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Distance labelings: a generalization of Langford sequences

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Abstract

A Langford sequence of order m and defect d can be identified with a labeling of the vertices of a path of order $2m$ in which each label from d up to $d+m-1$ appears twice and in which the vertices that have been labeled with k are at distance k. In this paper, we introduce two generalizations of this labeling that are related to distances. The basic idea is to assign nonnegative integers to vertices in such a way that if n vertices ($n > 1$) have been labeled with k then they are mutually at distance k. We study these labelings for some well known families of graphs. We also study the existence of these labelings in general. Finally, given a sequence or a set of nonnegative integers, we study the existence of graphs that can be labeled according to this sequence or set.

Keywords: Langford sequence, distance l-labeling, distance J-labeling, δ-sequence, δ-set.

Math. Subj. Class.: 11B99, 05C78

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1 Introduction

For the graph terminology not introduced in this paper we refer the reader to [14, 15]. For $m \leq n$, we denote the set $\{m, m+1, \dots, n\}$ by $[m, n]$. A *Skolem sequence* [8, 12] of order m is a sequence of $2m$ numbers $(s_1, s_2, \dots, s_{2m})$ such that (i) for every $k \in [1, m]$ there exist exactly two subscripts $i, j \in [1, 2m]$ with $s_i = s_j = k$, (ii) the subscripts i and j satisfy the condition $|i - j| = k$. The sequence $(4, 2, 3, 2, 4, 3, 1, 1)$ is an example of a Skolem sequence of order 4. It is well known that Skolem sequences of order m exist if and only if $m \equiv 0$ or 1 (mod 4).

Skolem introduced in [13] what is now called a *hooked Skolem sequence* of order m, where there exists a zero at the second to last position of the sequence containing $2m + 1$ elements. Later on, in 1981, Abrham and Kotzig [1] introduced the concept of *extended Skolem sequence*, where the zero is allowed to appear in any position of the sequence. An extended Skolem sequence of order m exists for every m. The following construction was given in [1]:

$$(p_m, p_m-2, \dots, 2, 0, 2, \dots, p_m-2, p_m, q_m, q_m-2, \dots, 3, 1, 1, 3, \dots, q_m-2, q_m), \quad (1.1)$$

where p_m and q_m are the largest even and odd numbers not exceeding m, respectively. Notice that from every Skolem sequence we can obtain two trivial extended Skolem sequences just by adding a zero either in the first or in the last position.

Let d be a positive integer. A *Langford sequence* of order m and defect d [11] is a sequence $(l_1, l_2, \dots, l_{2m})$ of $2m$ numbers such that (i) for every $k \in [d, d+m-1]$ there exist exactly two subscripts $i, j \in [1, 2m]$ with $l_i = l_j = k$, (ii) the subscripts i and j satisfy the condition $|i - j| = k$. Langford sequences, for $d = 2$, were introduced in [4] and they are referred to as *perfect Langford sequences*. Notice that, a Langford sequence of order m and defect $d = 1$ is a Skolem sequence of order m. Bermond, Brower and Germa on one side [2], and Simpson on the other side [11] characterized the existence of Langford sequences for every order m and defect d.

Theorem 1.1. *[2, 11] A Langford sequence of order m and defect d exists if and only if the following conditions hold: (i) $m \geq 2d - 1$, and (ii) $m \equiv 0$ or 1 (mod 4) if d is odd; $m \equiv 0$ or 3 (mod 4) if d is even.*

For a complete survey on Skolem-type sequences we refer the reader to [3]. For different constructions and applications of Langford type sequences we also refer the reader to [5, 6, 7, 9, 10].

1.1 Distance labelings

Let $L = (l_1, l_2, \dots, l_{2m})$ be a Langford sequence of order m and defect d. Consider a path P with $V(P) = \{v_i : i = 1, 2, \dots, 2m\}$ and $E(P) = \{v_i v_{i+1} : i = 1, 2, 2m-1\}$. Then, we can identify L with a labeling $f : V(P) \to [d, d+m-1]$ in such a way that, (i) for every $k \in [d, d+m-1]$ there exist exactly two vertices $v_i, v_j \in [1, 2m]$ with $f(v_i) = f(v_j) = k$, (ii) the distance $d(v_i, v_j) = k$. Motivated by this fact, we introduce two notions of distance labelings, one of them associated with a positive integer l and the other one associated with a set of positive integers J.

