Remarks on the thickness of $K_{n,n,n}$

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Abstract

The thickness $\theta(G)$ of a graph G is the minimum number of planar subgraphs into which G can be decomposed. In this paper, we provide a new upper bound for the thickness of the complete tripartite graphs $K_{n,n,n}$ ($n \geq 3$) and obtain $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$, when $n \equiv 3 \pmod{6}$.

Keywords: Thickness, complete tripartite graph, planar subgraphs decomposition.

Math. Subj. Class.: 05C10

1 Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs into which G can be decomposed. It was defined by Tutte [11] in 1963, derived from early work on biplanar graphs [2, 10]. It is a classical topological invariant of a graph and also has many applications to VLSI design, graph drawing, etc. Determining the thickness of a graph is NP-hard [7], so the results about thickness are few. The only types of graphs whose thicknesses have been determined are complete graphs [1, 3], complete bipartite graphs [4] and hypercubes [5]. The reader is referred to [6, 8] for more background on the thickness problems.

In this paper, we study the thickness of complete tripartite graphs $K_{n,n,n}$, ($n \geq 3$). When $n = 1, 2$, it is easy to see that $K_{1,1,1}$ and $K_{2,2,2}$ are planar graphs, so the thickness of both ones is one. Poranen proved $\theta(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$ in [9] which was the only result about the thickness of $K_{n,n,n}$, as far as the author knows. We will give a new upper bound for $\theta(K_{n,n,n})$ and provide the exact number for the thickness of $K_{n,n,n}$, when n is congruent to 3 mod 6, the main results of this paper are the following theorems.

Theorem 1.1. For $n \geq 3$, $\theta(K_{n,n,n}) \leq \lceil \frac{n+1}{3} \rceil + 1$.

Theorem 1.2. $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$ when $n \equiv 3 \pmod{6}$.

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2 The proofs of the theorems

In [4], Beineke, Harary and Moon determined the thickness of complete bipartite graph $K_{m,n}$ for almost all values of m and n .

Lemma 2.1. [4] *The thickness of $K_{m,n}$ is $\lceil \frac{mn}{2(m+n-2)} \rceil$ except possibly when m and n are odd, $m \leq n$ and there exists an integer k satisfying $n = \lfloor \frac{2k(m-2)}{m-2k} \rfloor$.*

Lemma 2.2. *For $n \geq 3$, $\theta(K_{n,n,n}) \geq \lceil \frac{n+1}{3} \rceil$.*

Proof. Since $K_{n,2n}$ is a subgraph of $K_{n,n,n}$, we have $\theta(K_{n,n,n}) \geq \theta(K_{n,2n})$. From Lemma 2.1, the thickness of $K_{n,2n}$ ($n \geq 3$) is $\lceil \frac{n+1}{3} \rceil$, so the lemma follows. $\square$

For the complete tripartite graph $K_{n,n,n}$ with the vertex partition (A, B, C) , where $A = \{a_0, \dots, a_{n-1}\}$, $B = \{b_0, \dots, b_{n-1}\}$ and $C = \{c_0, \dots, c_{n-1}\}$, we define a type of graphs, they are planar spanning subgraphs of $K_{n,n,n}$, denoted by $G[a_i b_{j+i} c_{k+i}]$, in which $0 \leq i, j, k \leq n-1$ and all subscripts are taken modulo n . The graph $G[a_i b_{j+i} c_{k+i}]$ consists of n triangles $a_i b_{j+i} c_{k+i}$ for $0 \leq i \leq n-1$ and six paths of length $n-1$, they are

$$a_0 b_{j+1} c_{k+2} a_3 b_{j+4} c_{k+5} \dots a_{3i} b_{j+3i+1} c_{k+3i+2} \dots,$$

$$c_k a_1 b_{j+2} c_{k+3} a_4 b_{j+5} \dots c_{k+3i} a_{3i+1} b_{j+3i+2} \dots,$$

$$b_j c_{k+1} a_2 b_{j+3} c_{k+4} a_5 \dots b_{j+3i} c_{k+3i+1} a_{3i+2} \dots,$$

