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Classification of some reflexible edge-transitive embeddings of complete bipartite graphs

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Abstract

In this paper, we classify some reflexible edge-transitive orientable embeddings of complete bipartite graphs. As a by-product, we classify groups Γ such that (i) $\Gamma = XY$ for some cyclic groups $X = \langle x \rangle$ and $Y = \langle y \rangle$ with $X \cap Y = \{1_\Gamma\}$ and (ii) there exists an automorphism of Γ which sends x and y to x^{-1} and y^{-1}, respectively.

Keywords: Complete bipartite graphs, reflexible edge-transitive embedding.

Math. Subj. Class.: 05C10, 05C30

1 Preliminaries

A *map* is a 2-cell embedding of a graph G in a compact, connected surface. A map is called *orientable* or *nonorientable* according to whether the supporting surface is orientable or nonorientable. In this paper, we only consider orientable maps.

For a simple connected graph G, an *arc* of G is an ordered pair (u, v) of adjacent vertices in G. The set of all arcs in G is denoted by $D(G)$. An orientable map $\mathcal{M}$ can be described by a pair $(G; R)$, where G is the underlying graph of $\mathcal{M}$ and R is a permutation of the arc set $D(G)$ whose orbits coincide with the sets of arcs emanating from the same vertex. The permutation R is called the *rotation* of the map $\mathcal{M}$.

For given two maps $\mathcal{M}_1 = (G_1; R_1)$ and $\mathcal{M}_2 = (G_2; R_2)$, a *map isomorphism* $\phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ is a graph isomorphism $\phi\colon G_1 \to G_2$ such that $\phi R_1(u, v) = R_2\phi(u, v)$ for any arc (u, v) in G_1. Furthermore if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, ϕ is called a *map automorphism* of $\mathcal{M}$. The set of all map automorphisms of $\mathcal{M}$ denoted by $\mathrm{Aut}(\mathcal{M})$ is a group under the composition operation, and it is called the *automorphism group* of $\mathcal{M}$. For a map $\mathcal{M} = (G; R)$,

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the group $\mathrm{Aut}(\mathcal{M})$ acts semi-regularly on the arc set $D(G)$, so $|\mathrm{Aut}(\mathcal{M})| \leq 2|E(G)|$. If this bound is attained, then $\mathrm{Aut}(\mathcal{M})$ acts regularly on the arc set, and the map is called a *regular map* or a *regular embedding*. The map $\mathcal{M}$ is said to be *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}(\mathcal{M})$ acts transitively on $V(G)$ or $E(G)$, respectively. For an orientable embedding $\mathcal{M}$ of a bipartite graph G, if the set of partite set preserving map automorphisms acts transitively on $E(G)$ then we call $\mathcal{M}$ an *edge-transitive map* or an *edge-transitive embedding* satisfying the Property (P) in this paper. For a map $\mathcal{M} = (G; R)$, if $\mathcal{M}$ and $\mathcal{M}^{-1} = (G; R^{-1})$ are isomorphic, $\mathcal{M}$ is called *reflexible*.

Classifying highly symmetric embeddings of graphs in a given class is an interesting problem in topological graph theory. In recent years, there has been particular interest in the regular embeddings of complete bipartite graphs $K_{n,n}$ by several authors [1, 2, 4, 5, 6, 7, 8, 10]. The reflexible regular embeddings and self-Petrie dual regular embeddings of $K_{n,n}$ have been classified by the authors [7]. Recently, G. Jones has completed the classification of regular embeddings of $K_{n,n}$ [5] and the authors have classified nonorientable regular embeddings of $K_{n,n}$ [8]. In [3], Graver and Watkins classified edge-transitive maps on closed surfaces into fourteen types. In this paper, we classify reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) which correspond to types 1 or 2 among 14 types. The following theorem is the main result in this paper.

Theorem 1.1. *For any integers*

$$m = 2^a p_1^{a_1} \cdots p_\ell^{a_\ell} p_{\ell+1}^{a_{\ell+1}} \cdots p_{\ell+f}^{a_{\ell+f}} \quad \textit{and}$$
$$n = 2^b p_1^{b_1} \cdots p_\ell^{b_\ell} q_{\ell+1}^{a_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}} \quad \textit{(prime decompositions)}$$

with $\gcd(m,n) = 2^c p_1^{c_1} \cdots p_\ell^{c_\ell}$ *and* $a \leq b$*, the number (up to isomorphism) of reflexible edge-transitive embeddings of* $K_{m,n}$ *satisfying the Property (P) is* 1 *if both* m *and* n *are odd;* $2^f(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})$ *if exactly one of* m *and* n *is even, namely, only* n *is even;* $A(a,b)2^{f+g+\ell}(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})$ *if both* m *and* n *are even, where*

$$A(a,b) = \begin{cases} 1 & \textit{if } (a,b) = (1,1), \\ 2 & \textit{if } (a,b) = (1,2), \\ 4 & \textit{if } (a,b) = (2,2) \textit{ or } (1,k) \textit{ with } k \geq 3, \\ 10 & \textit{if } (a,b) = (2,3), \\ 12 & \textit{if } (a,b) = (2,k) \textit{ with } k \geq 4, \\ 28 & \textit{if } (a,b) = (3,3), \\ 40 & \textit{if } (a,b) = (3,4), \\ 36 & \textit{if } (a,b) = (3,k) \textit{ with } k \geq 5, \\ 20(1+2^{a-2}) & \textit{if } a = b \geq 4, \\ 20 + 18 \cdot 2^{a-2} & \textit{if } b-1 = a \geq 4, \\ 20 + 16 \cdot 2^{a-2} & \textit{if } b-2 \geq a \geq 4. \end{cases}$$

Our paper is organized as follows. In the next section, we consider some relations between edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) and products of two cyclic groups. In Section 3, we classify reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) when at least one of m and n is odd. In Section 4, for even integers m and n, the classification of reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) is given. In the final section, we classify groups Γ satisfying the conditions:

(i) $\Gamma = XY$ for some cyclic groups $X = \langle x \rangle$ and $Y = \langle y \rangle$ with $X \cap Y = \{1_\Gamma\}$ and

(ii) there exists an automorphism of Γ which sends x and y to x^{-1} and y^{-1}.

2 (m, n)-bicyclic triples in $\mathrm{Aut}(K_{m,n})$

Regular embeddings of the complete bipartite graphs $K_{n,n}$ are related to groups Γ with two generators satisfying some conditions [4]. Using this relation, G. Jones classify regular embeddings of $K_{n,n}$ [5]. Similarly, we aim to find a relation between edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) and groups with two generators satisfying some conditions in this section.

In [4], G. Jones et al. showed that any finite group Γ is isomorphic to $\mathrm{Aut}(\mathcal{M})$ for some regular embedding $\mathcal{M}$ of $K_{n,n}$ if and only if Γ has cyclic subgroups $X = \langle x \rangle$ and $Y = \langle y \rangle$ of order n such that:

(i) $\Gamma = XY$

(ii) $X \cap Y = \{1_\Gamma\}$ and

(iii) there is an automorphism α of Γ transposing x and y.

They call the triple (Γ, x, y) satisfying these conditions the *n-isobicyclic* triple. In this relation, x and y correspond to rotations of $\mathcal{M}$ around two fixed adjacent vertices u and v, respectively. The automorphism α corresponds to the half-turn reversing the edge uv. For two n-isobicyclic triples (Γ_1, x_1, y_1) and (Γ_2, x_2, y_2), two corresponding regular embeddings $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic if and only if there exists a group isomorphism from Γ_1 to Γ_2 given by $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$. Using this, one can show that the regular embedding $\mathcal{M}$ induced by n-isobicyclic triple (Γ, x, y) is reflexible if and only if there exists an automorphism β of Γ which sends x and y to x^{-1} and y^{-1}, respectively. (For more information, the reader is referred to [4].)

Note that one can define an embedding of $K_{n,n}$ by using the first and second conditions of n-isobicyclic triple, and the induced map is edge-transitive map satisfying the Property (P) even though the third condition of n-isobicyclic triple is not satisfied. Conversely, any edge transitive embedding of $K_{n,n}$ satisfying the Property (P) is isomorphic to some induced map by such a triple (Γ, x, y). One can show that for different positive integers m and n, an edge-transitive embedding of $K_{m,n}$ satisfying the Property (P) can also be represented by a similar triple. For a group Γ containing cyclic subgroups $X = \langle x \rangle$ of order n and $Y = \langle y \rangle$ of order m, the triple (Γ, x, y) is called *(m, n)-bicyclic* if it satisfies:

(i) $\Gamma = XY$ and

(ii) $X \cap Y = \{1_\Gamma\}$.