Let G be a graph and let l be a nonnegative integer. Consider any function $f : V(G) \to [0, l]$. We say that f is a *distance labeling* of length l (or *distance l-labeling*) of G if the following two conditions hold, (i) either $f(V(G)) = [0, l]$ or $f(V(G)) = [1, l]$ and (ii) if there exist two vertices v_i, v_j with $f(v_i) = f(v_j) = k$ then $d(v_i, v_j) = k$. Clearly, a graph can have many different distance labelings. We denote by $\lambda(G)$, the *labeling length* of G, the minimum l for which a distance l-labeling of G exists. We say that a distance l-labeling of G is *proper* if for every $k \in [1, l]$ there exist at least two vertices v_i, v_j of G with $f(v_i) = f(v_j) = k$. We also say that a proper distance l-labeling of G is *regular* of degree r (for short r-*regular*) if for every $k \in [1, l]$ there exist exactly r vertices v_{i_1}, v_{i_2}, $\ldots$, v_{i_r} with $f(v_{i_1}) = f(v_{i_2}) = \ldots = f(v_{i_r}) = k$. Clearly, if a graph G admits a proper distance l-labeling then $l \leq D(G)$, where $D(G)$ is the diameter of G.

Let G be a graph and let J be a set of nonnegative integers. Consider any function $f : V(G) \to J$. We say that f is a *distance J-labeling* of G if the following two conditions hold, (i) $f(V(G)) = J$ and (ii) for any pair of vertices v_i, v_j with $f(v_i) = f(v_j) = k$ we have that $d(v_i, v_j) = k$. We say that a distance J-labeling is *proper* if for every $k \in J \setminus \{0\}$ there exist at least two vertices v_i, v_j with $f(v_i) = f(v_j) = k$. We also say that a proper distance J-labeling of G is *regular* of degree r (for short r-*regular*) if for every $k \in J \setminus \{0\}$ there exist exactly r vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ with $f(v_{i_1}) = f(v_{i_2}) = \ldots = f(v_{i_r}) = k$. Clearly, a distance l-labeling is a distance J-labeling in which either $J = [0, l]$ or $J = [1, l]$. Thus, the notion of a J-labeling is more general than the notion of a l-labeling.

In this paper, we provide the labeling length of some well known families of graphs. We also study the inverse problem, that is, for a given pair of positive integers l and r we ask for the existence of a graph of order lr with a regular l-labeling of degree r. Finally, we study a similar question when we deal with J-labelings. The organization of the paper is as follows. Section 2 is devoted to l-labelings; we start calculating the labeling length of complete graphs, paths, cycles and some others families. The inverse problem is studied in the second part of the section. Section 3 is devoted to the inverse problem in J-labelings. There are many open problems that remain to be solved, we end the paper by presenting some of them.

2 Distance l-labelings

We start this section by providing the labeling length of some well-known families of graphs. By definition, $\lambda(K_1) = 0$. In what follows, we only consider graphs of order at least 2.

Proposition 2.1. *Let $n \geq 2$. The complete graph K_n has $\lambda(K_n) = 1$.*

Proof. By assigning the label 1 to all vertices of K_n, we obtain a distance 1-labeling of it. □

Proposition 2.2. *Let $n \geq 2$. The path P_n has $\lambda(P_n) = \lfloor n/2 \rfloor$.*

Proof. By a previous comment, we know that a Skolem sequence of order m exists if $m \equiv 0$ or 1 (mod 4). This fact together with (1.1) guarantees the existence of a proper distance $\lfloor n/2 \rfloor$-labeling when $n \not\equiv 4, 6 \pmod 8$. By removing one of the end labels of (1.1), we obtain a (non proper) distance labeling of length $\lfloor n/2 \rfloor$. Thus, we have that $\lambda(P_n) \leq \lfloor n/2 \rfloor$. Since there are no three vertices in the path which are at the same distance, this lower bound turns out to be an equality. □

The sequence that appears in (1.1) also works for constructing proper distance labelings of cycles. Thus, we obtain the next result.