$$a_0 c_{k+1} b_{j+2} a_3 c_{k+4} b_{j+5} \dots a_{3i} c_{k+3i+1} b_{j+3i+2} \dots,$$

$$b_j a_1 c_{k+2} b_{j+3} a_4 c_{k+5} \dots b_{j+3i} a_{3i+1} c_{k+3i+2} \dots,$$

$$c_k b_{j+1} a_2 c_{k+3} b_{j+4} a_5 \dots c_{k+3i} b_{j+3i+1} a_{3i+2} \dots$$

Equivalently, the graph $G[a_i b_{j+i} c_{k+i}]$ is the graph with the same vertex set as $K_{n,n,n}$ and edge set

$$\{a_i b_{j+i-1}, a_i b_{j+i}, a_i b_{j+i+1}, a_i c_{k+i-1}, a_i c_{k+i}, a_i c_{k+i+1} \mid 1 \leq i \leq n-2\}$$

$$\cup \{b_{j+i} c_{k+i-1}, b_{j+i} c_{k+i}, b_{j+i} c_{k+i+1} \mid 1 \leq i \leq n-2\}$$

$$\cup \{a_0 b_j, a_0 b_{j+1}, a_{n-1} b_{j+n-2}, a_{n-1} b_{j+n-1}\}$$

$$\cup \{a_0 c_k, a_0 c_{k+1}, a_{n-1} c_{k+n-2}, a_{n-1} c_{k+n-1}\}$$

$$\cup \{b_j c_k, b_j c_{k+1}, b_{j+n-1} c_{k+n-2}, b_{j+n-1} c_{k+n-1}\}.$$

Figure 1(a) illustrates the planar spanning subgraph $G[a_i b_i c_i]$ of $K_{5,5,5}$.

Figure 1: A planar subgraphs decomposition of $K_{5,5,5}$

Theorem 2.3. When $n = 3p + 2$ (p is a positive integer), $\theta(K_{n,n,n}) \leq p + 2$.

Proof. When $n = 3p + 2$ (p is a positive integer), we will construct two different planar subgraphs decompositions of $K_{n,n,n}$ according to p is odd or even, in which the number of planar subgraphs is $p + 2$ in both cases.

Case 1. p is odd. Let $G_1, \dots, G_p$ be p planar subgraphs of $K_{n,n,n}$ where $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, for $1 \leq t \leq \frac{p+1}{2}$; and $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)+2}]$, for $\frac{p+3}{2} \leq t \leq p$ and $p \geq 3$. From the structure of $G[a_i b_{j+i} c_{k+i}]$, we get that no two edges in $G_1, \dots, G_p$ are repeated. Because subscripts in G_t , $1 \leq t \leq p$ are taken modulo n , $\{3(t-1) \pmod n \mid 1 \leq t \leq p\} = \{0, 3, 6, \dots, 3(p-1)\}$, $\{6(t-1) \pmod n \mid 1 \leq t \leq \frac{p+1}{2}\} = \{0, 6, \dots, 3(p-1)\}$ and $\{6(t-1)+2 \pmod n \mid \frac{p+3}{2} \leq t \leq p\} = \{3, 9, \dots, 3(p-2)\}$, the subscript sets of b and c in G_t , $1 \leq t \leq p$ are the same, i.e.,

$$\{i + 3(t-1) \pmod n \mid 1 \leq t \leq p\}$$

$$= \{i + 6(t-1) \pmod n \mid 1 \leq t \leq \frac{p+1}{2}\} \cup \{i + 6(t-1) + 2 \pmod n \mid \frac{p+3}{2} \leq t \leq p\}.$$

Furthermore, if there exists $t \in \{1, \dots, p\}$ such that $a_i b_j$ is an edge in G_t , then $a_i c_j$ is an edge in G_k for some $k \in \{1, \dots, p\}$. If the edge $a_i b_j$ is not in any G_t , then neither is the edge $a_i c_j$ in any G_t , for $1 \leq t \leq p$.

