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Classification of convex polyhedra by their rotational orbit Euler characteristic

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Abstract

Let $\mathcal{P}$ be a polyhedron whose boundary consists of flat polygonal faces on some compact surface $S(\mathcal{P})$ (not necessarily homeomorphic to the sphere S^2). Let $vo_R(\mathcal{P}), eo_R(\mathcal{P})$, $fo_R(\mathcal{P})$ be the numbers of rotational orbits of vertices, edges and faces, respectively, determined by the group $G = G_R(P)$ of all the rotations of the Euclidean space E^3 preserving $\mathcal{P}$. We define the *rotational orbit Euler characteristic* of $\mathcal{P}$ as the number $Eo_R(\mathcal{P}) = vo_R(\mathcal{P}) - eo_R(\mathcal{P}) + fo_R(\mathcal{P})$.

Using the Burnside lemma we obtain the lower and the upper bound for $Eo_R(\mathcal{P})$ in terms of the genus of the surface $S(P)$. We prove that $Eo_R \in \{2, 1, 0, -1\}$ for any convex polyhedron $\mathcal{P}$. In the non-convex case Eo_R may be arbitrarily large or small.

Keywords: Polyhedron, rotational orbit, Euler characteristic.

Math. Subj. Class.: 52B05, 52B10

1 Introduction

CONTEXT: Euler (1752) discovered the famous relation $v-e+f = 2$ between the numbers of vertices v, edges e and faces f of any convex polyhedron. This *Euler polyhedron formula* was implicitly stated in the formulas of Descartes (1630) $p = 2f + 2v - 4, p = 2e$, where p is the number of "plane angles" – corners of faces determined by pairs of adjacent edges ([5], p.469).

The number $\chi = v - e + f$ can be defined for any *map* (a graph cellularly embedded into a compact surface S) and is called its *Euler characteristic*. It is related to the *genus* g of the surface S as follows: $\chi = 2 - 2g$ ([5], p. 473.) and it may be used for the

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classification of surfaces by two parameters: one is χ and the other is orientability (or non-orientability) of the surface. In [4] we introduced the concept of *Euler orbit characteristic* $Eo = vo - eo + fo$, where vo, eo, fo denote the number of orbits of vertices, edges and faces, respectively, determined by the group of all rotations and reflections of the Euclidean space E^3, preserving the polyhedron $\mathcal{P}$, and we used it for the classification of the 92 Johnson solids ([4], p.258). In this paper we introduce a similar concept, called *rotational orbit Euler characteristic*, and we use it for the classification of convex polyhedra.

Definition 1.1. The rotational Euler orbit characteristic of the polyhedron $\mathcal{P}$ is defined as the number $Eo_R = vo_R - eo_R + fo_R$ where vo_R, eo_R, fo_R are the numbers of rotational orbits of the vertices, edges and faces, respectively, of P (these orbits are determined by the group $G_R(\mathcal{P})$ of all the rotations of the Euclidean space E^3 preserving $\mathcal{P}$).

Proposition 1.2. *$Eo_R = 1$ for all the Platonic solids and all the n-prisms and n-antiprisms, while $1 \leq Eo_R \leq 2$ for all the Archimedean solids.*

Proof. In the Table 1 the number of rotational orbits of Platonic and Archimedean solids are given. These values can be easily found for each solid directly or deduced from the symmetry-type graphs of Platonic and Archimedean solids [3]. The 5 Platonic solids have just one rotational orbit of vertices, edges and faces. The 13 Archimedean solids have at most two rotational orbits of vertices and at most three rotational orbits of edges and faces. The n-prisms and the n-antiprisms have just one rotational orbit of vertices and two rotational orbits of edges and faces. □