Proposition 2.3. *Let $n \geq 3$. The cycle C_n has*

$$\lambda(C_n) = \begin{cases} (n-2)/2, & n \neq 6, n \text{ is divisible by } 6, \\ \lfloor n/2 \rfloor, & \text{otherwise.} \end{cases}$$

Proof. Since, except for n divisible by 3, there are no three vertices in the cycle C_n which are at the same distance, we have that $\lambda(C_n) \geq \lfloor n/2 \rfloor$. The sequence that appears in (1.1) allows us to construct a (proper) distance $\lfloor n/2 \rfloor$-labeling of C_n when n is odd. Moreover, if n even not divisible by 3 we can obtain a distance $\lfloor n/2 \rfloor$-labeling of C_n from the sequence that appears in (1.1) just by removing the end odd label. Suppose now that n is divisible by 3. If n is odd or $n = 6$, at least $\lfloor n/2 \rfloor$ labels are needed to obtain a distance labeling of C_n. Thus, $\lambda(C_n) = \lfloor n/2 \rfloor$.

So, in what follows we will assume that n is divisible by 6. Since there are three vertices in the cycle which are at the same distance, we have that $\lambda(C_n) \geq (n-2)/2$. Let p_m and q_m be the largest even and odd numbers, respectively, not exceeding $(n-2)/2$. If $n \equiv 0, 4 \pmod 8$ then the sequence $(p_m, p_m - 2, \ldots, 2, q_m, 2, \ldots, p_m - 2, p_m, 0, q_m - 2, q_m - 4, \ldots, 3, q_m, n/3, 3, 5, \ldots, q_m - 2, 1, 1)$ defines a (proper) distance $(n-2)/2$-labeling of C_n. If $n \equiv 6 \pmod 8$ then $(p_m - 2, p_m - 4, \ldots, 2, p_m, 2, \ldots, p_m - 4, p_m - 2, 0, q_m, q_m - 2, \ldots, 3, p_m, n/3, 3, 5, \ldots, q_m - 2, q_m, 1, 1)$ defines a (proper) distance $(n-2)/2$-labeling of C_n. Finally, if $n \equiv 2 \pmod 8$, then the sequence $(p_m, p_m - 2, \ldots, 2, n/6 + \lceil n/12 \rceil, 2, \ldots, p_m, n/6 + \lceil n/12 \rceil, q_m, q_m - 2, \ldots, n/6 + \lceil n/12 \rceil + 2, n/6 + \lceil n/12 \rceil - 2, n/6 + \lceil n/12 \rceil - 4, \ldots, 3, 0, n/3, 3, 5, \ldots, n/6 + \lceil n/12 \rceil - 2, 1, 1, n/6 + \lceil n/12 \rceil + 2, n/6 + \lceil n/12 \rceil + 4, \ldots, q_m)$ defines a (proper) distance $(n-2)/2$-labeling of C_n. $\square$

Proposition 2.4. *The star $K_{1,k}$ has $\lambda(K_{1,k}) = 2$ when $k \geq 3$, and $\lambda(K_{1,k}) = 1$ otherwise.*

Proof. For $k \geq 3$, consider a labeling f that assigns the label 1 to the central vertex and to one of its leaves, and that assigns label 2 to the other vertices. Then f is a (proper) distance 2-labeling of $K_{1,k}$. For $1 \leq k \leq 2$, the sequences $1 - 1$ and $0 - 1 - 1$, where 0 is assigned to a leaf, give a (proper) distance 1-labeling of $K_{1,1}$ and $K_{1,2}$, respectively. $\square$

Proposition 2.5. *Let m and n be integers with $2 \leq m \leq n$. Then, $\lambda(K_{m,n}) = m$. In particular, the graph $K_{m,n}$ admits a proper distance l-labeling if and only if $m \in \{1, 2\}$.*

Proof. Let X and Y be the stable sets of $K_{m,n}$, with $|X| = m$ and $|Y| = n$. We have that $D(K_{m,n}) = 2$, however the maximum number of vertices that are mutually at distance 2 is n. Thus, by assigning label 2 to all vertices, except one, in Y, 1 to the remaining vertex in Y and to one vertex in X, 0 to another vertex of X we still have left $m - 2$ vertices in X to label. $\square$

