Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 12 (2017) 127–134

The strong metric dimension of generalized Sierpiński graphs with pendant vertices

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Received 17 February 2015, accepted 10 February 2016, published online 2 November 2016

Abstract

Let G be a connected graph of order n having $\varepsilon(G)$ end-vertices. Given a positive integer t, we denote by $S(G,t)$ the t-th generalized Sierpiński graph of G. In this note we show that if every internal vertex of G is a cut vertex, then the strong metric dimension of $S(G,t)$ is given by

$$\dim_s(S(G,t)) = \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right) - n + 1}{n-1}.$$

Keywords: Strong metric dimension, Sierpiński graphs.

Math. Subj. Class.: 05C12, 05C76

1 Introduction

For two vertices u and v in a connected graph G, the interval $I_G[u,v]$ between u and v is defined as the collection of all vertices that belong to some shortest $u-v$ path. A vertex w strongly resolves two vertices u and v if $v \in I_G[u,w]$ or $u \in I_G[v,w]$. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly resolved by some vertex of S. The smallest cardinality of a strong metric generator of G is called *strong metric dimension* and is denoted by $\dim_s(G)$. After the publication of

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the first paper [16], the strong metric dimension has been extensively studied. The reader is invited to read, for instance, the following works [10, 11, 12, 13, 15] and the references cited therein. For some basic graph classes, the strong metric dimension is easy to compute. For instance, $\dim_s(G) = n-1$ if and only if G is the complete graph of order n. For the cycle C_n of order n the strong dimension is $\dim_s(C_n) = \lceil n/2 \rceil$ and if T is a tree with $l(T)$ leaves, its strong metric dimension equals $l(T)-1$ (see [16]).

Given a connected graph G and two vertices $x, y \in V(G)$, we denote by $d_G(x,y)$ the distance from x to y. A vertex u of G is *maximally distant* from v if for every vertex w in the open neighborhood of u, $d_G(v,w) \leq d_G(u,v)$. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are *mutually maximally distant*. The *boundary* of $G=(V,E)$ is defined as $\partial(G) = \{u \in V :$ there exists $v \in V$ such that u, v are mutually maximally distant$\}$. For some basic graph classes, such as complete graphs K_n, complete bipartite graphs $K_{r,s}$, cycles C_n and hypercube graphs Q_k, the boundary is simply the whole vertex set. It is not difficult to see that this property holds for all 2-antipodal[1] graphs and also for all distance-regular graphs. Notice that the boundary of a tree consists exactly of the set of its leaves. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G. Notice that $\sigma(G) \subseteq \partial(G)$.

We use the notion of strong resolving graph introduced in [13]. The *strong resolving graph*[2] of G is a graph G_{SR} with vertex set $V(G_{SR}) = \partial(G)$ where two vertices u, v are adjacent in G_{SR} if and only if u and v are mutually maximally distant in G. There are some families of graph for which its resolving graph can be obtained relatively easily. For instance, we emphasize the following cases.

- If $\partial(G) = \sigma(G)$, then $G_{SR} \cong K_{|\partial(G)|}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T with $l(T)$ leaves, $(T)_{SR} \cong K_{l(T)}$.
- For any 2-antipodal graph G of order n, $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.
- $(C_{2k+1})_{SR} \cong C_{2k+1}$.

A set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of G. Oellermann and Peters-Fransen [13] showed that the problem of finding the strong metric dimension of a connected graph G can be transformed to the problem of finding the vertex cover number of G_{SR}.

Theorem 1.1. [13] *For any connected graph G,* $\dim_s(G) = \alpha(G_{SR})$.

It was shown in [13] that the problem of computing $\dim_s(G)$ is NP-hard. This suggests finding the strong metric dimension for special classes of graphs or obtaining good bounds on this invariant. In this note we study the problem of finding exact values or sharp bounds for the strong metric dimension of Sierpiński graphs with pendant vertices.

[1]The diameter of $G=(V,E)$ is defined as $D(G) = \max_{u,v \in V}\{d(u,v)\}$. We recall that $G=(V,E)$ is 2-antipodal if for each vertex $x \in V$ there exists exactly one vertex $y \in V$ such that $d_G(x,y) = D(G)$.