class	solid $\mathcal{P}$	vertex pattern	vo_R	eo_R	fo_R	Eo_R
I.	tetrahedron	(3.3.3)	1	1	1	1
I.	octahedron	(3.3.3.3)	1	1	1	1
I.	cube	(4.4.4)	1	1	1	1
I.	icosahedron	(3.3.3.3.3)	1	1	1	1
I.	dodecahedron	(5.5.5)	1	1	1	1
II.	cuboctahedron	(3.4.3.4)	1	1	2	2
II.	icosidodecahedron	(3.5.3.5)	1	1	2	2
III.	truncated tetrahedron	(3.6.6)	1	1	2	2
III.	truncated cube	(3.8.8)	1	1	2	2
III.	truncated octahedron	(4.6.6)	1	1	2	2
III.	truncated dodecahedron	(3.10.10)	1	1	2	2
III.	truncated icosahedron	(5.6.6)	1	1	2	2
IV.	rhombicuboctahedron	(3.4.4.4)	1	2	3	2
IV.	rhombicosidodecahedron	(3.4.5.4)	1	2	3	2
V.	truncated cuboctahedron	(4.6.8)	2	3	3	2
V.	truncated icosidodecahedron	(4.6.10)	2	3	3	2
VI.	snub cube	(3.3.3.3.4)	1	3	3	1
VI.	snub dodecahedron	(3.3.3.3.5)	1	3	3	1
VII.	n-prism	(4.4.n)	1	2	2	1
VIII.	n-antiprism	(3.3.n)	1	2	2	1

Table 1: Values of vo_R, eo_R, fo_R for Platonic and Archimedean solids and for the infinite families of n-prisms and n-antiprisms.

MOTIVATION: Similar bounds on Eo_R exist for the Johnson solids (i.e. convex polyhedra with regular polygonal faces and at least two orbits of vertices [2]). The direct motivation

for writing this paper came from the empirical observation that the values of Eo_R for the 92 Johnson solids are in a small range between -1 and 2. This was discovered during the process of constructing a table of 16 parameters of the Johnson solids presented in [4], while the range for Eo for the same solids turned out to be bigger: $0 \leq Eo \leq 5$.
COMPARISON OF Eo AND Eo_R: The two characteristics behave very differently on the set of all convex polyhedra: the main result of the paper (Theorem 2.1) states that the relation $-1 \leq Eo_R \leq 2$ holds for all convex polyhedra, while for Eo there is no fixed upper bound, we can obtain only the following estimate: $Eo = vo - eo + fo \leq vo + fo \leq vo_R + fo_R = \leq (vo_R - eo_R + fo_R) + eo_R = Eo_R + eo_R \leq 2 + eo_R$.

Definition 1.3. Let $G_R(\mathcal{P}) = \{R_1, R_2, ..., R_{n-1}, R_n = Id\}$ be the group of rotational symmetries of the polyhedron $\mathcal{P}$. The poles of the rotation R_i are the points in which the axis of the rotation R_i intersects the surface $S(\mathcal{P})$. Let $v_p(R_i)$, $e_p(R_i)$, $f_p(R_i)$ denote the numbers of poles of R_i in the vertices, edge centers and face centers of $\mathcal{P}$, respectively. The number $E_p(R_i) = v_p(R_i) - e_p(R_i) + f_p(R_i)$ is called the Euler polar characteristics of the rotation R_i.

Lemma 1.4. *Let n_i denote the number of poles of any non-trivial rotation R_i of the polyhedral map $\mathcal{P}$ on the surface S of genus g. Then*

$$\begin{gathered}
n_i \leq 2(g+1), \\
n_i \in \{0, 2, 4, \ldots, 2(g+1)\}, \\
v_p(R_i) + e_p(R_i) + f_p(R_i) = n_i, \\
0 \leq e_p(R_i) \leq n_i, \\
E_p(R_i) = n_i - 2e_p(R_i), \\
-2(g+1) \leq -n_i \leq E_p(R_i) \leq n_i \leq 2(g+1).
\end{gathered}$$

If the order of the rotation R_i is greater than 2 (i.e. $R_i^n = id$ and $n > 2$), then $e_p(R_i) = 0$ and $E_p(R_i) = n_i$.