Proposition 2.6. *Let n and k be positive integers with $n \geq 2$ and $k \geq 3$. Let S_k^n be the graph obtained from $K_{1,k}$ by replacing each edge with a path of n edges. Then*

$$\lambda(S_k^n) = \begin{cases} 2(n-1), & \text{if } k = n-1, \\ 2n-1, & \text{if } k = n, \\ 2n, & \text{if } k > n. \end{cases}$$

Moreover, for $k < n - 1$, the graph S_k^n admits an l-distance labeling, where $2(n-o) \leq l \leq 2(n-o)+1$, and $\lfloor (2n-1)/(2k+1) \rfloor \leq o \leq \lfloor (2n+2)/(2k+1) \rfloor$.

Proof. Suppose that S_k^n admits a distance l-labeling with $l < 2n$. Then, all the labels assigned to leaves should be different and they appear at most twice. Moreover, although each even label could appear k-times, one for each of the k paths that are joined to the star $K_{1,k}$, odd labels also appear at most twice (either in the same or in two of the original forming paths). Thus, once we fix the labels of leaves, we still have to assign a label to at least $(k-2)(n-2)+1$ vertices. Thus, at least $2n-2$ labels are needed for obtaining a distance labeling of S_k^n, when $k \geq n-1$. The following construction provides a distance $2(n-1)$-labeling of S_k^n, when $k = n-1$. Suppose that we label the central vertices of each path using the pattern $2-4-\ldots-2(n-1)$. Then, add odd labels to the leaves. For the case $k = n$, we need to introduce a new odd label, which corresponds to $2n-1$. Finally, when $k > n$, we cannot complete a distance l-labeling without using $2n$ labels. Fig. 1 provides a proper $2n$-labeling that can be generalized in that case.

The case $k < n-1$ requires a more detailed study. Consider the labeling of S_k^n obtained by assigning the labels in the sequence $0-2-4-\ldots-2(n-o)-s_1^i-s_2^i-\ldots-s_o^i$ to the vertices of the path P^i, $i = 1,\ldots,k$, where 0 is the label assigned to the central vertex of S_k^n, and $\{s_j^i\}_{i=1,\ldots,k}^{j=1,\ldots,o}$ is the (multi)set of odd labels, if necessary, we replace some of the even labels by the remaining odd labels. By considering the patern $1-1$, $3-1-1-3$, $5-3-1-1-3-5$ to the vertices of one of the paths, it can be checked that, the graph S_k^n admits an l-distance labeling with $l \in \{2(n-o), 2(n-o)+1\}$ and

$$\left\lfloor \frac{2n-1}{2k+1} \right\rfloor \leq o \leq \lfloor \frac{2n+2}{2k+1} \rfloor.$$

More specifically, if $\lfloor(2n-1)/(2k+1)\rfloor = \lfloor(2n+2)/(2k+1)\rfloor$ then $o = \lfloor(2n-1)/(2k+1)\rfloor$ and $l = 2(n-o)$. If $\lfloor(2n-1)/(2k+1)\rfloor + 1 = \lfloor(2n)/(2k+1)\rfloor$ then $o = \lfloor(2n)/(2k+1)\rfloor$ and $l = 2(n-o)+1$. Finally, if $\lfloor(2n)/(2k+1)\rfloor + 1 = \lfloor(2n+1)/(2k+1)\rfloor$ then $o = \lfloor(2n+1)/(2k+1)\rfloor$ and $l = 2(n-o)+1$.

□

Fig. 2 and Fig. 3 show proper distance labelings of S_4^5 and S_5^5, respectively, that have been obtained by using the above constructions, and then, combining pairs of paths (whose end odd labels sum up to 8) for obtaining a proper distance 8-labeling and 9-labeling, respectively.