For any (m, n)-bicyclic triple (Γ, x, y), one can define an embedding of $K_{m,n}$ by a similar way to define an embedding of $K_{n,n}$ using n-isobicyclic triple. We denote this embedding by $\mathcal{M}(\Gamma, x, y)$. One can see that $\mathcal{M}(\Gamma, x, y)$ is an edge-transitive embedding of $K_{m,n}$ satisfying the Property (P). Furthermore the following result holds.

Lemma 2.1 ([9]). *Let m, n be two positive integers (not necessarily distinct).*

(1) Any edge-transitive embedding $\mathcal{M}$ of $K_{m,n}$ satisfying the Property (P) is isomorphic to $\mathcal{M}(\Gamma, x, y)$ for some (m, n)-bicyclic triple (Γ, x, y).

(2) *For two (m, n)-bicyclic triples (Γ_1, x_1, y_1) and (Γ_2, x_2, y_2), two edge-transitive embeddings $\mathcal{M}(\Gamma_1, x_1, y_1)$ and $\mathcal{M}(\Gamma_2, x_2, y_2)$ are isomorphic if and only if there exists a group isomorphism from Γ_1 to Γ_2 given by $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$.*

For any (m, n)-bicyclic triple (Γ, x, y), there exists a subgroup H of the automorphism group $\mathrm{Aut}(K_{m,n})$ such that:

(i) H is isomorphic to Γ and

(ii) x and y in Γ correspond to elements in H which cyclically permute vertices in the partite sets of size n and m, respectively.

Hence it suffices to deal with such (m, n)-bicyclic triples in $\mathrm{Aut}(K_{m,n})$ to classify edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P).

For any positive integer m, denote the set $\{0, 1, \dots, m-1\}$ by $[m]$. Let

$$V = \{0, 1, \dots, (m-1)\} \cup \{0', 1', \dots, (n-1)'\} = [m] \cup [n]'$$

be the vertex set of $K_{m,n}$ as partite sets, and let

$$D = \{(i, j'), (j', i) : 0 \le i \le m-1 \text{ and } 0 \le j \le n-1\}$$

be the arc set, where (i, j') is the arc emanating from i to j' and (j', i) denotes its inverse. We denote the symmetric group on $[m]$ and $[n]'$ by S and S', respectively. Let S_0 and S_0' be their stabilizers of 0 and $0'$, respectively. Note that $\mathrm{Aut}(K_{m,n})$ is isomorphic to $S \times S'$ when $m \neq n$; $S \wr \mathbb{Z}_2$ when $m = n$. We identify integers $0, 1, 2, \dots$ with their residue classes modulo m or n according to the context.

Let (Γ, x, y) be an (m, n)-bicyclic triple such that Γ is a subgroup of $\mathrm{Aut}(K_{m,n})$. Now there exists an automorphism $\phi \in \mathrm{Aut}(K_{m,n})$ such that

$$x^\phi = \phi^{-1} x \phi = \alpha(0'\ 1'\ \cdots\ (n-1)') \quad \text{and} \quad y^\phi = \phi^{-1} y \phi = \beta(0\ 1\ \cdots\ m-1),$$

where $\alpha \in S_0$ and $\beta \in S_0'$. For any $\alpha \in S_0$ and $\beta \in S_0'$, let

$$x_\alpha = \alpha(0'\ 1'\ \cdots\ (n-1)') \quad \text{and} \quad y_\beta = \beta(0\ 1\ \cdots\ m-1).$$

From now on, we only consider triples $(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ as candidates of (m, n)-bicyclic triples.

Lemma 2.2 ([9]). *For any $\alpha \in S_0$ and $\beta \in S_0'$,*

1. *the group $\langle x_\alpha, y_\beta \rangle$ acts transitively on the edge set of $K_{m,n}$ and*
2. *the triple $(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is (m, n)-bicyclic if and only if $|\langle x_\alpha, y_\beta \rangle| = mn$.*

By Lemma 2.2, we need to characterize $\alpha \in S_0$ and $\beta \in S_0'$ satisfying $|\langle x_\alpha, y_\beta \rangle| = mn$ to classify edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P). To do this, we denote

$$\mathrm{ET}_{m,n} = \{(\alpha, \beta) : \alpha \in S_0,\ \beta \in S_0' \text{ and } |\langle x_\alpha, y_\beta \rangle| = mn\}.$$

Note that for any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, $(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is an (m, n)-bicyclic triple and hence $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$ is an edge-transitive embedding of $K_{m,n}$ satisfying the Property (P). Conversely for any edge-transitive embedding $\mathcal{M}$ of $K_{m,n}$ satisfying the Property (P), there exists $(\alpha, \beta) \in \mathrm{ET}_{m,n}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{M}(\langle x_\alpha, y_\beta \rangle, x_\alpha, y_\beta)$.

Remark 2.3.

(1) For any $(\alpha,\beta)\in \mathrm{ET}_{m,n}$,

$$\langle x_\alpha, y_\beta\rangle = \{x_\alpha^i y_\beta^j \mid i\in[n],\ j\in[m]\} = \{y_\beta^j x_\alpha^i \mid i\in[n],\ j\in[m]\}.$$

Hence in many cases, if α satisfies some properties then β also satisfies the same properties and vice versa.

(2) Note that for different positive integers m and n and for an orientable embedding $\mathcal{M}$ of $K_{m,n}$, any automorphism of $\mathcal{M}$ is partite set preserving. Let $m=n$ be odd and let $\mathcal{M}$ be an orientable edge-transitive embedding of $K_{n,n}$. If a subgroup Γ of $\mathrm{Aut}(\mathcal{M})$ acts regularly on the edge set then $|\Gamma| = m^2$ is odd and hence there exists no partite set reversing element in Γ. Hence for odd n, every edge-transitive embedding of $K_{n,n}$ is an edge-transitive embedding of $K_{n,n}$ satisfying the Property (P). On the other hand, for even n, we do not know whether the above statement is true or not.

The next lemma shows that for different $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in \mathrm{ET}_{m,n}$, two induced edge-transitive embeddings are non-isomorphic.

Lemma 2.4 ([9]). *For any $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in \mathrm{ET}_{m,n}$, the induced edge-transitive embeddings $\mathcal{M}(\langle x_{\alpha_1}, y_{\beta_1}\rangle, x_{\alpha_1}, y_{\beta_1})$ and $\mathcal{M}(\langle x_{\alpha_2}, y_{\beta_2}\rangle, x_{\alpha_2}, y_{\beta_2})$ are isomorphic if and only if $(\alpha_1,\beta_1)=(\alpha_2,\beta_2)$.*

By Lemma 2.4, distinct pairs in $\mathrm{ET}_{m,n}$ give non-isomorphic edge-transitive embeddings of $K_{m,n}$ and the number of edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) equals to the cardinality $|\,\mathrm{ET}_{m,n}\,|$. But for distinct pairs $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in \mathrm{ET}_{m,n}$, two groups $\langle x_{\alpha_1}, y_{\beta_1}\rangle$ and $\langle x_{\alpha_2}, y_{\beta_2}\rangle$ may possibly be isomorphic. We do not know a necessary and sufficient condition for $\langle x_{\alpha_1}, y_{\beta_1}\rangle \simeq \langle x_{\alpha_2}, y_{\beta_2}\rangle$. So we propose the following problem.

Problem 2.5. For any positive integers m and n and for any $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in \mathrm{ET}_{m,n}$, find a necessary and sufficient condition for $\langle x_{\alpha_1}, y_{\beta_1}\rangle \simeq \langle x_{\alpha_2}, y_{\beta_2}\rangle$.

From now on, we aim to characterize the set $\mathrm{ET}_{m,n}$. Note that for any $(\alpha,\beta)\in \mathrm{ET}_{m,n}$, the stabilizers $\langle x_\alpha, y_\beta\rangle_0$ and $\langle x_\alpha, y_\beta\rangle_{0'}$ are cyclic groups $\langle x_\alpha\rangle$ of order n and $\langle y_\beta\rangle$ of order m, respectively.

