A note on automorphisms of halved Cayley graphs of Coxeter systems

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Received 10 February 2015, accepted 12 June 2015, published online 24 September 2015

Abstract

We consider the halved Cayley graphs of Coxeter systems and show that every automorphism of such a graph can be uniquely extended to an automorphism of the corresponding Cayley graph.

Keywords: Coxeter system, Cayley graph.
Math. Subj. Class.: 20F55, 05C12

1 Introduction

Let $W$ be a group generated by a finite set $S$ whose elements are involutions. For distinct $s, s' \in S$ we denote by $m(s, s')$ the order of the element $ss'$. Then $m(s, s') = m(s', s)$ and the condition $m(s, s') = 2$ is equivalent to the commuting of $s$ and $s'$. Suppose that $(W, S)$ is a Coxeter system, i.e $W$ is the quotient of the free group over $S$ by the normal subgroup generated by all $(ss')^{m(s, s')}$ with $m(s, s') < \infty$.

The Cayley graph $C(W, S)$ is the graph whose vertex set is $W$ and $w, v \in W$ are adjacent vertices of the graph if $v = sw$ for a certain $s \in S$ (since $S$ consists of involutions, the adjacency relation is symmetric). For the dihedral Coxeter system $I_2(n)$ this graph is the $(2n)$-cycle and we get an infinite path if $n = \infty$. The Cayley graph of $A_n$ is the permutohedron [6]. See [1, Figures 3.3] for the Cayley graph of $H_3$. Also, $C(W, S)$ can be identified with the graph whose vertices are maximal simplices of the Coxeter complex $\Sigma(W, S)$ and two maximal simplices are adjacent vertices if their intersection consists of $|S| - 1$ elements [5].

In almost all cases, the automorphism group of $C(W, S)$ is known. For every $w \in W$ the right multiplication $R_w : v \to vw$ is an automorphism of the graph. If the diagram of our Coxeter system does not contain adjacent edges labeled by $\infty$ then the automorphism
group of $C(W,S)$ is the semidirect product of $W$ and the automorphism group of the diagram [1, Corollary 3.2.6].

The length $l(w)$ of an element $w \in W$ is the smallest number $m$ such that $w$ has an expression

$$w = s_1 \ldots s_m, \quad s_1, \ldots, s_m \in S.$$  \hspace{1cm} (1.1)

It is clear that $l(w)$ is the distance between 1 and $w$ in the Cayley graph. Since every right multiplication is an automorphism of the graph, the distance $d(w,v)$ between $w, v \in W$ is equal to $l(wv^{-1}) = l(vw^{-1})$. Recall the following remarkable property of Coxeter systems called the exchange condition: if (1.1) is a reduced expression, i.e. $l(w) = m$, then for every $s \in S$ satisfying $l(sw) \leq m$ there exists $k \in \{1, \ldots, m\}$ such that

$$sw = s_1 \ldots \hat{s}_k \ldots s_m$$

(the symbol $\hat{\ }$ means that the corresponding term is omitted).

The group $W$ can be presented as the disjoint union of the following subsets

$$W_1 := \{ w \in W : l(w) \text{ is odd} \} \quad \text{and} \quad W_2 := \{ w \in W : l(w) \text{ is even} \}.$$ 

Using the exchange condition we establish the following:

- the distance between any two elements of $W_i, i \in \{1, 2\}$ is even,
- the distance between every element of $W_1$ and every element of $W_2$ is odd.

Note that $W_2$ is a subgroup of $W$. Consider the graph $\Gamma_i, i \in \{1, 2\}$ whose vertex set is $W_i$ and two elements of $W_i$ are adjacent vertices if the distance between them (in the Cayley graph) is equal to 2. The right multiplication $R_w$ preserves both $W_i$ in the case when $w \in W_2$. If $w \in W_1$ then $R_w$ transfers $W_1$ to $W_2$ and conversely. The latter implies that $\Gamma_1$ and $\Gamma_2$ are isomorphic.

The main result of the note is the following.

**Theorem 1.1.** If $|S| \geq 5$ then every isomorphism between $\Gamma_i$ and $\Gamma_j, i, j \in \{1, 2\}$ can be uniquely extended to an automorphism of the Cayley graph.