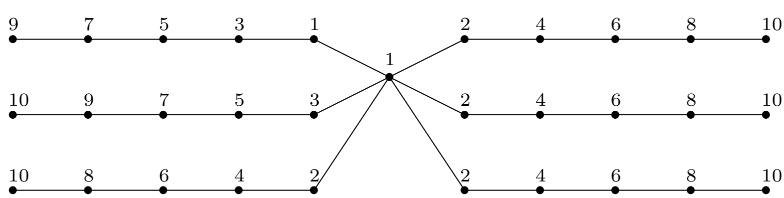

Figure 1: A proper distance 10-labeling of S_6^5.

Proposition 2.7. *For $n \geq 3$, let W_n be the wheel of order $n+1$. Then $\lambda(W_n) = \lceil n/2 \rceil$.*

Proof. Except for W_3, all wheels have $D(W_n) = 2$. The maximum number of vertices that are mutually at distance 2 is $\lfloor n/2 \rfloor$ and all of them are in the cycle. Thus, by assigning

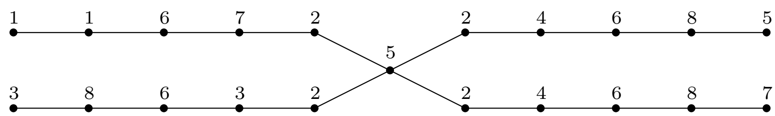

Figure 2: A proper distance 8-labeling of S_4^5.

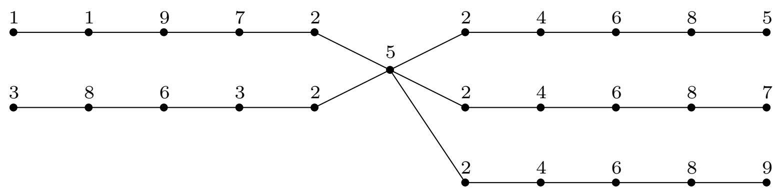

Figure 3: A proper distance 9-labeling of S_5^5.

label 2 to all these vertices, 0 to one vertex of the cycle and 1 to the central vertex and to one vertex of the cycle, we still have to label $\lceil n/2 \rceil - 2$ vertices.

Proposition 2.8. *For $n \geq 2$, let F_n be the fan of order $n+1$. Then $\lambda(F_n) = \lfloor n/2 \rfloor$.*

Proof. Except for F_2, all fans have $D(F_n) = 2$. The maximum number of vertices that are mutually at distance 2 is $\lceil n/2 \rceil$ and all of them are in the path. Thus, by assigning label 2 to all these vertices, 0 to one vertex of the path, 1 to the central vertex and to one vertex of the path when n is even and to two vertices when n is odd, we still have to label $\lfloor n/2 \rfloor - 2$ vertices.

2.1 The inverse problem

For every positive integer l, there exists a graph G of order l with a trivial l-labeling that assigns a different label in $[1, l]$ to each vertex. In this section, we are interested in the existence of a graph G that admits a proper distance l-labeling.

We are now ready to state and prove the next result.

Theorem 2.9. *For every pair of positive integers l and r, $r \geq 2$, there exists a graph G of order lr with a regular l-labeling of degree r.*

Proof. We give a constructive proof. Assume first that l is odd. Let G be the graph obtained from the complete graph K_r by identifying $r-1$ vertices of K_r with one of the end vertices of a path of length $\lfloor l/2 \rfloor$ and the remaining vertex of K_r with the central vertex of the graph $S_{r+1}^{\lfloor l/2 \rfloor}$. That is, G is obtained from K_r by attaching $2r$ paths of length $\lfloor l/2 \rfloor$ to its vertices, $r+1$ to a particular vertex v_1 of K_r and exactly one path to each of the remaining vertices $F = \{v_2, v_3, \ldots, v_r\}$ of K_r. Now, consider the labeling f of G that assigns 1 to the vertices of K_r, the sequence $1 - 3 - \ldots - l$ to the vertices of the paths attached to F and one of the paths attached to v_1, and the sequence $1 - 2 - 4 - \ldots - (l-1)$ to the remaining paths. Then f is a regular l-labeling of degree r of G. Assume now that l is even. Let G be the graph obtained in the above construction for $l-1$. Then, by adding a leaf to each vertex of G labeled with $l-2$ we obtain a new graph G' that admits a regular l-labeling f' of degree r. The labeling f' can be obtained from the labeling f of G, defined above, just by assigning the label l to the new vertices. $\square$

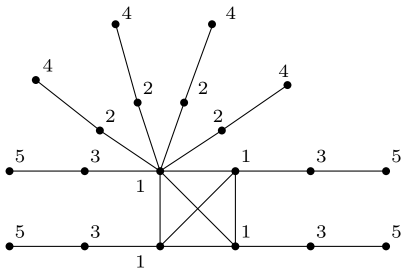

Figure 4: A regular 5-labeling of degree 4 of a graph G.