[2]In fact, according to [13] the strong resolving graph G'_{SR} of a graph G has vertex set $V(G'_{SR}) = V(G)$ and two vertices u, v are adjacent in G'_{SR} if and only if u and v are mutually maximally distant in G. So, the strong resolving graph defined here is a subgraph of the strong resolving graph defined in [13] and can be obtained from the latter graph by deleting its isolated vertices.

2 Preliminaries on generalized Sierpiński graphs

Let G be a non-empty graph of order n and vertex set $V(G)$. We denote by $V^t(G)$ the set of words of size t on alphabet $V(G)$. The letters of a word u of length t are denoted by $u_1u_2\dots u_t$. The concatenation of two words u and v is denoted by uv. Klavžar and Milutinović introduced in [6] the graph $S(K_n,t)$ whose vertex set is $V^t(K_n)$, where $\{u,v\}$ is an edge if and only if there exists $i \in \{1,\dots,t\}$ such that:

$$\text{(i) } u_j = v_j, \text{ if } j < i; \text{ (ii) } u_i \neq v_i; \text{ (iii) } u_j = v_i \text{ and } v_j = u_i \text{ if } j > i.$$

When $n = 3$, those graphs are exactly Tower of Hanoi graphs. Later, those graphs have been called Sierpiński graphs in [7] and they were studied by now from numerous points of view. The reader is invited to read, for instance, the following recent papers [2, 5, 4, 7, 8, 9] and references therein. This construction was generalized in [3] for any graph G, by defining the t-th *generalized Sierpiński graph* of G, denoted by $S(G,t)$, as the graph with vertex set $V^t(G)$ and edge set defined as follows. $\{u,v\}$ is an edge if and only if there exists $i \in \{1,\dots,t\}$ such that:

(i) $u_j = v_j$, if $j < i$;

(ii) $u_i \neq v_i$ and $\{u_i,v_i\} \in E(G)$;

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

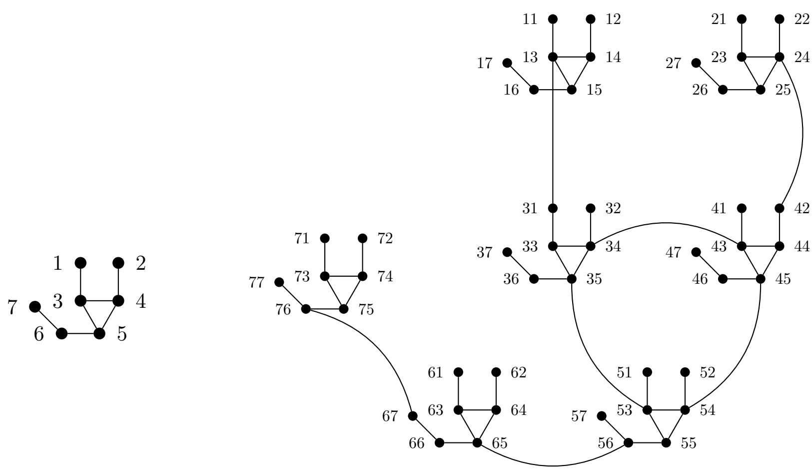

Figure 1: A graph G and the generalized Sierpiński graph $S(G,2)$

Figure 1 shows a graph G and the Sierpiński graph $S(G,2)$, while Figure 2 shows the Sierpiński graph $S(G,3)$.

Notice that if $\{u,v\}$ is an edge of $S(G,t)$, there is an edge $\{x,y\}$ of G and a word w such that $u = wxyy\dots y$ and $v = wyxx\dots x$. In general, $S(G,t)$ can be constructed recursively from G with the following process: $S(G,1) = G$ and, for $t \geq 2$, we copy n times $S(G,t-1)$ and add the letter x at the beginning of each label of the vertices belonging to the copy of $S(G,t-1)$ corresponding to x. Then for every edge $\{x,y\}$ of

G, add an edge between vertex $xyy\dots y$ and vertex $yxx\dots x$. See, for instance, Figure 2. Vertices of the form $xx\dots x$ are called *extreme vertices*. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G,t)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx\dots x$ of $S(G,t)$ also has degree $d(x)$. Moreover, the degrees of two vertices $yxx\dots x$ and $xyy\dots y$, which connect two copies of $S(G,t-1)$, are equal to $d(x)+1$ and $d(y)+1$, respectively.