Proof. Any line intersecting $\mathcal{P}$ has at most $2g$ intersecting points with S. If P is a convex polyhedron then any nontrivial rotation R_i has exactly two poles, hence $n_i = 2$. If $n_i > 2$ then each segment of the rotational axis r_i not lying in the interior of P contributes two poles (hence n_i is an even number!) and at least one new handle. Thus it "increases" the genus of S for 1 (since it is well known that the genus counts the numbers of "handles" of a surface), therefore it must be $n_i \leq 2(g+1)$. The poles can be only in vertices, edge centers or face centers, hence $v_p(R_i) + e_p(R_i) + f_p(R_i) = n_i$ and $0 \leq e_p(R_i) \leq n_i$. Obviously $E_p(R_i) = v_p(R_i) - e_p(R_i) + f_p(R_i) = n_i - 2e_p(R_i)$. Therefore the upper bound for $E_p(R_i)$ is n_i and the lower bound is $-n_i$. □

Corollary 1.5. *If $\mathcal{P}$ is a convex polyhedron, then $E_p(R_i) = 2 - 2e_p(R_i) \in \{2, 0, -2\}$. If $e_p = 0$ then $E_p = 2$, if $e_p = 1$ then $E_p = 1$, and if $e_p = 2$ then $E_p = -2$. If the order of the rotation R_i is greater than 2, then $E_p(R_i) = 2$.*

Proof. Every convex polyhedron is homeomorphic (by a radial projection from any point of its interior) to a sphere, which has genus $g = 0$. Now $n_i = 2$ and the formulas follow from the Lemma 1.4. □

The next tool we need (in order to prove the main result, Theorem 2.1) is the Burnside lemma, a standard tool for calculating the number of orbits.

Lemma 1.6. *(Burnside lemma) Let a group G act on some set Q. Let $|G| = n$ denote the number of elements of G and let $|Fix(g)|$ denote the number of elements a of the set Q, preserved by the given element g of the group: $g(a) = a$. Then the number of orbits Qo of the set Q is given by the formula*

$$Qo = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|.$$

If the group of rotational symmetries of a convex polyhedron is the cyclical group C_n then the exact value for the Euler orbit characteristic $Eo_R(\mathcal{P})$ can be obtained by a straightforward application of the Burnside lemma. This is a generalization of the similar result for the spherical polyhedra ([4], p.253).

Proposition 1.7. *Let $\mathcal{P}$ be a convex polyhedron. If $G_R(P) = C_n$ where C_n is generated by a rotation R_1 (thus $R_1^n = I$), then $Eo_R(\mathcal{P}) \in \{0, 1, 2\}$.*

Proof. The identity transformation fixes all vertices, edges and faces while the other $n-1$ rotations fix only the poles. Hence we get by the Burnside lemma and using the Euler formula $v - e + f = 2$ (valid for any convex polyhedron) the following formulas for the numbers of rotational orbits:

$$vo_R = \frac{1}{n}(v + (n-1)v_p),$$

$$eo_R = \frac{1}{n}(e + (n-1)e_p),$$

$$fo_R = \frac{1}{n}(f + (n-1)f_p),$$

$$Eo_R = \frac{1}{n}(2 + (n-1)E_p(R_1)),$$

and using $E_p(R_1) = 2 - 2e_p(R_1)$ we see: if $e_p(R_1) = 0$ then $E_p(R_1) = 2$ and $Eo_R = 2$; if $e_p(R_1) = 1$ then $n = 2$, $E_p(R_1) = 0$ and $Eo_R = 1$; if $e_p(R_1) = 2$ then $n = 2$, $E_p(R_1) = -2$ and $Eo_R = 0$. Thus, if $n > 2$ then $Eo_R = 2$. □

2 The main result

Theorem 2.1. *Let $\mathcal{P}$ be a polyhedron with faces on the surface S of genus g. Then*

$$Eo_R(\mathcal{P}) = \frac{1}{n}(\chi(\mathcal{P}) + \sum_{i=1}^{n-1} E_p(R_i)),$$

and we get the following bounds on $Eo_R(\mathcal{P})$:

$$\frac{1}{n}(\chi(\mathcal{P}) - \sum_{i=1}^{n-1} 2(g+1)) \leq Eo_R(\mathcal{P}) \leq \frac{1}{n}(\chi(\mathcal{P}) + \sum_{i=1}^{n-1} 2(g+1)).$$

If $\mathcal{P}$ is a convex polyhedron, then

$$-1 \leq Eo_R(\mathcal{P}) \leq 2.$$

Proof. Let n be the number of elements in the group $G_R(P)$.