Lemma 2.6. *For any $(\alpha,\beta)\in \mathrm{ET}_{m,n}$, $\langle\alpha\rangle$ and $\langle\beta\rangle$ are cyclic groups of order $|\{\alpha^i(1) : i\in[n]\}|$ and $|\{\beta^i(1') : i\in[m]\}|$, the lengths of the orbit containing 1 and $1'$, respectively. Furthermore they are divisors of n and m, respectively.*

Proof. Let $d_1 = |\{\alpha^i(1) : i\in[n]\}|$ and $d_2 = |\{\beta^i(1') : i\in[m]\}|$. Now d_1 and d_2 are divisors of the orders $|\langle x_\alpha\rangle| = n$ and $|\langle y_\beta\rangle| = m$, respectively. Note that

$$\alpha^{d_1}(1) = 1 \quad \text{and} \quad y_\beta^{-1} x_\alpha^{d_1} y_\beta(0) = 0,$$

which implies that, as a conjugate of $x_\alpha^{d_1}$, $y_\beta^{-1} x_\alpha^{d_1} y_\beta$ belongs to the vertex stabilizer $\langle x_\alpha, y_\beta\rangle_0 = \langle x_\alpha\rangle$. Since d_1 is a divisor of n, $y_\beta^{-1} x_\alpha^{d_1} y_\beta = x_\alpha^{rd_1}$ for some $r\in[n]$ such that $\gcd(r, \frac{n}{d_1}) = 1$, where $\gcd(r, \frac{n}{d_1})$ is the greatest common divisor of r and $\frac{n}{d_1}$. Now, suppose to the contrary that $|\langle\alpha\rangle| \neq d_1$. Then there exists $k \in [m]$ such that $\alpha^{d_1}(k) \neq k$. Let q be the largest element in $[m]$ such that $\alpha^{d_1}(q) \neq q$. On the other hand,

$$\alpha^{rd_1}(q) = x_\alpha^{rd_1}(q) = y_\beta^{-1} x_\alpha^{d_1} y_\beta(q) = y_\beta^{-1} x_\alpha^{d_1}(q+1) = y_\beta^{-1}(q+1) = q,$$

contradictory to $\alpha^{rd_1}(q) \neq q$. Therefore $|\langle\alpha\rangle| = d_1$. Similarly, one can show that $|\langle\beta\rangle| = d_2$. □

For any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, it follows from Lemma 2.6 that the length of each cycle in α (β, resp.) is a divisor of the length d_1 (d_2, resp.) of the cycle containing 1 ($1'$, resp.).

From now on we denote i', $[n]'$ and $\beta(i')$ simply by i, $[n]$ and $\beta(i)$ for any $i' \in [n]'$, respectively. The following lemma is related to a characterization of the set $\mathrm{ET}_{m,n}$.

Lemma 2.7 ([9]). *Let $\alpha \in S_0$ and $\beta \in S_0'$. Then $(\alpha, \beta) \in \mathrm{ET}_{m,n}$ if and only if for each $i \in [n]$, there exist $a(i) \in [n]$ and $b(i) \in [m]$ such that $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ for all $k \in [m]$ and $\beta(t + i) = \beta^{b(i)}(t) + a(i)$ for all $t \in [n]$. In this case, we have $a(i) = \beta(i)$ and $b(i) = -\alpha^{-i}(-1)$.*

Note that the equations in Lemma 2.7 is equivalent to $y_\beta x_\alpha^i = x_\alpha^{a(i)} y_\beta^{b(i)}$. The next lemma gives a characterization of $(\alpha, \beta) \in \mathrm{ET}_{m,n}$ whose induced edge-transitive embedding contains a partite set preserving reflection.

Lemma 2.8 ([9]). *For any $(\alpha, \beta) \in \mathrm{ET}_{m,n}$, $\mathcal{M}(\langle x_\alpha, y_\beta\rangle, x_\alpha, y_\beta)$ contains a partite set preserving reflection if and only if $\alpha^{-1}(-k) = -\alpha(k)$ for any $k \in [m]$ and $\beta^{-1}(-t) = -\beta(t)$ for any $t \in [n]$.*

For our convenience, we denote

$$\mathrm{RET}_{m,n} = \{(\alpha, \beta) \in \mathrm{ET}_{m,n} : \alpha^{-1}(-k) = -\alpha(k) \text{ for any } k \in [m] \text{ and } \beta^{-1}(-t) = -\beta(t) \text{ for any } t \in [n]\}.$$

We call an edge-transitive embedding of $K_{m,n}$ satisfying the Property (P) which also contains a partite set preserving reflection a *reflexible edge-transitive embedding of $K_{m,n}$ satisfying the Property (P)*. By Lemmas 2.4 and 2.8, the number (up to isomorphism) of reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) equals to the cardinality $|\mathrm{RET}_{m,n}|$. Note that if $\alpha \in S$ and $\beta \in S'$ are the identity permutations, then (α, β) belongs to $\mathrm{RET}_{m,n}$ by Lemma 2.8. So for any two positive integers m and n, there exists at least one reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P).

By Lemma 2.8, for any $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ and for any $j \in [m]$ and $i \in [n]$

$$\alpha^{-i}(-j) = \alpha^{-i+1}(-\alpha(j)) = \alpha^{-i+2}(-\alpha^2(j)) = \cdots = \alpha^{-1}(-\alpha^{i-1}(j)) = -\alpha^i(j)$$

and similarly $\beta^{-j}(-i) = -\beta^j(i)$.

Lemma 2.9. *For any $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ and for any $j \in [m]$ and $i \in [n]$,*

$$y_\beta^j x_\alpha^i = x_\alpha^{\beta^j(i)} y_\beta^{\alpha^i(j)}.$$

Proof. Since $\langle x_\alpha, y_\beta \rangle = \langle x_\alpha \rangle \langle y_\beta \rangle$, for any $j \in [m]$ and $i \in [n]$, there exist $a(i,j) \in [n]$ and $b(i,j) \in [m]$ such that $y_\beta^j x_\alpha^i = x_\alpha^{a(i,j)} y_\beta^{b(i,j)}$. By taking their values of $k \in [m]$ and $t \in [n]$, we have

$$\alpha^i(k) + j = \alpha^{a(i,j)}(k + b(i,j)) \quad \text{and} \quad \beta^j(t+i) = \beta^{b(i,j)}(t) + a(i,j).$$

Inserting $k = -b(i,j)$ and $t = 0$ to the equation $\alpha^i(k) + j = \alpha^{a(i,j)}(k + b(i,j))$ and $\beta^j(t+i) = \beta^{b(i,j)}(t) + a(i,j)$, respectively, we have

$$b(i,j) = -\alpha^{-i}(-j) = \alpha^i(j) \quad \text{and} \quad a(i,j) = \beta^j(i). \qquad \square$$

Lemma 2.10. *Let* $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ *and let* $d_1 = |\langle\alpha\rangle|$ *and* $d_2 = |\langle\beta\rangle|$. *It holds that* $\alpha(k) \equiv -k \pmod{d_2}$ *for any* $k \in [m]$ *and* $\beta(t) \equiv -t \pmod{d_1}$ *for any* $t \in [n]$.

Proof. By Lemma 2.7, for each $i \in [n]$, there exist $a(i) \in [n]$ and $b(i) \in [m]$ such that $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ for all $k \in [m]$ and $\beta(t+i) = \beta^{b(i)}(t) + a(i)$ for all $t \in [n]$. Furthermore $a(i) = \beta(i)$ and $b(i) = -\alpha^{-i}(-1) = \alpha^i(1)$. Inserting $k = 0$ to the equation $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$, we have $b(i) = \alpha^{-a(i)}(1) = \alpha^{-\beta(i)}(1)$. Hence $\alpha^i(1) = \alpha^{-\beta(i)}(1)$ for any $i \in [n]$. Since the order of α equals to the length of the orbit containing 1 by Lemma 2.6, $\beta(i) \equiv -i \pmod{d_1}$. By symmetry between α and β, it also holds that $\alpha(k) \equiv -k \pmod{d_2}$ for any $k \in [m]$. $\square$

By Lemmas 2.7 and 2.10, $b(i) = -\alpha^{-i}(-1) = \alpha^i(1) \equiv (-1)^i \pmod{d_2}$. Hence for any $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ with $d_1 = |\langle\alpha\rangle|$ and $d_2 = |\langle\beta\rangle|$, we have

$$\beta(t+i) = \beta^{b(i)}(t) + a(i) = \beta^{\alpha^i(1)}(t) + \beta(i) = \beta^{(-1)^i}(t) + \beta(i)$$

for all $i, t \in [n]$. By symmetry, it also holds $\alpha(k+j) = \alpha^{(-1)^j}(k) + \alpha(j)$ for all $j, k \in [m]$.

Lemma 2.11. *Let* $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ *and let* $d_1 = |\langle\alpha\rangle|$ *and* $d_2 = |\langle\beta\rangle|$. *Now*

(1) *if one of* d_1 *and* d_2 *is 1, say* $d_1 = 1$, *then either* $d_2 = 1$ *or (*m *is even and* $d_2 = 2$*);*

(2) *if one of* d_1 *and* d_2 *is at least 3, say* $d_1 \geq 3$, *then both* m *and* d_2 *are even;*

(3) *if* m*(*n*, resp.) is even then* α *(*β*, resp.) is parity preserving. Furthermore there exists* $s, t \in [m]$ *such that* $\alpha(2k) = 2kt$, $\alpha(2k+1) = 2kt + 2s + 1$ *and* $2t^2 = 2$*;*

(4) *if both* d_1 *and* d_2 *are at least 3 then they are divisors of* $\gcd(m,n)$.