The same fails if $|S| \in \{3, 4\}$ (Remark 3.2), but the statement holds for $|S| = 2$ (the Cayley graph is a cycle or an infinite path) and the case $|S| = 1$ is trivial.

Theorem 1.1 easily follows from the description of maximal 2-cliques of Cayley graph, i.e. maximal cliques of the halved Cayley graph, given in Lemma 2.5.

If $S$ consists of $n$ mutually commuting involutions then $C(W,S)$ is the $n$-dimensional hypercube graph $H_n$ and every $\Gamma_i$ is the half-cube graph $\frac{1}{2}H_n$. So, it is natural to ask which properties of the hypercube and half-cube graphs can be extended to $C(W,S)$ and $\Gamma_i$, respectively?

## 2 Maximal 2-cliques

Two vertices in a graph are said to be 2-adjacent if the distance between them is equal to 2. Recall that a clique is a subset of the vertex set, where any two distinct vertices are adjacent. We say that a subset in the vertex set is a 2-clique if any two distinct elements of this subset are 2-adjacent vertices.

Consider examples of 2-cliques in $C(W,S)$. 
Example 2.1 (First type). Any two distinct elements of $S$ are 2-adjacent and $S$ is a 2-clique. Since the right multiplication $R_w$ is an automorphism of the Cayley graph, $Sw$ is a 2-clique for every $w \in W$.

Remark 2.2. Suppose that $S = Sw$. Then for any $s_1, s_2 \in S$ there exist $s'_1, s'_2 \in S$ such that $s_1 = s'_1 w$ and $s_2 = s'_2 w$. If $w \neq 1$ then $s_1 \neq s'_1$ and $s_2 \neq s'_2$. We have

$$s'_1 s_1 = w = s'_2 s_2 \quad \text{and} \quad s'_2 s'_1 s_1 = s_2$$

Since $W$ cannot be generated by a proper subset of $S$, the latter means that $s_2 = s'_1$. Therefore, $S = \{s_1, s_2\}$ and $s_1 s_2 = s_2 s_1$. So, the equality $Sw = Sw'$ implies that $w = w'$ except the case when our Coxeter system is $I_2(2)$.

Example 2.3 (Second type). Let $s, s', s''$ be three distinct mutually commuting elements of $S$. Then $ss' s''$ is 2-adjacent to $s', s''$ and $\{sw, s'w, s''w, ss' s''w\}$ is a 2-clique for every $w \in W$.

Example 2.4 (Third type). Suppose that $s, s' \in S$ and $m(s, s') = 3$. Then $ss's = s'ss'$ and we denote this element by $w(s, s')$. It is 2-adjacent to $s, s'$ and for every $w \in W$ the set $\{sw, s'w, w(s, s')w\}$ is a 2-clique.

Lemma 2.5. Every maximal 2-clique of $C(W, S)$ is one of the 2-cliques described above.

Remark 2.6. The $n$-dimensional hypercube graph contains only maximal 2-cliques of the first and second types if $n \geq 4$ [3]. In the case when $n = 3$, there are precisely two maximal 2-cliques of the second type and 2-cliques of the first type are not maximal.

To prove Lemma 2.5 we use the following properties of Coxeter systems:

(P1) for every $w \in W$ there is a subset $S_w \subset S$ such that every reduced expression of $w$ is formed by all elements of $S_w$.

(P2) the group $W$ cannot be generated by a proper subset of $S$.

Lemma 2.7. If $u \in W \setminus S$ is 2-adjacent to three distinct $s, s', s'' \in S$ then $s, s', s''$ are mutually commuting and $u = ss' s''$.