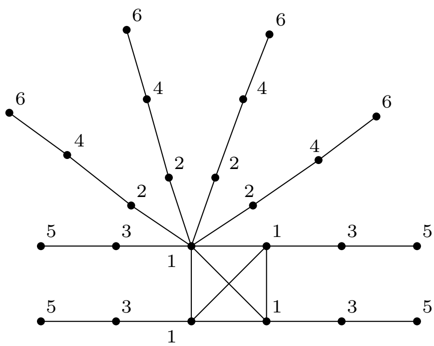

Figure 5: A regular 6-labeling of degree 4 of a graph G'.

Notice that, the graph provided in the proof of Theorem 2.9 also has $\lambda(G) = l$. Figs 4 and 5 show examples for the above construction. The pattern provided in the proof of the above theorem, for $r = 2$, can be modified in order to obtain the following lower bound for the size of a graph G as in Theorem 2.9.

Proposition 2.10. *For every positive integer l there exists a graph of order $2l$ and size $(l+2)(l+1)/2-2$ that admits a regular distance l-labeling of degree 2.*

Proof. Let G be the graph of order $2l$ and size $(l+1)l/2+l-1$, obtained from K_{l+1} and the path P_l by identifying one of the end vertices u of P_l with a vertex v of K_{l+1}. Let f be the labeling of G that assigns the sequence $1-2-3-\ldots-l$ to the vertices of P_l and $1-1-2-\ldots-l$ to the verticalces of K_{l+1} in such a way that the vertex obtained by identifying u and v is labeled 1. Then, f is a 2-regular l-labeling of G. $\square$

Thus, a natural question appears.

Question 2.11. Can we find graphs that admit a regular distance l-labeling of degree 2 which have bigger density (where by density we refer the number of edges in relation to the number of vertices) than the one of Proposition 2.10?

We end this section by introducing an open question related to complexity.

Question 2.12. What is the algorithmic complexity of computing $\lambda(G)$ for a general graph G? What about for a tree?

3 Distance J-labelings

It is clear from the definition that to say that a graph admits a (proper) distance l-labeling is the same as to say that the graph admits either a (proper) distance $[0, l]$-labeling or a (proper) distance $[1, l]$-labeling. That is, we relax the condition on the labels, the set of labels is not necessarily a set of consecutive integers. In this section, we study which kind of sets J can appear as the set of labels of a graph that admits a distance J-labeling.

The following easy fact is obtained from the definition.

Lemma 3.1. *Let G be a graph with a proper distance J-labeling f. Then $J \subset [0, D(G)]$, where $D(G)$ is the diameter of G.*

3.1 The inverse problem: distance J-labelings obtained from sequences.

We start with a definition. Let $S = (s_1, s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots, s_l, \ldots, s_l)$ be a sequence of nonnegative integers where, (i) $s_i < s_j$ whenever $i < j$ and (ii) each number s_i appears k_i times, for $i = 1, 2, \ldots, l$. We say that S is a *δ-sequence* if there is a simple graph G that admits a partition of the vertices $V(G) = \cup_{i=1}^{l} V_i$ such that, for all $i \in \{1, 2, \ldots, l\}$, $|V_i| = k_i$, and if $u, v \in V_i$ then $d_G(u, v) = s_i$. The graph G is said to *realize* the sequence S.

Let $\Sigma = \{s_1 < s_2 < \ldots < s_l\}$ be a set of nonnegative integers. We say that Σ is a δ-set with n degrees of freedom or a *δ_n-set* if there is a δ-sequence S of the form $S = (s_1, s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots, s_l, \ldots, s_l)$, in which the following conditions hold: (i) all, except n numbers different from zero, appear at least twice, and (ii) if $s_1 = 0$ then 0 appears exactly once in S. We say that any graph realizing S also *realizes* Σ. If $n = 0$ we simply say that Σ is a *δ-set*. Let us notice that an equivalent definition for a δ-set is the following: Σ is a *δ-set* if there exists a graph G that admits a proper distance Σ-labeling.