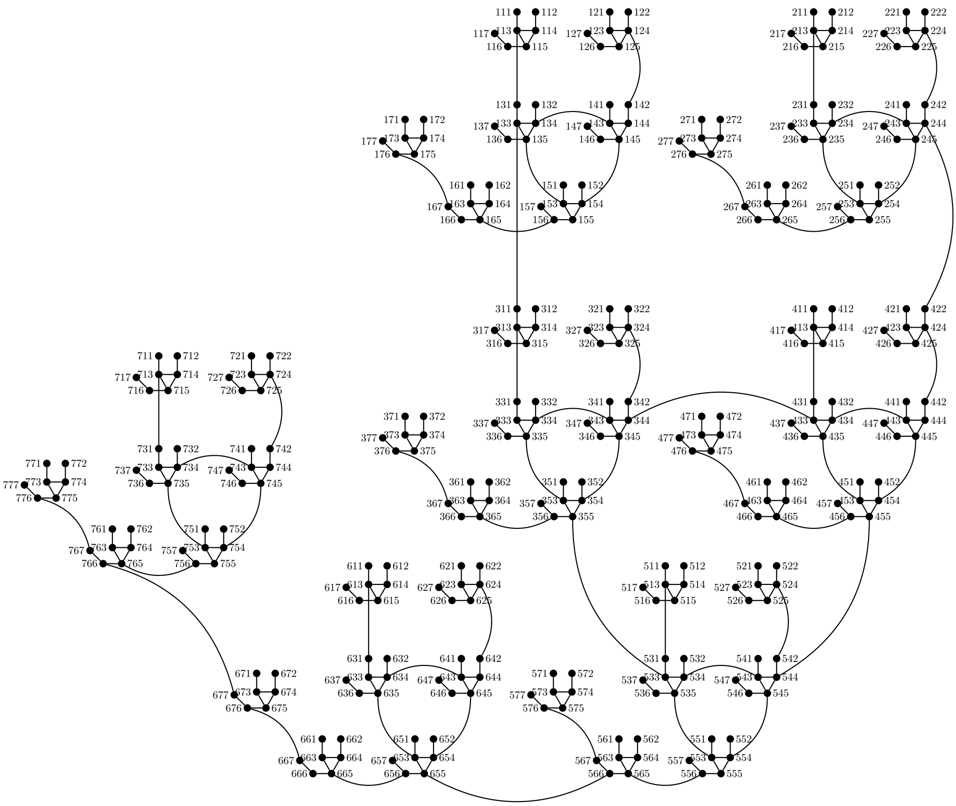

Figure 2: The generalized Sierpiński graph $S(G,3)$ with the base graph G shown in Figure 1.

To the best of our knowledge, [14] is the first published paper studying the generalized Sierpiński graphs. In that article, the authors obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs. In this note we consider the case where every internal vertex of G is a cut vertex and we obtain a closed formula for the strong metric dimension of $S(G,t)$.

3 The strong metric dimension of $S(G,t)$

The following basic lemma will become an important tool to prove our main results.

Lemma 3.1. *Let G be a connected graph. If v is a cut vertex of G, then $v \notin \partial(G)$.*

Proof. Let $v \in V(G)$ be a cut vertex and $x \in V(G)-\{v\}$. Let G_1 be the connected component of $G-\{v\}$ containing x and let G_2 be a connected component of $G-\{v\}$ different from G_1. Since there exists $y \in V(G_2)$ which is adjacent to v in G and $d_G(x,v) < d_G(x,y)$, we conclude that x and v are not mutually maximally distant in G. □

An *end-vertex* is a vertex of a graph that has exactly one edge incident to it, while a *support vertex* is a vertex adjacent to an end-vertex.

Theorem 3.2. *Let G be a connected graph and let $\varepsilon(G)$ be the number of end-vertices of G. Then,*

$$\dim_s(G) \geq \varepsilon(G) - 1.$$

Moreover, if every vertex of degree greater than one is a cut vertex, then the bound is achieved.

Proof. Let G be a connected graph. Since the set $\Omega(G)$ of end-vertices of G is a subset of $\partial(G)$ and the subgraph of G_{SR} induced by $\Omega(G)$ is a clique, we conclude that $\alpha(G_{SR}) \geq \varepsilon(G) - 1$. Hence, by Theorem 1.1 we obtain the lower bound.