Proof. Since $u$ is 2-adjacent to $s, s', s''$ and $u \not\in S$, there are three reduced expressions

$$u = s_1 s_2 s, \quad u = s'_1 s'_2 s', \quad u = s''_1 s''_2 s'''$$

where $s_1, s_2, s'_1, s'_2, s''_1, s''_2 \in S$. By (P1), we have $S_u = \{s, s', s''\}$ and

$$\{s_1, s_2\} = \{s', s''\}, \quad \{s'_1, s'_2\} = \{s, s''\}, \quad \{s''_1, s''_2\} = \{s, s'\}.$$ 

Thus there are the following possibilities for the first and second expressions:

1. $u = s'' s' s = s'' s s'$.
2. $u = s'' s' s = s s'' s'$.
3. $u = s'' s' s = s s'' s'$.
4. $u = s' s'' s = s s' s'$. 
Case (1). The involutions $s, s'$ are commuting and the third expression is

$$u = ss' s'' = s' ss''.$$  \hspace{1cm} (2.1)

Then $s'' s' = u = s' ss''$ and $s'' s' s = ss''$. We apply the exchange condition to $w = s'' s' s$ and get the following three possibilities:

- $s' s = ss''$,
- $s'' s = ss''$,
- $s'' s' = ss''$.

The first and third contradict (P2). So, $s$ and $s''$ are commuting. Similarly, the equality $s'' s' s = u = ss' s''$ shows that $ss'' s' = s' s''$. Using the above arguments we establish that $s'$ and $s''$ are commuting.

Case (2). The equality $s'' s' s = ss'' s'$ implies that $s' s = s'' ss'' s'$. As in the previous case, we show that $s$ and $s'$ are commuting. Then the third expression is (2.1) which implies that $ss'' s' = u = ss' s''$ and $s', s''$ are commuting. The equality

$$s'' ss' = s'' s' s = u = ss'' s'$$

guarantees that $s$ and $s''$ are commuting.

Case (3). We have $s' s'' s = ss'' s'$ and $s'' s' s'' s = ss'$. As above, this means that $s, s'$ are commuting and the third expression is (2.1). Then $s' s'' s = u = ss'' s'$ and $s, s''$ are commuting. The equality

$$s' s'' s = u = ss'' s' = s'' s' s$$

shows that $s'$ and $s''$ are commuting.

Case (4). Since $s' s'' s = ss'' s'$, we have $s' s = s' ss'' s'$ and $ss'' s' s = s' s'' s$. By the standard arguments, $s''$ is commuting with both $s$ and $s'$. Then

$$s'' ss' = ss'' s' = u = ss'' s' = s'' s' s$$

which implies that $s$ and $s'$ are commuting.

\[\square\]

**Remark 2.8.** If $s, s', s''$ are distinct elements of $S$ then each of the equalities

$$s'' s' s = ss'' s', \quad s'' s = s'' ss', \quad s'' s'' s = ss'' s'$$

implies that $s$ and $s'$ are commuting; moreover, the third equality guarantees that $s, s', s''$ are mutually commuting. See the cases (2)–(4) in the proof of Lemma 2.7.

Lemma 2.7 shows that for any three distinct mutually commuting $s, s', s'' \in S$ the 2-clique formed by $s, s', s''$ and $ss's'$ is maximal. Therefore, every 2-clique of the second type is maximal.

**Lemma 2.9.** If $u \in W \setminus S$ is 2-adjacent to distinct $s, s' \in S$ then one of the following possibilities is realized:
\[ m(s, s') = 3 \text{ and } u = w(s, s'), \]
\[ s, s' \text{ are commuting and } u = s''s's \text{ for a certain } s'' \in S. \]

**Proof.** Since \( u \) is 2-adjacent to \( s, s' \) and \( u \not\in S \), there are two reduced expressions
\[ u = s_1s_2s \text{ and } u = s'_1s'_2s', \]
where \( s_1, s_2, s'_1, s'_2 \in S \). By (P1), we have \( \{s, s_1, s_2\} = S_u = \{s', s'_1, s'_2\} \). If \( |S_u| = 2 \)
then \( S_u = \{s, s'\} \) and \( u = ss's = s'ss' \) which implies that \( m(s, s') = 3 \), i.e. the first possibility is realized.

If \( |S_u| = 3 \) then \( S_u = \{s, s', s''\} \) and, as in the proof of Lemma 2.7, we have the following possibilities for the above expressions:

1. \( u = s''s's = s''ss' \),
2. \( u = s''s's = ss''s' \),
3. \( u = s's''s = s''ss' \),
4. \( u = s's''s = ss''s' \).