Proposition 3.2. *Let $\Sigma = \{1 = s_1 < s_2 < \ldots < s_l\}$ be a set such that $s_i - s_{i-1} \leq 2$, for $i = 1, 2, \ldots, l$. Then Σ is a δ-set. Furthermore, there is a caterpillar of order $2l$ that realizes Σ.*

Proof. We claim that for each set $\Sigma = \{1 = s_1 < s_2 < \ldots < s_l\}$ such that $s_i - s_{i-1} \leq 2$ there is a caterpillar of order $2l$ that admits a 2-regular distance Σ-labeling in which the label s_l is assigned to exactly two leaves. The proof is by induction on l. For $l = 1$, the path P_2 admits a 2-regular distance $\{1\}$-labeling, and for $l = 2$, the star $K_{1,3}$ and the path P_4 admit a 2-regular distance $\{1, 2\}$-labeling and a 2-regular distance $\{1, 3\}$-labeling, respectively. Assume that the claim is true for l and let $\Sigma = \{1 = s_1 < s_2 < \ldots < s_{l+1}\}$ such that $s_i - s_{i-1} \leq 2$. Let $\Sigma' = \Sigma \setminus \{s_{l+1}\}$. By the induction hypothesis, there is a caterpillar G' of order $2l$ that admits a regular distance Σ'-labeling of degree 2 in which the label s_l is assigned to leaves, namely, u_1 and u_2. Let $u \in V(G')$ be the (unique) vertex in G' adjacent to u_1. Then, if $s_{l+1} - s_l = 2$, the caterpillar obtained from G' by adding two new vertices v_1 and v_2 and the edges $u_i v_i$, for $i = 1, 2$, admits a regular distance Σ-labeling of degree 2 in which the label s_{l+1} is assigned to leaves $\{v_1, v_2\}$. Otherwise, if $s_{l+1} - s_l = 1$ then the caterpillar obtained from G' by adding two new vertices v_1 and v_2 and the edges uv_1 and $u_2 v_2$ admits a regular distance Σ-labeling of degree 2 in which the label s_{l+1} is assigned to leaves $\{v_1, v_2\}$. This proves the claim. To complete the proof, we only have to consider the vertex partition of G defined by the vertices that receive the same label. $\square$

Proposition 3.2 provides us with a family of δ-sets, in which, if we order the elements of each δ-set, we get that the differences between consecutive elements are at most 2. This fact may lead us to get the idea that the differences between consecutive elements in δ-sets cannot be too large. This is not true in general and we show it in the next result.

Theorem 3.3. *Let $\{k_1, k_2, \ldots, k_n\}$ be a set of positive integers. Then there exists a δ-set $\Sigma = \{s_1 < s_2 < \ldots < s_l\}$ and a set of indices $\{1 \leq j_1 < j_2 < \ldots < j_n\}$, with $j_n < l-1$, such that*

$$s_{j_1+1} - s_{j_1} = k_1,\ s_{j_2+1} - s_{j_2} = k_2, \ldots,\ s_{j_n+1} - s_{j_n} = k_n.$$

Moreover, s_1 can be chosen to be any positive integer.

Proof. Choose any number $d_1 \in \mathbb{N}$ and choose any Langford sequence of defect d_1 (such a sequence exists by Theorem 1.1. We let $d_1 = s_1$. (Notice that if $d_1 = 1$ then the sequence is actually a Skolem sequence). Let this Langford sequence be L_1. Next, choose a Langford sequence L_2 with defect $\max L_1 + k_1$. Next, choose a Langford sequence L_3 with defect $\max L_2 + k_2$. Continue this procedure until we have used all the values $k_1, k_2, \ldots, k_n$. At this point create a new sequence L, where L is the concatenation of $L_1, L_2, \ldots, L_{n+1}$ and label the vertices of the path P_r, $r = \sum_{i=1}^{n+1} |L_i|$, with the elements of L keeping the order in the labeling induced by the sequence L. This shows the result. $\square$

The next result shows that there are sets that are not δ-sets.