Proof. (1): Let $d_1 = 1$ and $d_2 \geq 2$. By Lemma 2.10, $\alpha(1) \equiv -1 \pmod{d_2}$. Since α is the identity, $1 \equiv -1 \pmod{d_2}$. By the assumption $d_2 \geq 2$, $d_2 = 2$. By Lemma 2.6, d_2 is a divisor of m, and hence m is even.

(2): Let $d_1 \geq 3$. By lemma 2.10, $\beta(k) \equiv -k \pmod{d_1}$, which implies that the order d_2 of β is even. Since d_2 is a divisor of m, m is also even.

(3): Let m be even. If $d_1 = 1$ then α is the identity and hence α is parity preserving. If $d_1 = 2$ then $\alpha^{-1} = \alpha$ and

$$\alpha(k) = \alpha(k-1+1) = \alpha(k-1) + \alpha(1) = \alpha(k-2) + 2\alpha(1) = \cdots = k\alpha(1)$$

for all $k \in [m]$. Since $\alpha^2(1) = \alpha(\alpha(1)) = (\alpha(1))^2 = 1$ and m is even, $\alpha(1)$ should be odd. Hence α is parity preserving. Assume that $d_1 \geq 3$. Then, d_2 is even by (2). Since $\alpha(k) \equiv -k \pmod{d_2}$, α is parity preserving.

For any $2k \in [m]$,

$$\alpha(2k) = \alpha(2(k-1)) + \alpha(2) = \alpha(2(k-2)) + 2\alpha(2) = \cdots = k\alpha(2) \quad \text{and}$$
$$\alpha(2k+1) = \alpha(2(k-1)+1) + \alpha(2) = \cdots = \alpha(1) + k\alpha(2).$$

Let $\alpha(1) = 2s+1$ and $\alpha(2) = 2t$. Now $\alpha(2k) = k\alpha(2) = 2kt$ and $\alpha(2k+1) = k\alpha(2) + \alpha(1) = 2kt + 2s + 1$. Note that for any $2k \in [m]$, $\alpha(1) + \alpha(2k) = \alpha(2k+1) = \alpha^{-1}(2k) + \alpha(1)$. Hence $\alpha^{-1}(2k) = \alpha(2k)$, namely, $\alpha^2(2k) = 2k$. So we have $\alpha^2(2) = \alpha(2t) = 2t^2 = 2$.

(4): Let $d_1, d_2 \geq 3$. Now all of d_1, d_2, m and n are even by (2). Hence there exist $s, t \in [m]$ such that $\alpha(2k) = 2kt$, $\alpha(2k+1) = 2kt + 2s + 1$ and $2t^2 = 2$ by (3). Since d_1 is even and

$$\alpha^{2i}(1) = \alpha^{2i-1}(2s+1) = \alpha^{2i-2}(2st + 2s + 1) = \cdots = 2is(t+1) + 1,$$

d_1 is the smallest positive integer such that $d_1 s(t+1) \equiv 0 \pmod{m}$ by Lemma 2.6. Hence d_1 is a divisor of m and consequently a divisor of $\gcd(m, n)$. Similarly d_2 is a divisor of $\gcd(m, n)$. □

3 At least one of m and n is odd

In this section, we classify reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) when at least one of m and n is odd. Note that when at least one of m and n is odd, any orientable edge-transitive embedding of $K_{m,n}$ is an edge-transitive embedding satisfying the Property (P). In [9], the second author counted $|\operatorname{RET}_{m,n}|$ when both m and n are odd as follows.

Theorem 3.1 ([9]). *If both m and n are odd then $|\operatorname{RET}_{m,n}| = 1$, namely, there exists only one reflexible edge-transitive embedding of $K_{m,n}$ satisfying the Property (P) up to isomorphism.*

In the next theorem, we count $|\operatorname{RET}_{m,n}|$ when exactly one of m and n is odd. By symmetry, we assume that m is odd.

Theorem 3.2. *Let*

$$m = p_1^{a_1} \cdots p_\ell^{a_\ell} p_{\ell+1}^{a_{\ell+1}} \cdots p_{\ell+f}^{a_{\ell+f}} \quad \textit{(prime factorization)}$$

be odd and

$$n = 2^b p_1^{b_1} \cdots p_\ell^{b_\ell} q_{\ell+1}^{b_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}} \quad \textit{(prime factorization)}$$

be even. Let $\gcd(m, n) = p_1^{c_1} \cdots p_\ell^{c_\ell}$ with $c_i \geq 1$ for any $i = 1, \ldots, \ell$. Now

$$|\operatorname{RET}_{m,n}| = 2^f (1 + p_1^{c_1}) \cdots (1 + p_\ell^{c_\ell}),$$

namely, there exist $2^f (1 + p_1^{c_1}) \cdots (1 + p_\ell^{c_\ell})$ reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) up to isomorphism.

Proof. Let $(\alpha, \beta) \in \operatorname{RET}_{m,n}$ and let $d_1 = |\langle \alpha \rangle|$ and $d_2 = |\langle \beta \rangle|$. Suppose that $d_1 \geq 3$. Then both d_2 and m are even by Lemma 2.11(2), which is a contradiction. Hence $d_1 = 1$ or 2. Furthermore for any $k \in [m]$,

$$\alpha(k) = \alpha^{-1}(k-1) + \alpha(1) = \alpha(k-1) + \alpha(1) = \cdots = k\alpha(1).$$

Let $\alpha(1) = r$. Now $\alpha(k) = rk$ and $\alpha^2(1) = \alpha(r) = r^2 \equiv 1 \pmod m$.

Since n is even, β is parity preserving and there exists $s, t \in [n]$ such that $\beta(2k) = 2kt$, $\beta(2k+1) = 2kt + 2s + 1$ and $2t^2 = 2$ for any $2k \in [n]$ by Lemma 2.11(3). If $2t \neq 2$ then the length of the orbit containing 2 is 2 and hence d_2 is even. But it can not happen because m is odd. Hence for any $2k \in [n]$, $\beta(2k) = 2k$, $\beta(2k+1) = 2k + 2s + 1$ and for any $i \in [m]$,

$$\beta^i(1) = \beta^{i-1}(2s+1) = \beta^{i-2}(2s+2s+1) = \cdots = 2is + 1.$$

Therefore d_2 is the smallest positive integer such that $2d_2 s \equiv 0 \pmod n$, which implies that d_2 is a divisor of n, and hence d_2 is a divisor of $\gcd(m,n) = p_1^{c_1} \cdots p_\ell^{c_\ell}$.

If $r \equiv 1 \pmod{p_i^{a_i}}$ for some $i = 1, 2, \ldots, \ell$, then the fact $\alpha(1) = r \equiv -1 \pmod{d_2}$ implies that p_i can not be a divisor of d_2. Hence $p_i^{b_i}$ should divide s, namely, $s \equiv 0 \pmod{p_i^{b_i}}$. If $r \equiv -1 \pmod{p_j^{a_j}}$ for some $j = 1, 2, \ldots, \ell$, then $s \equiv x \cdot p_j^{b_j - c_j} \pmod{p_j^{b_j}}$ for some x with $0 \leq x \leq p_j^{c_j} - 1$ because d_2 is a divisor of $\gcd(m,n)$. Therefore, for any $j = 1, \ldots, \ell$, the pair $(r \pmod{p_j^{a_j}}, s \pmod{p_j^{b_j}})$ is $(1, 0)$ or $(-1, x \cdot p_j^{b_j - c_j})$ for some x with $0 \leq x \leq p_j^{c_j} - 1$.

Because $d_2 \mid \gcd(m,n)$, we have $2s \equiv 0 \pmod{2^b}$ and for any $k = 1, 2, \ldots, g$, $s \equiv 0 \pmod{q_{\ell+k}^{b_{\ell+k}}}$. Since $r^2 \equiv 1 \pmod m$, $r \equiv \pm 1 \pmod{p_{\ell+j}^{a_{\ell+j}}}$ for any $j = 1, 2, \ldots f$.