It is clear that \( s \) and \( s' \) are commuting in the case (1). By Remark 2.8, the same holds for the cases (2) – (4) and \( s, s', s'' \) are mutually commuting in the case (4). So, we get the second possibility.

By Lemma 2.9, for any \( s, s' \in S \) satisfying \( m(s, s') = 3 \) the 2-clique formed by \( s, s' \) and \( w(s, s') \) is maximal. Thus every 2-clique of the third type is maximal.

**Proof of Lemma 2.5.** Let \( C \) be a maximal 2-clique. For any distinct \( u, u' \in C \) there exist \( w \in W \) and \( s, s' \in S \) such that \( u = sw \) and \( u' = s'w \). The maximal 2-clique \( Cw^{-1} \)
contains \( s \) and \( s' \). Thus we can suppose that \( C \) contains at least two distinct elements of \( S \).
Let \( s \) and \( s' \) be elements of \( S \) belonging to \( C \). Suppose that \( C \neq S \), i.e. there is \( u \in C \setminus S \).

If there is a third element \( s'' \in S \) contained in \( C \) then, by Lemma 2.7, \( s, s', s'' \) are mutually commuting and \( C \) is the 2-clique of the second type formed by \( s, s', s'' \) and \( u = ss's'' \). In the case when \( C \) contains precisely two elements of \( S \), Lemma 2.9 shows that \( m(s, s') = 3 \) and \( C = \{s, s', w(s, s')\} \) or \( s, s' \) are commuting and \( u = s''s's \) for a certain \( s'' \in S \). The latter means that the maximal 2-clique \( Cs's \) contains \( s, s', s'' \), i.e. it coincides with \( S \) or \( \{s, s', s'', ss's'\} \). Then \( C \) is a 2-clique of the first type or the second type.

### 3 Proof of Theorem 1.1

We consider the case when \( i = j = 1 \). Let \( f : W_1 \to W_1 \) be an automorphism of \( \Gamma_1 \). Then \( f \) preserves the family of maximal cliques of \( \Gamma_1 \). Every maximal clique of \( \Gamma_1 \) is a maximal 2-clique of \( C(W, S) \) contained in \( W_1 \). By Lemma 2.5, there are precisely three types of such subsets. They contain \(|S| \) vertices, 4 vertices and 3 vertices, respectively. The condition \(|S| \geq 5 \) guarantees that \( f \) preserves the types of maximal cliques.

If \( w \in W_2 \) then \( Sw \) is a maximal clique of \( \Gamma_1 \) and \( f(Sw) = Sw' \) for a certain \( w' \in W_2 \). We set \( f(w) := w' \) and get a bijective transformation of \( W \).

If \( w, v \in W \) are adjacent vertices of the Cayley graph then one of these vertices belongs to \( W_1 \) and the other is an element of \( W_2 \). Suppose that \( v \in W_1 \) and \( w \in W_2 \). Then \( v \in Sw \).
and $f(v) \in f(Sw) = Sf(w)$ which implies that $f(v)$ and $f(w)$ are adjacent vertices of the Cayley graph. The apply the same arguments to $f^{-1}$ and establish that $f$ is an automorphism of $C(W, S)$.

The uniqueness of such extension follows from the fact that $w$ is the unique vertex of the Cayley graph adjacent to all vertices from $Sw$ (Remark 2.2).

**Remark 3.1.** A similar idea was exploited in [4, Section 4.8] for an alternative proof of Cooperstein–Kasikova–Shult’s characterization of apartments in half-spin Grassmannians [2].

**Remark 3.2.** If $C(W, S)$ is $H_4$ then there are automorphisms of $\Gamma_i = \frac{1}{2}H_4$ which change the types of 2-cliques contained in $W_i$. Such automorphisms are not extendable to automorphisms of $H_4$. Similarly, if $(W, S)$ is the direct product of $I_2(3)$ and the group spanned by an involution then every maximal 2-clique of $C(W, S)$ is of the first or of the third type and there are automorphisms of $\Gamma_i$ changing the types of 2-cliques contained in $W_i$.

**References**