From the construction of G_t , the edges that belong to $K_{n,n,n}$ but not to any G_t , $1 \leq t \leq p$, are

$$a_0 b_{3(t-1)-1}, a_0 c_{3(t-1)-1}, \quad 1 \leq t \leq p \quad (1)$$

$$a_{n-1} b_{3(t-1)}, a_{n-1} c_{3(t-1)}, \quad 1 \leq t \leq p \quad (2)$$

$$a_i b_{i-3}, a_i b_{i-2}, \quad 0 \leq i \leq n-1 \quad (3)$$

$$a_i c_{i-3}, a_i c_{i-2}, \quad 0 \leq i \leq n-1 \quad (4)$$

$$b_i c_{i+3(t-1)-1}, b_i c_{i+3(t-1)}, \quad 0 \leq i \leq n-1 \text{ and } t = \frac{p+3}{2} \quad (5)$$

$$b_{3(t-1)} c_{6(t-1)-1}, b_{3(t-1)-1} c_{6(t-1)}, \quad 1 \leq t \leq \frac{p+1}{2} \quad (6)$$

$$b_{3(t-1)} c_{6(t-1)+1}, b_{3(t-1)-1} c_{6(t-1)+2}, \quad \frac{p+3}{2} \leq t \leq p \text{ and } p \geq 3 \quad (7)$$

Let G_{p+1} be the graph whose edge set consists of the edges in (3) and (5), and G_{p+2} be the graph whose edge set consists of the edges in (1), (2), (4), (6) and (7). In the following, we will describe plane drawings of G_{p+1} and G_{p+2} .

(a) A planar embedding of G_{p+1} .

Place vertices $b_0, b_1, \dots, b_{n-1}$ on a circle, place vertices a_{i+3} and $c_{i+\frac{n+1}{2}}$ in the middle of b_i and b_{i+1} , join each of a_{i+3} and $c_{i+\frac{n+1}{2}}$ to both b_i and b_{i+1} , we get a planar embedding of G_{p+1} . For example, when $p = 1$, $n = 5$, Figure 1(b) shows the subgraph G_2 of $K_{5,5,5}$.

(b) A planar embedding of G_{p+2} .

Firstly, we place vertices $c_0, c_1, \dots, c_{n-1}$ on a circle, join vertex a_{i+3} to c_i and c_{i+1} , for $0 \leq i \leq n-1$, so that we get a cycle of length $2n$. Secondly, join vertex a_{n-1} to $c_{3(t-1)}$ for $1 \leq t \leq p$, with lines inside of the cycle. Let ℓ_t be the line drawn inside the cycle joining a_{n-1} with $c_{6(t-1)-1}$ if $1 \leq t \leq \frac{p+1}{2}$ or with $c_{6(t-1)+1}$ if $\frac{p+3}{2} \leq t \leq p$ ($p \geq 3$). For $1 \leq t \leq p$, insert the vertex $b_{3(t-1)}$ in the line ℓ_t . Thirdly, join vertex a_0 to $c_{3(t-1)-1}$ for $1 \leq t \leq p$, with lines outside of the cycle. Let ℓ'_t be the line drawn outside the cycle joining a_0 with $c_{6(t-1)}$ if $1 \leq t \leq \frac{p+1}{2}$ or with $c_{6(t-1)+2}$ if $\frac{p+3}{2} \leq t \leq p$ ($p \geq 3$). For $1 \leq t \leq p$, insert the vertex $b_{3(t-1)-1}$ in the line ℓ'_t . In this way, we can get a planar embedding of G_{p+2} . For example, when $p = 1$, $n = 5$, Figure 1(c) shows the subgraph G_3 of $K_{5,5,5}$.

Summarizing, when p is an odd positive integer and $n = 3p+2$, we get a decomposition of $K_{n,n,n}$ into $p+2$ planar subgraphs $G_1, \dots, G_{p+2}$.