The identity transformation fixes each vertex, edge or face.

Every rotation R_i fixes $v_p(R_i)$ vertices, $e_p(R_i)$ edges and $f_p(R_i)$ faces.

Therefore, by the Burnside lemma:

$$vo_R = \frac{1}{n}(v + v_p(R_1) + \cdots v_p(R_{n-1})),$$

$$eo_R = \frac{1}{n}(e + e_p(R_1) + \cdots e_p(R_{n-1})),$$

$$fo_R = \frac{1}{n}(f + f_p(R_1) + \cdots f_p(R_{n-1})),$$

$$Eo_R = \frac{1}{n}(\chi + E_p(R_1) + \cdots E_p(R_{n-1})).$$

Using $-2(g+1) \leq E_p(R_i) \leq 2(g+1)$ (proved in Lemma 1.4) we get

$$\frac{1}{n}(\chi - \sum_{i=1}^{n-1} 2(g+1)) \leq Eo_R \leq \frac{1}{n}(\chi + \sum_{i=1}^{n-1} 2(g+1)).$$

If $\mathcal{P}$ is a convex polyhedron, then $g = 0$, $\chi = 2$, hence

$$Eo_R \leq \frac{1}{n}(2 + (n-1)2) = 2,$$

$$Eo_R \geq \frac{1}{n}(2 + (n-1)(-2)) = \frac{1}{n}(4 + n(-2)) \geq -2 + \frac{4}{n} \geq -1,$$

because $\frac{4}{n} > 0$ and Eo_R must be an integer. □

Thus there are 4 classes C_2, C_1, C_0, C_{-1} of convex polyhedra, whose Eo_R are $2, 1,$ $0, -1$, respectively.

Is the lower bound $Eo_R = -1$ actually obtained, and (if it is so) for which convex polyhedra? And is there any simple description of these four classes?

Proposition 2.2. *Let a, b, c be the numbers of rotations R_i in the group $G_R(\mathcal{P})$ of a convex polyhedron for which $E_p(R_i)$ equals 2, 0 and -2, respectively, and let n be the number of elements in $G_R(\mathcal{P})$. Then*

$$Eo_R(\mathcal{P}) = \frac{1}{n}(2 + a \cdot 2 + c(-2)) = \frac{2}{n}(1 + a - c).$$

Thus the number $1 + a - c$ is an integer multiple of $\frac{n}{2}$. The numbers a and c can be (for each of the 4 possible values of Eo_R) expressed by b, n and Eo_R.

Proof. This formula follows immediately from the Burnside lemma. Also, it is clear that

$$a + b + c + 1 = n,$$

hence

$$a + c = n - b - 1.$$

The equation $\frac{2}{n}(1+a-c) = Eo_R$ implies $2(1+a-c) = Eo_R \cdot n$ and

$$c - a = 1 - \frac{Eo_R \cdot n}{2}.$$

Then $(a+c)+(c-a) = 2c = n - b - 1 + 1 - \frac{Eo_R \cdot n}{2} = \frac{(2-Eo_R)\cdot n}{2} - b$ and

$$c = \frac{(2-Eo_R)n - 2b}{4}.$$

Similarly, $2a = n - b - 1 - 1 + \frac{Eo_R \cdot n}{2}$, hence

$$a = \frac{n \cdot (2 + Eo_R) - (2b+4)}{4}.$$

For example, if $Eo_R = -1$ and $b = 0$ then $a = \frac{n}{4} - 1$ and $c = \frac{3n}{4}$. In that case n must be divisible by 4. □

Example 2.3. To find such a solid with 4 symmetries we have to look for one having three rotations with poles in edge centers! The lower bound $Eo_R = -1$ is really obtained for the Johnson solid J84 (Snub Disphenoid, see Figure 1), where $vo_R = 2, eo_R = 6, fo_R = 3$, hence $Eo_R = 2 - 6 + 3 = -1$. Here the number of symmetries is 4 (the identity transformation and 3 rotations of order two with axes going through edge centers), $b = 0, a = 0$ and $c = 3$. Thus this lower bound -1 is sharp.