Proposition 3.4. *The set $\Sigma = \{2, 3\}$ is not a δ-set.*

Proof. The proof is by contradiction. Assume to the contrary that $\Sigma = \{2, 3\}$ is a δ-set. That is to say, we assume that there exists a sequence S consisting of k_1 copies of 2 and k_2 copies of 3 that is a δ-sequence. Let G be a graph that realizes S and $V_1 \cup V_2$ the partition of $V(G)$ defined as follows: if $u, v \in V_i$ then $d_G(u, v) = i + 1$, for $i = 1, 2$. It is clear that V_1 must be formed by the leaves of a star with center some vertex $a \in V$. Since a is at distance 1 of any vertex in V_1, it follows that a must be in V_2 and furthermore, all vertices adjacent to a must be in V_1. Thus, there are no two adjacent vertices in the neighborhood of a. At this point, let $b \in V_2 \setminus \{a\}$. Then, there is a path of the form a, u_1, u_2, b, where $u_1 \in V_1$ and hence, $u_2, b \in V_2$. This contradicts the fact that $d_G(u_2, b) = 1$. $\square$

The above proof works for any set of the form $\Sigma = \{2, n\}$, for $n \geq 3$. Thus, in fact, Proposition 3.4 can be generalized as follows.

Proposition 3.5. *The set $\Sigma = \{2, n\}$ is not a δ-set.*

Notice that, although $\Sigma = \{2, n\}$ is not a δ-set, it is a δ_1-set, since we can consider a star in which the center is labeled with n and the leaves with 2.

The next result gives a lower bound on the size of δ-sets in terms of the maximum of the set.

Theorem 3.6. *Let Σ be a δ-set with $s = \max \Sigma$. Then, $|\Sigma| \geq \lceil (s+1)/2 \rceil$.*

Proof. Let G be a graph that realizes Σ and let $V(G) = \cup_{i\in\Sigma} V_i$ be the partition defined as follows: if $u, v \in V_i$ then $d_G(u, v) = i$. Let $a_1, a_2 \in V_s$. At this point, let $P = b_1 b_2 \ldots b_{s+1}$ be a path of length s starting at a_1 and ending at a_2. We claim that there are no three vertices in $V(P)$ belonging to the same set V_j, $j \in \Sigma$. Assume to the contrary that there exist vertices u, v and $w \in V(P)$ such that $d_G(u, v) = d_G(u, w) = d_G(v, w)$. Then, we also obtain that $d_P(u, v) = d_P(v, w) = d_P(v, w)$ (since they are on a shortest path between two points), a contradiction. Hence, each set in the partition of $V(G)$ can contain at most two vertices of P. Since $|V(P)| = s + 1$, it follows that we need at least $\lceil (s+1)/2 \rceil$ sets in the partition of $V(G)$. Therefore, we obtain that $|\Sigma| \geq \lceil (s+1)/2 \rceil$. $\square$

It is clear that the above proof cannot be improved in general, since from Proposition 3.2 we get that the any set of the form $\{1, 3, 5, \ldots, 2n+1\}$ is a δ-set and $|\{1, 3, 5, \ldots, 2n + 1\}| = \lceil (2n+2)/2 \rceil$. Furthermore, Proposition 3.5 for $n \geq 4$ is an immediate consequence of the above result. It is also worth to mention that there are sets which meet the bound provided in Theorem 3.6, however they are not δ-sets. For instance, the set $\{2, 3\}$ considered in Proposition 3.4. From this fact, it seems that we cannot characterize δ-sets just from a density point of view. Next we want to propose the following open problem.

Problem 3.7. Characterize δ-sets.

Let Σ be a set. By construction, a path of order $|\Sigma|$ in which each vertex receives a different labeling of Σ defines a distance $|\Sigma|$-labeling. That is, every set is a $\delta_{|\Sigma|}$-set. So, according to that, we propose the next problem.

Problem 3.8. Given a set Σ is there a construction that provides the minimum r such that Σ is a δ_r-set?

Thus, the above problem is a bit more general than Open problem 3.7.

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