Conversely for any $r \in [m]$ and $s \in [n]$ satisfying the conditions

(i) for any $j = 1, \ldots, \ell$, the pair $(r \pmod{p_j^{a_j}}), s \pmod{p_j^{b_j}})$ is $(1, 0)$ or $(-1, x \cdot p_j^{b_j - c_j})$ for some integer x with $0 \leq x \leq p_j^{c_j} - 1$,

(ii) $2s \equiv 0 \pmod{2^b q_{\ell+1}^{b_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}}}$ and

(iii) for any $j = 1, 2, \ldots f$, $r \equiv \pm 1 \pmod{p_{\ell+j}^{a_{\ell+j}}}$,

define $\alpha(k) = rk$ for any $k \in [m]$ and $\beta(2t) = 2t$, $\beta(2t+1) = 2t + 2s + 1$ for any $2t \in [n]$. Note that $\alpha \in S_0$ and $\beta \in S_0'$. Let $d_1' = |\langle \alpha \rangle|$ and $d_2' = |\langle \beta \rangle|$. Now $d_1' = 1$ or 2 depending on the value of r and d_2' is the smallest positive integer satisfying $2d_2' s \equiv 0 \pmod n$. Note that d_2' divides $\gcd(m,n)$ and $r \equiv -1 \pmod{d_2'}$. For any $i \in [n]$, let $a(i) = \beta(i)$ and $b(i) = \alpha^i(1) = r^i$. For the first case, let i be even. Now $a(i) = \beta(i) = i$ and $b(i) = \alpha^i(1) = 1$. For any $2t \in [n]$,

$$\beta(2t+i) = 2t + i \quad \text{and}$$
$$\beta^{b(i)}(2t) + a(i) = \beta(2t) + \beta(i) = 2t + i$$

and

$$\beta(2t+1+i) = 2t + i + 2s + 1 \quad \text{and}$$
$$\beta^{b(i)}(2t+1) + a(i) = \beta(2t+1) + \beta(i) = 2t + 2s + 1 + i.$$

Hence $\beta(t+i) = \beta^{b(i)}(t) + a(i)$ for any $t \in [n]$. For any $k \in [m]$,

$$\alpha^i(k) = k \quad \text{and}$$
$$\alpha^{a(i)}(k + b(i)) - 1 = k.$$

Hence $\alpha^i(k) = \alpha^{a(i)}(k + b(x)) - 1$ for any $k \in [m]$.

For the remaining case, let i be odd. Now $a(i) = \beta(i) = i + 2s$ and $b(i) = \alpha^i(1) = r \equiv -1 \pmod{d_2'}$. For any $2t \in [n]$,

$$\beta(2t+i) = 2t+i+2s \quad \text{and}$$
$$\beta^{b(i)}(2t) + a(i) = \beta^{-1}(2t) + \beta(i) = 2t+i+2s$$

and

$$\beta(2t+1+i) = 2t+i+1 \quad \text{and}$$
$$\beta^{b(i)}(2t+1) + a(i) = \beta^{-1}(2t+1) + \beta(i) = 2t+1-2s+i+2s = 2k+i+1.$$

Hence $\beta(t+i) = \beta^{b(i)}(t) + a(i)$ for any $t \in [n]$. For any $k \in [m]$,

$$\alpha^i(k) = rk \quad \text{and}$$
$$\alpha^{a(i)}(k+b(i)) - 1 = \alpha(k+r) - 1 = rk + r^2 - 1 = rk.$$

Hence $\alpha^i(k) = \alpha^{a(i)}(k+b(i)) - 1$ for any $k \in [m]$. By Lemma 2.7, $(\alpha, \beta) \in \mathrm{ET}_{m,n}$. Furthermore one can easily check that $\alpha^{-1}(-k) = -\alpha(k)$ for any $k \in [m]$ and $\beta^{-1}(-t) = -\beta(t)$ for any $t \in [n]$. Hence $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ by Lemma 2.8.

Therefore

$$|\,\mathrm{RET}_{m,n}\,| = 2^f(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell}). \qquad \square$$

4 Both m and n are even

In this section, we classify reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) when both m and n are even, and consequently prove Theorem 1.1. For the classification, we give the following lemma.

Lemma 4.1. *Let m and n be even and let $\alpha \in S_0$ and $\beta \in S_0'$ with $d_1 = |\langle\alpha\rangle|$ and $d_2 = |\langle\beta\rangle|$. Now $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ if and only if α and β are defined by*

$$\alpha(2k) = 2kt_1 \quad \textit{and}$$
$$\alpha(2k+1) = 2kt_1 + 2s_1 + 1$$

for any $2k \in [m]$ and

$$\beta(2k) = 2kt_2 \quad \textit{and}$$
$$\beta(2k+1) = 2kt_2 + 2s_2 + 1$$

for any $2k \in [n]$ for some quadruple $(s_1, t_1; s_2, t_2) \in [\frac{m}{2}] \times [\frac{m}{2}] \times [\frac{n}{2}] \times [\frac{n}{2}]$ satisfying the following conditions;

(i) $d_1 \mid \gcd(m,n)$ *and* $d_2 \mid \gcd(m,n)$;

(ii) $2t_1^2 \equiv 2 \pmod{m}$ *and* $2t_2^2 \equiv 2 \pmod{n}$;

(iii) $2(s_1+1) \equiv 0 \pmod{d_2}$, $2(t_1+1) \equiv 0 \pmod{d_2}$, $2(s_2+1) \equiv 0 \pmod{d_1}$, *and* $2(t_2+1) \equiv 0 \pmod{d_1}$;

(iv) $2(s_1+1)(t_1-1) \equiv 0 \pmod{m}$ *and* $2(s_2+1)(t_2-1) \equiv 0 \pmod{n}$.

Proof. ($\Leftarrow$): Assume that $2t_1 = 2$, namely, $t_1 = 1$. Then $\alpha(2k) = 2k$ and $\alpha(2k+1) = 2k+2s_1+1$ for any $2k \in [m]$. Since for any $i \in [n]$, $\alpha^i(2k+1) = 2k+2is_1+1$, d_1 is the smallest positive integer such that $2d_1s_1 \equiv 0 \pmod m$. Now assume that $2t_1 \neq 2$. Then d_1 should be even because $\alpha^2(2) = 2t_1^2 = 2$. Since for any $2i \in [n]$ and for any $2k \in [m]$, $\alpha^{2i}(2k+1) = 2k + 2is_1(t_1+1) + 1$, d_1 is the smallest positive even integer such that $d_1s_1(t_1+1) \equiv 0 \pmod m$. Similarly one can show that d_2 is the smallest positive integer such that $2d_2s_2 \equiv 0 \pmod n$ if $t_2 = 1$; and the smallest positive even integer such that $d_2s_2(t_2+1) \equiv 0 \pmod n$ if $t_2 \neq 1$.

For any $i \in [n]$, let $a(i) = \beta(i)$ and $b(i) = \alpha^i(1)$. For the first case, let i be even. Then $a(i) = \beta(i) = it_2 \equiv -i \pmod{d_1}$ and $b(i) = \alpha^i(1) = is_1(t_1+1)+1 \equiv 1 \pmod{d_2}$. For any $2k \in [n]$,

$$\beta(2k+i) = 2kt_2 + it_2 \quad \text{and}$$
$$\beta^{b(i)}(2k) + a(i) = \beta(2k) + \beta(i) = 2kt_2 + it_2$$

and

$$\beta(2k+1+i) = 2kt_2 + it_2 + 2s_2 + 1 \quad \text{and}$$
$$\beta^{b(i)}(2k+1) + a(i) = \beta(2k+1) + \beta(i) = 2kt_2 + 2s_2 + 1 + it_2.$$

Hence $\beta(k+i) = \beta^{b(i)}(k) + a(i)$ for any $k \in [n]$. For any $2k \in [m]$,

$$\begin{aligned}\alpha^i(2k) &= 2k \quad \text{and}\\ \alpha^{a(i)}(2k+b(i)) - 1 &= \alpha^{-i}(2k + is_1(t_1+1)+1) - 1\\ &= (2k + is_1(t_1+1) - is_1(t_1+1) + 1) - 1 = 2k\end{aligned}$$

and

$$\begin{aligned}\alpha^i(2k+1) &= 2k + is_1(t_1+1) + 1, \quad \text{and}\\ \alpha^{a(i)}(2k+1+b(i)) - 1 &= \alpha^{-i}(2k + is_1(t_1+1) + 2) - 1\\ &= (2k + is_1(t_1+1) + 2) - 1 = 2k + is_1(t_1+1) + 1.\end{aligned}$$

Hence $\alpha^i(k) = \alpha^{a(i)}(k+b(i)) - 1$ for any $k \in [m]$.