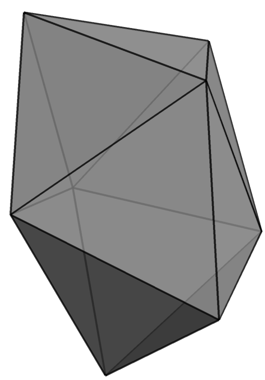

Figure 1: The Johnson solid J84, also known as the Snub Disphenoid.

Remark 2.4. A rotational axis of a non-convex polyhedron may have more than 2 ”poles”. As a consequence, there is no upper or lower bound for Eo_R in the non-convex case.

3 Classification of convex polyhedra

As an immediate consequence of the formulas in Proposition 3 we get:

Corollary 3.1. *The four classes C_2, C_1, C_0, C_{-1} of convex polyhedra (whose Eo_R are $2, 1, 0, -1$, respectively) can be characterized as follows:*

C_2: $a - c = n - 1$

C_1: $a - c = -1 + \frac{n}{2}$

C_0: $a - c = -1$

C_{-1}: $a - c = -1 - \frac{n}{2}$, *all poles in edge centers,* $a = 0$.

Corollary 3.2. *If $a \geq n/2$ then $Eo_R \in \{1, 2\}$.*

Proof. The relation $a \geq n/2$ is a sufficient condition for $a - c > 0$ (since there is also the identity transformation in the group $G_R(\mathcal{P})$) that holds only for polyhedra from C_2 and C_1. □

Corollary 3.3. *Let q be the number of all rotations $R_i \in G_R(\mathcal{P})$ with the property that R_i has the same rotational axis as some k-fold rotation for any $k > 2$. If $q \geq n/2$, where n is the order of the group G_R, then $Eo_R(\mathcal{P}) \in \{1, 2\}$.*

Proof. No rotation with such an axis can have any of its two poles in an edge center, hence $a \geq q \geq n/2$, therefore $Eo_R(\mathcal{P}) \in \{1, 2\}$. □

Now we can classify convex polyhedra with respect to their rotational symmetry groups and their rotational orbit Euler characteristic.

The only possible rotational groups of the Euclidean space E^3 are the rotational groups of 1) the n-gonal pyramid, 2) the n-gonal dipyramid or prism, 3) the regular tetrahedron, 4) the cube or the regular octahedron, 5) the regular dodecahedron or the regular icosahedron ([1], p.34).

Theorem 3.4. *Convex polyhedra with at least one rotational symmetry can be classified by their G_R and by their Eo_R into 13 classes (in Table 2 the impossible cases are marked with $\emptyset$):*

	C_2	C_1	C_0	C_{-1}
C_n cyclical group				$\emptyset$
D_n dihedral group				
T tetrahedron group			$\emptyset$	$\emptyset$
O octahedron group			$\emptyset$	$\emptyset$
D dodecahedron group			$\emptyset$	$\emptyset$

Table 2: Classification of convex polyhedra by G_R and Eo_R.

Proof. If $G_R = C_n$ then $Eo_R \neq -1$ (by Proposition 2). If $G_R \in \{T, O, D\}$ then $q \geq n/2$ (Table 3) hence $\mathcal{P}$ cannot be in C_0 or C_{-1} (by Corollary 3.3). □

	2	3	4	5	n	q
T tetrahedron	3	4			3 + 4.2 + 1 = 12	9
O cube	6	4	3		6 + 4.2 + 3.3 + 1 = 24	17
D dodecahedron	15	10		6	15 + 10.2+12.4 + 1= 60	44

Table 3: Numbers of the 2-,3-,4-,5-fold axes in the solids $\mathcal{P}$ with the rotational groups T, O or D, orders n of the groups T, O, D and the numbers q of all rotations $R_i \in G_R(\mathcal{P})$ with the property that R_i has the same rotational axis as some k-fold rotation for any $k > 2$.

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