} = 0$ for $1 \leq i < n$. In other words, Fibonacci strings are binary strings that contain no consecutive ones. The Fibonacci cube Γ_n , for $n \geq 1$ is the subgraph of Q_n induced by the Fibonacci strings of length n . A Fibonacci string $b_1 b_2 \dots b_n$ is a Lucas string if $b_1 \cdot b_n = 0$. In other words, Lucas strings are binary strings

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that contain no consecutive ones circularly. The Lucas cube Λ_n , for $n \geq 1$ is the subgraph of Q_n induced by the Lucas strings of length n . It is well-known that $|V(\Gamma_n)| = F_{n+2}$, where F_n are the Fibonacci numbers: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Similarly, $|V(\Lambda_n)| = L_n$ for $n \geq 1$, where L_n are the Lucas numbers: $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.

Let G be a graph. Set $D \subseteq V(G)$ is a dominating set if every vertex from $V(G)$ either belongs to D or is adjacent to some vertex from D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . A set $X \subseteq V(G)$ is called a 2-packing if $d(u, v) > 2$ for any two different vertices u and v of X . The 2-packing number $\rho(G)$ is the maximum cardinality of a 2-packing of G . It is well-known that for any graph G holds $\gamma(G) \geq \rho(G)$.

An independent set or stable set is a set of vertices in a graph, no two of which are adjacent. The independent domination number $i(G)$ of a graph G is the size of the smallest independent dominating set (or, equivalently, the size of the smallest maximal independent set). The minimum dominating set in a graph will not necessarily be independent, but the size of a minimum dominating set is always less than or equal to the size of a minimum maximal independent set, $\gamma(G) \leq i(G)$.

Pike and Zou in [9] obtained a lower bound for the domination number of Fibonacci cube of order n and determined the exact value of the domination number of Fibonacci cubes of order at most 8. Castro et al. in [1] obtained upper and lower bounds for the domination and 2-packing number of Fibonacci and Lucas cubes. Furthermore, the authors obtained the exact values for $\gamma(\Gamma_n)$ and $\gamma(\Lambda_n)$ for $n \leq 9$ and for $\rho(\Gamma_n)$ and $\rho(\Lambda_n)$ for $n \leq 10$.

In this paper we use integer programming method to compute the exact values of the domination, 2-packing and independent domination number of Fibonacci and Lucas cubes for $n \leq 13$, which resolves the conjecture from [1].

2 Main results

For each subset of the vertex set $S \subseteq V(G)$ define

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in V \setminus S. \end{cases}$$

The neighborhood $N(v)$ of a vertex v in a graph G is the induced subgraph of G consisting of all vertices adjacent to v and all edges connecting two such vertices. Let $N[v] = N(v) \cup \{v\}$ denote the closed neighborhood of the vertex v .

The domination number of G can be formulated as the following 0–1 integer programming problem:

$$\gamma(G) = \min \sum_{i=1}^n x_i \tag{2.1}$$

subject to

$$\sum_{j \in N[v]} x_j \geq 1, \tag{2.2}$$

$$x_i \in \{0, 1\}, \quad \text{for all } 1 \leq i \leq n. \tag{2.3}$$

It is easy to see that the conditions (2.2) and (2.3) define dominating set S and vice versa [3]. For Fibonacci cube Γ_n this formulation has F_{n+2} variables and $2F_{n+2}$ constraints,

while each condition from (2.2) contains at most n variables. For Lucas cube Λ_n this formulation has L_n variables and $2L_n$ constraints, while each condition from (2.2) contains at most n variables.

The 2-packing number of G can be formulated as the following 0 – 1 integer programming problem:

$$\rho(G) = \max \sum_{i=1}^n x_i \quad (2.4)$$

subject to

$$\sum_{j \in N[i]} x_j \leq 1, \quad (2.5)$$

$$x_i \in \{0, 1\}, \quad \text{for all } 1 \leq i \leq n. \quad (2.6)$$

We will prove that the conditions (2.5) and (2.6) define 2-packing set S and vice versa. Let S be a 2-packing set. Since S does not contain two vertices on distance 1 or 2, for each $v \in V(G)$ there is at most one vertex from the closed neighborhood $N[v]$ which belongs to S . Assume now that the set S satisfies the condition (2.5) and let u and v be two vertices from S on distance 2. In that case for the shortest path vwu , we have $\sum_{j \in N[w]} x_j \geq 2$, which is impossible. Therefore, S is a 2-packing set.

The independent domination number G can be formulated as the following 0 – 1 integer programming problem:

$$i(G) = \min \sum_{i=1}^n x_i \quad (2.7)$$

subject to

$$\sum_{j \in N[i]} x_j \geq 1, \quad (2.8)$$

$$(n-1)x_i + \sum_{j \in N(i)} x_j \leq n-1, \quad (2.9)$$

$$x_i \in \{0, 1\}, \quad \text{for all } 1 \leq i \leq n. \quad (2.10)$$

The conditions (2.8) and (2.10) define domination set S , while the condition (2.9) ensures the independence. For $x_i = 0$ we have always true $\sum_{j \in N(i)} x_j \leq n-1$, while for $x_i = 1$ we have $\sum_{j \in N(i)} x_j \leq 0$ which is equivalent to $\sum_{j \in N[i]} x_j = 1$. This proves that the formulation is correct. For Fibonacci cube Γ_n this formulation has F_{n+2} variables and $3F_{n+2}$ constraints, while each conditions from (2.8) and (2.9) contain at most n variables. For Lucas cube Λ_n this formulation has L_n variables and $3L_n$ constraints, while each condition from (2.8) and (2.9) contain at most n variables.

The tests were performed on the Intel Core 2 Duo T5800 2.0 GHz with 2 GB RAM running the Linux operating system and using CPLEX 8.1. The results are summarized in Tables 1 and 2. In Tables 3 and 4 we give some examples of dominating sets and 2-packing sets that were obtained during the computation of these values.

These results resolve the conjecture from [1] and support Problem 5.1 for $n \leq 12$.

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