For the remaining case, let i be odd. Now $a(i) = \beta(i) = (i-1)t_2 + 2s_2 + 1 \equiv -i \pmod{d_1}$ and $b(i) = \alpha^i(1) = (i-1)s_1(t_1+1) + 2s_1 + 1 \equiv -1 \pmod{d_2}$. For any $2k \in [n]$,

$$\beta(2k+i) = 2kt_2 + (i-1)t_2 + 2s_2 + 1 \quad \text{and}$$
$$\beta^{b(i)}(2k) + a(i) = \beta^{-1}(2k) + \beta(i) = 2kt_2 + (i-1)t_2 + 2s_2 + 1$$

and

$$\begin{aligned}\beta(2k+1+i) &= (2k+i+1)t_2 \quad \text{and}\\ \beta^{b(i)}(2k+1) + a(i) &= \beta^{-1}(2k+1) + \beta(i)\\ &= (2kt_2 - 2s_2t_2 + 1) + (i-1)t_2 + 2s_2 + 1\\ &= (2k+i+1)t_2 - 2(s_2+1)(t_2-1) = (2k+i+1)t_2.\end{aligned}$$

Hence $\beta(k+i) = \beta^{b(i)}(k) + a(i)$ for any $k \in [n]$. For any $2k \in [m]$,

$$\begin{aligned}\alpha^i(2k) &= 2kt_1 \quad \text{and} \\ \alpha^{a(i)}(2k + b(i)) - 1 &= \alpha^{-i}(2k + (i-1)s_1(t_1+1) + 2s_1 + 1) - 1 \\ &= (2k + (i-1)s_1(t_1+1) + 2s_1)t_1 - (i+1)s_1(t_1+1) + 2s_1 \\ &= 2kt_1 - 2s_1(t_1+1) + 2s_1t_1 + 2s_1 = 2kt_1\end{aligned}$$

and

$$\begin{aligned}\alpha^i(2k+1) &= 2kt_1 + (i-1)s_1(t_1+1) + 2s_1 + 1 \quad \text{and} \\ \alpha^{a(i)}(2k+1+b(i)) - 1 &= \alpha^{-i}(2k + (i-1)s_1(t_1+1) + 2s_1 + 2) - 1 \\ &= (2k + (i-1)s_1(t_1+1) + 2s_1 + 2)t_1 - 1 \\ &= 2kt_1 + (i-1)s_1(t_1+1) + 2s_1 + 1 + 2(s_1+1)(t_1-1) \\ &= 2kt_1 + (i-1)s_1(t_1+1) + 2s_1 + 1.\end{aligned}$$

Hence $\alpha^i(k) = \alpha^{a(i)}(k + b(i)) - 1$ for any $k \in [m]$. By Lemma 2.7, $(\alpha, \beta) \in \mathrm{ET}_{m,n}$. Furthermore one can easily check that $\alpha^{-1}(-k) = -\alpha(k)$ for any $k \in [m]$ and $\beta^{-1}(-k) = -\beta(k)$ for any $k \in [n]$. Hence $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ by Lemma 2.8.

($\Rightarrow$): Since m and n are even, both α and β are parity preserving. For any $2k \in [m]$,

$$\begin{aligned}\alpha(2k) &= \alpha(2(k-1)) + \alpha(2) \\ &= \alpha(2(k-2)) + 2\alpha(2) = \cdots = k\alpha(2) \quad \text{and} \\ \alpha(2k+1) &= \alpha(2(k-1)+1) + \alpha(2) \\ &= \alpha(2(k-2)+1) + 2\alpha(2) = \cdots = \alpha(1) + k\alpha(2).\end{aligned}$$

Let $\alpha(1) = 2s_1 + 1$ and $\alpha(2) = 2t_1$ for some $s_1, t_1 \in [\frac{m}{2}]$. Then $\alpha(2k) = 2kt_1$ and $\alpha(2k+1) = 2kt_1 + 2s_1 + 1$ for any $2k \in [m]$. Note that for any $2k \in [m]$, $\alpha(1) + \alpha(2k) = \alpha(2k+1) = \alpha^{-1}(2k) + \alpha(1)$. Hence $\alpha^{-1}(2k) = \alpha(2k)$, namely, $\alpha^2(2k) = 2k$. It implies that $\alpha^2(2) = \alpha(2t_1) = 2t_1^2 \equiv 2 \pmod{m}$. Assume that $2t_1 = 2$, namely, $t_1 = 1$. Then by Lemma 2.6, the order $|\langle\alpha\rangle|$ is the smallest positive integer d_1 such that

$$\alpha^{d_1}(1) = \alpha^{d_1-1}(2s_1+1) = \alpha^{d_1-2}(2s_1+2s_1+1) = \cdots = 2d_1s_1 + 1 \equiv 1.$$

Now assume that $2t_1 \neq 2$. Then the order $|\langle\alpha\rangle|$ is even and it is the smallest positive even integer d_1 such that

$$\begin{aligned}\alpha^{d_1}(1) &= \alpha^{d_1-1}(2s_1+1) = \alpha^{d_1-2}(2s_1t_1+2s_1+1) = \alpha^{d_1-3}(2s_1t_1+4s_1+1) \\ &= \alpha^{d_1-4}(4s_1t_1+4s_1+1) = \cdots = d_1s_1(t_1+1) + 1 \equiv 1.\end{aligned}$$

Hence d_1 is a divisor of m and consequently a divisor of $\gcd(m,n)$.

By a similar reason, there exist $s_2, t_2 \in [\frac{n}{2}]$ such that $\beta(2k) = 2kt_2$ and $\beta(2k+1) = 2kt_2 + 2s_2 + 1$ for any $2k \in [n]$. Furthermore $2t_2^2 \equiv 2 \pmod{n}$ and d_2 is a divisor of $\gcd(m,n)$. By Lemma 2.10, $\alpha(1) = 2s_1 + 1 \equiv -1 \pmod{d_2}$, namely, $2(s_1+1) \equiv 0 \pmod{d_2}$ and $\alpha(2) = 2t_1 \equiv -2 \pmod{d_2}$, namely, $2(t_1+1) \equiv 0 \pmod{d_2}$. Similarly it holds that $2(s_2+1) \equiv 2(t_2+1) \equiv 0 \pmod{d_1}$. Note that

$$2t_1 = \alpha(2) = \alpha^{-1}(1) + \alpha(1) = (-2s_1t_1 + 1) + 2s_1 + 1.$$

Hence $2(s_1+1)(t_1-1) \equiv 0 \pmod{m}$. By a similar reason, it holds that $2(s_2+1)(t_2-1) \equiv 0 \pmod{n}$. □

For even m and n, let $\mathcal{Q}(m,n)$ be the set of quadruples $(s_1,t_1;s_2,t_2) \in [\frac{n}{2}] \times [\frac{n}{2}] \times [\frac{m}{2}] \times [\frac{m}{2}]$ satisfying the conditions in Lemma 4.1. By Lemma 4.1, the classification of reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) is equivalent to the classification of $\mathcal{Q}(m,n)$, and the number $|\,\mathrm{RET}_{m,n}\,|$ equals to the cardinality $|\mathcal{Q}(m,n)|$.

In this section, let

$$\begin{aligned} m &= 2^a p_1^{a_1} \cdots p_\ell^{a_\ell} p_{\ell+1}^{a_{\ell+1}} \cdots p_{\ell+f}^{a_{\ell+f}} \quad \text{and} \\ n &= 2^b p_1^{b_1} \cdots p_\ell^{b_\ell} q_{\ell+1}^{a_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}} \quad \text{(prime decompositions)} \end{aligned}$$

and let $\gcd(m,n) = 2^c p_1^{c_1} \cdots p_\ell^{c_\ell}$ with $c_i \geq 1$ for any $i = 1,\ldots,\ell$. Without any loss of generality, assume that $a \leq b$, namely, $a = c$. By Chinese Remainder Theorem, it suffices to consider quadruples $(s_1,t_1;s_2,t_2)$ modulo prime powers dividing m and n, respectively. So we have the following lemma.

Lemma 4.2. *For a quadruple* $(s_1,t_1;s_2,t_2) \in [\frac{n}{2}] \times [\frac{n}{2}] \times [\frac{m}{2}] \times [\frac{m}{2}]$, $(s_1,t_1;s_2,t_2)$ *belongs to* $\mathcal{Q}(m,n)$ *if and only if:*

(1) for $i = 1,\ldots,\ell$, $(s_1 \pmod{p_i^{a_i}}, t_1 \pmod{p_i^{a_i}}; s_2 \pmod{p_i^{b_i}}, t_2 \pmod{p_i^{b_i}})$ *is one of* $(-1,-1;-1,-1)$, $(-1,-1; y\cdot p_i^{b_i-c_i},1)$, $(x\cdot p_i^{a_i-c_i},1;-1,-1)$ *and* $(0,1;0,1)$, *where* $x,y = 0,1,\ldots,p_i^{c_i}-1$*;*

(2) for any $j = 1,2,\ldots,f$, $(s_1 \pmod{p_{\ell+j}^{a_{\ell+j}}}, t_1 \pmod{p_{\ell+j}^{a_{\ell+j}}})$ *is* $(0,1)$ *or* $(-1,-1)$*;*

(3) for any $k = 1,2,\ldots,g$, $(s_2 \pmod{q_{\ell+k}^{b_{\ell+k}}}, t_2 \pmod{q_{\ell+k}^{b_{\ell+k}}})$ *is* $(0,1)$ *or* $(-1,-1)$*;*

(4) $(s_1 \pmod{2^a}, t_1 \pmod{2^a}; s_2 \pmod{2^b}, t_2 \pmod{2^b})$ *belongs to* $\mathcal{Q}(2^a,2^b)$.

Proof. Assume that $(s_1,t_1;s_2,t_2)$ belongs to $\mathcal{Q}(m,n)$. Then $t_1^2 \equiv 1 \pmod{\frac{m}{2}}$ and $t_2^2 \equiv 1 \pmod{\frac{n}{2}}$.