Case 2. p is even. Let $G_1, \dots, G_p$ be p planar subgraphs of $K_{n,n,n}$ where $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)+3}]$, for $1 \leq t \leq \frac{p}{2}$; and $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)+2}]$, for $\frac{p+2}{2} \leq t \leq p$. With a similar argument to the proof of Case 1, we can get that the subscript sets of b and c in G_t , $1 \leq t \leq p$ are the same, i.e.,

$$\{i + 3(t-1) \pmod n \mid 1 \leq t \leq p\}$$

$$= \{i + 6(t-1) + 3 \pmod n \mid 1 \leq t \leq \frac{p}{2}\} \cup \{i + 6(t-1) + 2 \pmod n \mid \frac{p+2}{2} \leq t \leq p\}.$$

From the construction of G_t , $G_{\frac{p}{2}}$ and $G_{\frac{p+2}{2}}$ have $n - 2$ edges in common, they are $b_{i+3(\frac{p+2}{2}-1)}c_{i+6(\frac{p+2}{2}-1)+1}$, $1 \leq i \leq n - 1$ and $i \neq n - 4$, we can delete them in one of these two graphs to avoid repetition.

The edges that belong to $K_{n,n,n}$ but not to any G_t , $1 \leq t \leq p$, are

$$a_0b_{3(t-1)-1}, \quad a_0c_{3(t-1)-1}, \quad 1 \leq t \leq p \quad (8)$$

$$a_{n-1}b_{3(t-1)}, \quad a_{n-1}c_{3(t-1)}, \quad 1 \leq t \leq p \quad (9)$$

$$a_ib_{i-3}, \quad a_ib_{i-2}, \quad 0 \leq i \leq n - 1 \quad (10)$$

$$a_ic_{i-3}, \quad a_ic_{i-2}, \quad 0 \leq i \leq n - 1 \quad (11)$$

$$b_ic_{i-1}, \quad b_ic_i, \quad b_ic_{i+1}, \quad 0 \leq i \leq n - 1 \quad (12)$$

$$b_{3(t-1)}c_{6t-4}, \quad 1 \leq t \leq \frac{p}{2} \quad (13)$$

$$b_{3(t-1)}c_{6t-5}, \quad \frac{p+2}{2} < t \leq p \quad (14)$$

$$b_{3(t-1)-1}c_{6t-3}, \quad 1 \leq t < \frac{p}{2} \quad (15)$$

$$b_{3(t-1)-1}c_{6t-4}, \quad \frac{p+2}{2} \leq t \leq p \quad (16)$$

Let G_{p+1} be the graph whose edge set consists of the edges in (10), (11) and (12), and G_{p+2} be the graph whose edge set consists of the edges in (8), (9), (13), (14), (15) and (16). We draw G_{p+1} in the following way. Firstly, place vertices $b_0, c_0, b_1, c_1, \dots, b_{n-1}, c_{n-1}$ on a circle C , join vertex c_i to b_i and b_{i+1} , we get a cycle of length $2n$. Secondly, place vertices $a_0, a_2, \dots, a_{n-2}$ on a circle C' in the unbounded region defined by the circle C such that C is contained in the closed disk defined by C' , place vertices $a_1, a_3, \dots, a_{n-1}$ on a circle C'' contained in the bounded region of C . Join a_i to $b_{i-3}, b_{i-2}, c_{i-3}$, and c_{i-2} , join b_i to c_{i+1} . We can get a planar embedding of G_{p+1} , so it is a planar graph. G_{p+2} is also planar because it is a subgraph of a graph homeomorphic to a dipole (two vertices joined by some edges). For example, when $p = 2$, $n = 8$, Figure 2(c) and Figure 2(d) show the subgraphs G_3 and G_4 of $K_{8,8,8}$ respectively.

Summarizing, when p is an even positive integer and $n = 3p + 2$, we obtain a decomposition of $K_{n,n,n}$ into $p + 2$ planar subgraphs $G_1, \dots, G_{p+2}$.

Theorem follows from Cases 1 and 2. $\square$

From the proof of Theorem 2.3, we draw planar subgraphs decompositions of $K_{5,5,5}$ and $K_{8,8,8}$ as illustrated in Figure 1 and Figure 2 respectively.