Now, if every vertex of degree greater than one is a cut vertex, by Lemma 3.1 we have that $\partial(G)$ is equal to the set of end-vertices of G. Then $G_{SR} \cong K_{|\varepsilon(G)|}$ and so Theorem 1.1 leads to $\dim_s(G) = \varepsilon(G) - 1$. □

From now on, we will say that a vertex of degree greater than one in a graph G is an *internal vertex* of G. We shall show that if every internal vertex of G is a cut vertex, then the bound above is achieved for $S(G,t)$. To begin with, we state the following lemma.

Lemma 3.3. *Let G be a graph of order n having $\varepsilon(G)$ end-vertices. For any positive integer t, the number of end-vertices of $S(G,t)$ is*

$$\varepsilon(S(G,t)) = \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right)}{n-1}.$$

Proof. In this proof, we denote by $\mathrm{Sup}(G)$ the set of support vertices of G. Also, if $x \in \mathrm{Sup}(G)$, then $\varepsilon_G(x)$ will denote the number of end-vertices of G which are adjacent to x.

Let $t \geq 2$. For any $x \in V(G)$, we denote by $S_x(G,t-1)$ the copy of $S(G,t-1)$ corresponding to x in $S(G,t)$, *i.e.*, $S_x(G,t-1)$ is the subgraph of $S(G,t)$ induced by the set $\{xw : \ w \in V^{t-1}(G)\}$, which is isomorphic to $S(G,t-1)$. To obtain the result, we only need to determine the contribution of $S_x(G,t-1)$ to the number of end-vertices of $S(G,t)$, for all $x \in V(G)$. By definition of $S(G,t)$, there exists an edge of $S(G,t)$ connecting the vertex $xy\dots y$ of $S_x(G,t-1)$ with the vertex $yx\dots x$ of $S_y(G,t-1)$ if and only if x and y are adjacent in G. Hence, an end-vertex $xy\dots y$ of $S_x(S(G,t-1)$ is adjacent in $S(G,t)$ to a vertex $yx\dots x$ of $S_y(G,t-1)$ if and only if y is an end-vertex of G and x is its support vertex. Thus, if $x \in \mathrm{Sup}(G)$, then the contribution of $S_x(G,t-1)$ to the number of end-vertices of $S(G,t)$ is $\varepsilon(S(G,t-1)) - \varepsilon_G(x)$ and, if $x \notin \mathrm{Sup}(G)$, then the contribution of $S_x(G,t-1)$ to the number of end-vertices of $S(G,t)$ is $\varepsilon(S(G,t-1))$. Then we obtain,

$$\begin{aligned}\varepsilon(S(G,t)) &= (n - |\mathrm{Sup}(G)|)\varepsilon(S(G,t-1)) + \sum_{x\in \mathrm{Sup}(G)} (\varepsilon(S(G,t-1)) - \varepsilon_G(x)) \\ &= n\varepsilon(S(G,t-1)) - \varepsilon(G).\end{aligned}$$

Now, since $\varepsilon(S(G,1)) = \varepsilon(G)$, we have that

$$\varepsilon(S(G,t)) = \varepsilon(G)\left(n^{t-1} - n^{t-2} - \cdots - n - 1\right) = \varepsilon(G)\left(n^{t-1} - \frac{\left(n^{t-1}-1\right)}{n-1}\right).$$

Therefore, the result follows. □

The following result is a direct consequence of Theorem 3.2 and Lemma 3.3.

Theorem 3.4. *Let G be a connected graph of order n having $\varepsilon(G)$ end-vertices and let t be a positive integer. Then*

$$\dim_s(S(G,t)) \geq \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right) - n + 1}{n-1}.$$

As we will show in Theorem 3.6, the bound above is tight.

Lemma 3.5. *Let G be a connected graph and let t be a positive integer. If every internal vertex of G is a cut vertex, then every internal vertex of $S(G,t)$ is a cut vertex.*

Proof. As above, for any $x \in V(G)$, we denote by $S_x(G,t-1)$ the copy of $S(G,t-1)$ corresponding to x in $S(G,t)$. We proceed by induction on t. Let $S(G,1) = G$ be a connected graph such that every internal vertex is a cut vertex and assume that every internal vertex of $S(G,t-1)$ is a cut vertex. We differentiate two cases for any internal vertex xw of $S(G,t)$, where $x \in V(G)$ and $w \in V^{t-1}(G)$.