(1): First let us consider the quadruple modulo $p_i^{a_i}$ and $p_i^{b_i}$ for $i = 1,\ldots,\ell$. Note that $t_1 \equiv \pm 1 \pmod{p_i^{a_i}}$ and $t_2 \equiv \pm 1 \pmod{p_i^{b_i}}$.

If $t_1 \equiv -1 \pmod{p_i^{a_i}}$, then s_1 should be -1 modulo $p_i^{a_i}$ to satisfy

$$2(s_1+1)(t_1-1) \equiv 0 \pmod{p_i^{a_i}}.$$

By similar reason, if $t_2 \equiv -1 \pmod{p_i^{b_i}}$, then $s_2 \equiv -1 \pmod{p_i^{b_i}}$.

Let $(s_1,t_1) \equiv (-1,-1) \pmod{p_i^{a_i}}$. Since d_1 is the smallest positive even integer satisfying $d_1 s_1(t_1+1) \equiv 0 \pmod m$, p_i does not divide d_1. If $t_2 \equiv -1 \pmod{p_i^{b_i}}$ then s_2 should be -1 modulo $p_i^{b_i}$. If $t_2 \equiv 1 \pmod{p_i^{b_i}}$, then $s_2 \equiv y\cdot p_i^{b_i-c_i} \pmod{p_i^{b_i}}$ for some $y = 0,1,\ldots,p_i^{c_i}-1$ because $d_2 \mid \gcd(m,n)$. By a similar reason, one can say that if $(s_2,t_2) \equiv (-1,-1) \pmod{p_i^{b_i}}$, then $(s_1,t_1) \equiv (-1,-1)$ or $(x\cdot p_i^{a_i-c_i},1) \pmod{p_i^{a_i}}$ for some $x = 0,1,\ldots,p_i^{c_i}-1$.

Let $(s_1,t_1) \equiv (0,1) \pmod{p_i^{a_i}}$. By the condition (iii) in Lemma 4.1, p_i does not divide d_2. Note that if $t_2 = 1$ then d_2 is the smallest positive integer satisfying $2d_2 s_2 \equiv 0 \pmod n$, and if $t_2 \neq 1$ then d_2 is the smallest positive even integer such that $d_2 s_2(t_2+1) \equiv 0 \pmod n$. Hence $s_2 = 0$ or $t_2 = -1$ modulo $p_i^{b_i}$, which implies that $(s_2,t_2) \equiv (0,1)$ or $(-1,-1) \pmod{p_i^{b_i}}$.

Let $t_1 \equiv 1 \pmod{p_i^{a_i}}$ and $s_1 \not\equiv 0 \pmod{p_i^{a_i}}$. One can see that p_i divides d_1. By the condition (iii) in Lemma 4.1, $t_2 \equiv -1 \pmod{p_i^{b_i}}$ and $s_2 \equiv -1 \pmod{p_i^{b_i}}$.

Therefore

$$(s_1 \pmod{p_i^{a_i}}, t_1 \pmod{p_i^{a_i}}; s_2 \pmod{p_i^{b_i}}, t_2 \pmod{p_i^{b_i}}) = \\ (-1,-1;-1,-1), (-1,-1; y\cdot p_i^{b_i-c_i}, 1), (x\cdot p_i^{a_i-c_i}, 1; -1,-1) \text{ or } (0,1;0,1),$$

where $x, y = 0, 1, \ldots, p_i^{c_i} - 1$.

(2): For any $j = 1, 2, \ldots, f$, $t_1 \equiv \pm 1 \pmod{p_{\ell+j}^{a_{\ell+j}}}$. If $t_1 \equiv 1 \pmod{p_{\ell+j}^{a_{\ell+j}}}$ then $s_1 \equiv 0 \pmod{p_{\ell+j}^{a_{\ell+j}}}$ because $p_{\ell+j}$ does not divide d_1. If $t_1 \equiv -1 \pmod{p_{\ell+j}^{a_{\ell+j}}}$ then $s_1 \equiv -1 \pmod{p_{\ell+j}^{a_{\ell+j}}}$ to satisfy $2(s_1+1)(t_1-1) \equiv 0 \pmod{p_{\ell+j}^{a_{\ell+j}}}$.

(3): By the similar reason with (2), for any $k = 1, 2, \ldots, g$, $(s_2 \pmod{q_{\ell+k}^{b_{\ell+k}}}, t_2 \pmod{q_{\ell+k}^{b_{\ell+k}}})$ is $(0,1)$ or $(-1,-1)$.

(4): If a quadruple $(s_1, t_1; s_2, t_2) \in [\frac{n}{2}] \times [\frac{n}{2}] \times [\frac{m}{2}] \times [\frac{m}{2}]$ satisfies all conditions in Lemma 4.1, then it also satisfies these conditions modulo 2^a and 2^b. Hence

$$(s_1 \pmod{2^a}, t_1 \pmod{2^a}; s_2 \pmod{2^b}, t_2 \pmod{2^b}) \in \mathcal{Q}(2^a, 2^b).$$

By Chinese Remainder Theorem, one can show that if (1), (2), (3) and (4) hold, then $(s_1, t_1; s_2, t_2) \in \mathcal{Q}(m, n)$. □

Corollary 4.3. *The number of reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) up to isomorphism is $2^{f+g+\ell}(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})|\mathcal{Q}(2^a, 2^b)|$.*

Proof. By Lemma 4.2, the number of reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) up to isomorphism is

$$(2+2p_1^{c_1})\cdots(2+2p_\ell^{c_\ell})2^f 2^g |\mathcal{Q}(2^a, 2^b)| = \\ 2^{f+g+\ell}(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})|\mathcal{Q}(2^a, 2^b)|. \qquad \square$$

By Lemma 4.2, it suffices to classify $\mathcal{Q}(2^a, 2^b)$ to classify reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P). Let $\mathcal{P}(2) = \{(0,1)\}$ and for a 2-power 2^a ($a > 1$), let $\mathcal{P}(2^a)$ be the set of all pairs $(s,t) \in [2^{a-1}] \times [2^{a-1}]$ satisfying the conditions:

(i) $2t^2 \equiv 2 \pmod{2^a}$ and

(ii) $2(s+1)(t-1) \equiv 0 \pmod{2^a}$.

For any $(s,t) \in \mathcal{P}(2^a) \setminus \{(0,1)\}$, let $d(s,t)$ be the smallest positive even number d such that $ds(t+1) \equiv 0 \pmod{2^a}$ and let $e(s,t)$ be the largest number 2^j with $2^j \le 2^a$ satisfying $2(s+1) \equiv 0 \pmod{2^j}$ and $2(t+1) \equiv 0 \pmod{2^j}$. Let $d(0,1) = 1$ and $e(0,1) = 2$. Now we have the following lemma.

Lemma 4.4. *For 2-powers 2^a ($a \ge 1$) and 2^b ($b \ge 1$), a quadruple $(s_1, t_1; s_2, t_2)$ belongs to $\mathcal{Q}(2^a, 2^b)$ if and only if $(s_1, t_1; s_2, t_2)$ satisfies the conditions*

(a) $(s_1, t_1) \in \mathcal{P}(2^a)$ and $(s_2, t_2) \in \mathcal{P}(2^b)$,

(b) $d(s_1, t_1) \le e(s_2, t_2)$ and $d(s_2, t_2) \le e(s_1, t_1)$.

Proof. The conditions (i) and (ii) in the definition of $\mathcal{P}(2^a)$ correspond to the conditions (ii) and (iv) in Lemma 4.1.

Suppose that $d(s_1,t_1) \leq e(s_2,t_2)$ and $d(s_2,t_2) \leq e(s_1,t_1)$. Since $d(s_1,t_1) \leq 2^a$ and $e(s_2,t_2) \leq 2^b$, $d(s_1,t_1)$ divides $\gcd(2^a,2^b)$, the minimum of 2^a and 2^b. Similarly $d(s_2,t_2)$ also divides $\gcd(2^a,2^b)$. Furthermore it holds that

$$\begin{aligned} 2(s_1+1) &\equiv 0 \pmod{d(s_2,t_2)}, \\ 2(t_1+1) &\equiv 0 \pmod{d(s_2,t_2)}, \\ 2(s_2+1) &\equiv 0 \pmod{d(s_1,t_1)} \quad \text{and} \\ 2(t_2+1) &\equiv 0 \pmod{d(s_1,t_1)}. \end{aligned}$$

Therefore the conditions (i) and (iii) in Lemma 4.1 hold, and hence $(s_1,t_1;s_2,t_2)$ belongs to $\mathcal{Q}(2^a,2^b)$.