(a) The subgraph $G_1 = G[a_i b_i c_{i+3}]$ of $K_{8,8,8}$

(b) The subgraph $G_2 - b_4 c_0 - b_5 c_1 - b_6 c_2 - b_0 c_4 - b_1 c_5 - b_2 c_6$ of $K_{8,8,8}$ in which $G_2 = G[a_i b_i c_{i+3}]$

(c) The subgraph G_3 of $K_{8,8,8}$ (d) The subgraph G_4 of $K_{8,8,8}$ Figure 2: A planar subgraphs decomposition of $K_{8,8,8}$

Proof of Theorem 1.1. Because graph $K_{n-1,n-1,n-1}$ is a subgraph of $K_{n,n,n}$, $\theta(K_{n-1,n-1,n-1}) \leq \theta(K_{n,n,n})$, by Theorem 2.3, $\theta(K_{n,n,n}) \leq p + 2$ also holds, when $n = 3p$ or $n = 3p + 1$ (p is a positive integer), the theorem follows. $\square$

Proof of Theorem 1.2. When $n = 3p$ is odd, i.e., $n \equiv 3 \pmod{6}$, we decompose $K_{n,n,n}$ into $p + 1$ planar subgraphs $G_1, \dots, G_{p+1}$, where $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, for $1 \leq t \leq p$. With a similar argument to the proof of Theorem 2.3, we can get that the subscript sets of b and c in G_t , $1 \leq t \leq p$ are the same, i.e.,

$$\{i + 3(t - 1) \pmod{n} \mid 1 \leq t \leq p\} = \{i + 6(t - 1) \pmod{n} \mid 1 \leq t \leq p\}.$$

If the edge $a_i b_j$ is in G_t for some $t \in \{1, \dots, p\}$, then there exists $k \in \{1, \dots, p\}$ such that $a_i c_j$ is in G_k . If the edge $a_i b_j$ is not in any G_t , then neither is the edge $a_i c_j$ in any G_t , for $1 \leq t \leq p$.

From the construction of $G_t = G[a_i b_{i+3(t-1)} c_{i+6(t-1)}]$, we list the edges that belong to $K_{n,n,n}$ but not to any G_t , $1 \leq t \leq p$, as follows.

$$a_0 b_{3(t-1)-1}, \quad a_0 c_{6(t-1)-1}, \quad 1 \leq t \leq p \quad (17)$$

$$a_{n-1} b_{3(t-1)}, \quad a_{n-1} c_{6(t-1)}, \quad 1 \leq t \leq p \quad (18)$$

$$b_{3(t-1)} c_{6(t-1)-1}, \quad b_{3(t-1)-1} c_{6(t-1)}, \quad 1 \leq t \leq p \quad (19)$$

Let G_{p+1} be the graph whose edge set consists of the edges in (17), (18) and (19). It is easy to see that G_{p+1} is homeomorphic to a dipole and it is a planar graph.

Summarizing, when p is an odd positive integer and $n = 3p$, we obtain a decomposition of $K_{n,n,n}$ into $p + 1$ planar subgraphs $G_1, \dots, G_{p+1}$, therefore $\theta(K_{n,n,n}) \leq p + 1$. Combining this fact and Lemma 2.2, the theorem follows. $\square$

Figure 3: A planar subgraphs decomposition of $K_{3,3,3}$

According to the proof of Theorem 1.2, we draw a planar subgraphs decomposition of $K_{3,3,3}$ as shown in Figure 3.

For some other $\theta(K_{n,n,n})$ with small n , combining Lemma 2.2 and Poranen's result mentioned in Section 1, we have $\theta(K_{4,4,4}) = 2$, $\theta(K_{6,6,6}) = 3$. Since there exists a decomposition of $K_{7,7,7}$ with three planar subgraphs as shown in Figure 4, Lemma 2.2 implies that $\theta(K_{7,7,7}) = 3$. We also conjecture that the thickness of $K_{n,n,n}$ is $\lceil \frac{n+1}{3} \rceil$ for all $n \geq 3$.

(a) The subgraph $G_1 = G[a_i b_i c_i]$ of $K_{7,7,7}$

(b) The subgraph

$G_2 - a_1b_2 - a_2b_3 - a_3b_4 - a_4b_5 - a_5b_6 - b_0c_1 - b_1c_2 - b_3c_4 - b_4c_5 - b_5c_6$ of $K_{7,7,7}$ in which $G_2 = G[a_i b_{i+2} c_{i+4}]$

(c) The subgraph G_3 of $K_{7,7,7}$

Figure 4: A planar subgraphs decomposition of $K_{7,7,7}$

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