Case 1. w **has degree one in** $S(G,t-1)$**.** In this case xw has degree two in $S(G,t)$. Hence, xw is adjacent to x_1w', for some $x_1 \in V(G)-\{x\}$, and then $w = x_1x_1\ldots x_1$, $w' = xx\ldots x$, x_1 is an end-vertex of G and x is the support of x_1. As a result, $\{xw, x_1w'\}$ is the only edge connecting vertices in $S_{x_1}(G,t-1)$ to vertices outside the subgraph $S_{x_1}(G,t-1)$. Therefore, xw is a cut vertex of $S(G,t)$.

Case 2. w **is a cut vertex of** $S(G,t-1)$**.** In this case, we take two connected components C_1 and C_2 obtained by removing w from $S(G,t-1)$. Suppose, for contradiction purposes, that xw is not a cut vertex of $S(G,t)$. Then there exist two neighbours x_1, x_k of x and a sequence of subgraphs $S_{x_1}(G,t-1)$, $S_{x_2}(G,t-1), \ldots, S_{x_k}(G,t-1)$ such that $x_1\ldots x_1 \in V(C_1)$, $x_k\ldots x_k \in V(C_2)$ and there exists an edge of $S(G,t)$ connecting $S_{x_i}(G,t-1)$ to $S_{x_{i+1}}(G,t-1)$, for all $i \in \{1,2,\ldots,k\}$. Note that the only vertices connecting $S_{x_i}(G,t-1)$ and $S_{x_{i+1}}(G,t-1)$ are $x_ix_{i+1}x_{i+1}\ldots x_{i+1}$ and $x_{i+1}x_ix_i\ldots x_i$, where x_i and x_{i+1} are adjacent in G. Hence, $x, x_1, x_2, \ldots, x_k, x$ is a cycle in G, and so there is a cycle in $S(G,t-1)$ of the form P_{xx_1},$P_{x_1x_2}$,$P_{x_2x_3}, \ldots, P_{x_{k-1}x_k}$, P_{x_kx}, where $P_{x_ix_{i+1}}$ is the path of order 2^{t-1} from $x_ix_i\ldots x_i$ to $x_{i+1}\, x_{i+1}\ldots x_{i+1}$ composed by binary words on alphabet $\{x_i, x_{i+1}\}$ (the paths P_{xx_1} and P_{x_kx} are defined by analogy) and we identify the vertex $x_ix_i\ldots x_i$ of two consecutive paths $P_{x_{i-1}x_i}$ and $P_{x_ix_{i+1}}$ to form the cycle. As a result, there are two disjoint paths from $x_1x_1\ldots x_1$ to $x_kx_k.\ldots x_k$, which contradicts the fact that $x_1x_1\ldots x_1 \in V(C_1)$ and $x_kx_k.\ldots x_k \in C_2$. Therefore, xw is a cut vertex of $S(G,t)$.

According to the two cases above, we conclude the proof by induction. □

Our next result is obtained from Theorem 3.2 and Lemma 3.5.

Theorem 3.6. *Let G be a connected graph of order n having $\varepsilon(G)$ end-vertices and let t be a positive integer. If every internal vertex of G is a cut vertex, then*

$$\dim_s(S(G,t)) = \frac{\varepsilon(G)\left(n^t - 2n^{t-1} + 1\right) - n + 1}{n-1}.$$

Obviously, if the base graph is a tree, then we can apply the formula above. In particular, we would emphasize the following particular case of this result, where $K_{1,r}$ denotes the star graph of r leaves and P_r denotes the path graph of order r.

Corollary 3.7. *For any integers $r, t \geq 2$,*

- $\dim_s(S(K_{1,r},t)) = (r+1)^{t-1}(r-1)$.
- $\dim_s(S(P_r,t)) = \dfrac{2r^t - 4r^{t-1} - r + 3}{r-1}$.

Let G be a graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of graphs. The corona product graph $G \odot \mathcal{H}$ is defined as the graph obtained from G and $\mathcal{H}$ by taking one copy of G and joining by an edge each vertex of H_i with the i^{th}-vertex of G. These graphs were defined by Frucht and Harary in [1].

Corollary 3.8. *Let G be a graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of empty graphs of order n_i, respectively. Then for any positive integer t,*

$$\dim_s(S(G \odot \mathcal{H},t)) = \frac{n'(n+n')^{t-1}(n+n'-2) - n + 1}{n+n'-1},$$

where $n' = \sum_{i=1}^{n} n_i$.

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