Let $(s_1,t_1;s_2,t_2)$ belong to $\mathcal{Q}(2^a,2^b)$. Now the condition (iii) in Lemma 4.1 is equivalent to the condition $d(s_1,t_1) \leq e(s_2,t_2)$ and $d(s_2,t_2) \leq e(s_1,t_1)$. □

By Lemma 4.4, the calculation of $d(s,t)$ and $e(s,t)$ for each $(s,t) \in \mathcal{P}(2^a)$ is helpful to calculate $|\mathcal{Q}(2^a,2^b)|$. The following lemma gives full list of $(s,t) \in \mathcal{P}(2^a)$ and corresponding $d(s,t)$ and $e(s,t)$.

Lemma 4.5. *For a* 2*-power* 2^a $(a>1)$*, the set* $\{(s,t,d(s,t),e(s,t)) : (s,t) \in \mathcal{P}(2^a)\}$ *is the following:*

$$\begin{cases} \{(0,1,1,2),(1,1,2,4)\}, & \text{if } a=2 \\ \{(0,1,1,2),(1,1,4,4),(2,1,2,2),(3,1,4,4),(1,3,2,4),(3,3,2,8)\}, & \text{if } a=3 \\ \{(0,1,1,2),(2^{a-2}-1,2^{a-2}-1,4,2^{a-1}),(2^{a-1}-1,2^{a-2}-1,4,2^{a-1}), & \\ (2^{a-2}-1,2^{a-1}-1,2,2^{a-1}),(2^{a-1}-1,2^{a-1}-1,2,2^a)\} & \\ \cup\{(x,1,2^{a-1},4),(x,2^{a-2}+1,2^{a-1},4) : x=1,3,\ldots,2^{a-1}-1\} & \\ \cup\{(2^iy,1,2^{a-i-1},2) : i=1,\ldots,a-2,\;\; y=1,3,\ldots,2^{a-i-1}-1\} & \text{if } a\geq 4. \end{cases}$$

Proof. Let $(s,t) \in \mathcal{P}(2^a)$.

For $a=2$, t should be 1 and both $s=0$ and $s=1$ satisfy the conditions for $(s,t) \in \mathcal{P}(2^a)$. Hence $(s,t,d(s,t),e(s,t)) = (0,1,1,2)$ or $(1,1,2,4)$. Let $a=3$. Then $t=1$ and $t=3$. If $t=1$, then $s=i$ for some $i=0,1,2,3$. If $t=3$, then $s=1$ or $s=3$. In any possible pair (s,t), one can easily calculate $d(s,t)$ and $e(s,t)$.

Now assume that $a \geq 4$. Then $t = 1, 2^{a-2}-1, 2^{a-2}+1$ or $2^{a-1}-1$. For $t=1$, any number $0,1,2,\ldots,2^{a-1}-1$ is possible for s to satisfy the condition (ii) in the definition of $\mathcal{P}(2^a)$. Note that if $(s,t)=(0,1)$, then $(d(0,1),e(0,1)) = (1,2)$. One can easily show that if $(s,t)=(x,1)$ for any $x=1,3,\ldots,2^{a-1}-1$ then $(d(s,t),e(s,t)) = (2^{a-1},4)$. If $(s,t)=(2^iy,1)$ for any $i=1,\ldots,a-2$ and for any $y=1,3,\ldots,2^{a-i-1}-1$, then $(d(s,t),e(s,t)) = (2^{a-i-1},2)$.

For $t=2^{a-2}-1$, both $s=2^{a-2}-1$ and $s=2^{a-1}-1$ satisfy the conditions for $(s,t) \in \mathcal{P}(2^a)$. If $(s,t) = (2^{a-2}-1,2^{a-2}-1)$ or $(2^{a-1}-1,2^{a-2}-1)$ then we have $(d(s,t),e(s,t)) = (4,2^{a-1})$.

Let $t=2^{a-2}+1$. Then any number $s=1,3,\ldots,2^{a-1}-1$ satisfies the condition (ii) in the definition of $\mathcal{P}(2^a)$. For any $(s,t)=(x,2^{a-2}+1)$ with $x=1,3,\ldots,2^{a-1}-1$, we have $(d(s,t),e(s,t)) = (2^{a-1},4)$.

For the final case, let $t = 2^{a-1}-1$. Then $s = 2^{a-2}-1$ or $2^{a-1}-1$. If $(s,t) = (2^{a-2}-1, 2^{a-1}-1)$ then we have $(d(s,t), e(s,t)) = (2, 2^{a-1})$; if $(s,t) = (2^{a-1}-1, 2^{a-1}-1)$ then $(d(s,t), e(s,t)) = (2, 2^a)$. □

Theorem 4.6. *For any 2-powers 2^a and 2^b with $a \le b$, the number $|\mathcal{Q}(2^a, 2^b)|$ of reflexible edge-transitive embeddings of $K_{m,n}$ satisfying the Property (P) up to isomorphism is the following:*

$$|\mathcal{Q}(2^a,2^b)| = \begin{cases} 1 & \text{if } (a,b) = (1,1), \\ 2 & \text{if } (a,b) = (1,2), \\ 4 & \text{if } (a,b) = (2,2) \text{ or } (1,k) \text{ with } k \ge 3, \\ 10 & \text{if } (a,b) = (2,3), \\ 12 & \text{if } (a,b) = (2,k) \text{ with } k \ge 4, \\ 28 & \text{if } (a,b) = (3,3), \\ 40 & \text{if } (a,b) = (3,4), \\ 36 & \text{if } (a,b) = (3,k) \text{ with } k \ge 5, \\ 20(1+2^{a-2}) & \text{if } a = b \ge 4, \\ 20 + 18 \cdot 2^{a-2} & \text{if } b-1 = a \ge 4, \\ 20 + 16 \cdot 2^{a-2} & \text{if } b-2 \ge a \ge 4. \end{cases}$$

Proof. By Lemma 4.4, it suffices to find all $(s_1, t_1; s_2, t_2)$ satisfying the conditions

(a) $(s_1, t_1) \in \mathcal{P}(2^a)$ and $(s_2, t_2) \in \mathcal{P}(2^b)$,

(b) $d(s_1, t_1) \le e(s_2, t_2)$ and $d(s_2, t_2) \le e(s_1, t_1)$.

By Lemma 4.5, one can get all the lists of $(s_1, t_1; s_2, t_2)$ satisfying the conditions as Table 1. □

Proof of Theorem 1.1. For odd m and n, the number $|\operatorname{RET}_{m,n}|$ of reflexible edge-transitive embeddings of $K_{m,n}$ up to isomorphism is 1 by Theorem 3.1. When exactly one of m and n is odd, then the number $|\operatorname{RET}_{m,n}|$ is counted in Theorem 3.2.

Assume that both m and n are even. Let

$$m = 2^a p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell} p_{\ell+1}^{a_{\ell+1}} \cdots p_{\ell+f}^{a_{\ell+f}} \quad \text{and}$$
$$n = 2^b p_1^{b_1} p_2^{b_2} \cdots p_\ell^{b_\ell} q_{\ell+1}^{a_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}} \quad \text{(prime decompositions)}$$

and let $\gcd(m,n) = 2^c p_1^{c_1} p_2^{c_2} \cdots p_\ell^{c_\ell}$ with $c_i \ge 1$ for any $i = 1, \dots, \ell$. Without any loss of generality, assume that $a \le b$, namely, $a = c$. By Corollary 4.3, the number $|\operatorname{RET}_{m,n}| = |\mathcal{Q}(m,n)|$ is

$$2^{f+g+\ell}(1+p_1^{c_1}) \cdots (1+p_\ell^{c_\ell}) |\mathcal{Q}(2^a, 2^b)|.$$

Theorem 4.6 completes the proof. □

Table 1: All lists of $\mathcal{Q}(2^a, 2^b)$.

(a,b)	$\mathcal{Q}(2^a, 2^b)$
$(1,1)$	$(0,1;0,1)$
$(1,2)$	$(0,1;0,1)$, $(0,1;1,1)$
$(1,\geq 3)$	$(0,1;0,1)$, $(0,1;2^{b-2},1)$, $(0,1;2^{b-2}-1,2^{b-1}-1)$, $(0,1;2^{b-1}-1,2^{b-1}-1)$
$(2,2)$	$(0,1;0,1)$, $(0,1;1,1)$, $(1,1;0,1)$, $(1,1;1,1)$
$(2,3)$	$(0,1;0,1)$, $(0,1;2,1)$, $(0,1;1,3)$, $(0,1;3,3)$, $(1,1;0,1)$, $(1,1;1,1)$, $(1,1;2,1)$, $(1,1;3,1)$, $(1,1;1,3)$, $(1,1;3,3)$
$(2,\geq 4)$	$(0,1;0,1)$, $(0,1;2^{b-2},1)$, $(0,1;2^{b-2}-1,2^{b-1}-1)$, $(0,1;2^{b-1}-1,2^{b-1}-1)$, $(1,1;0,1)$, $(1,1;2^{b-3},1)$, $(1,1;2^{b-2},1)$, $(1,1;3\cdot 2^{b-3},1)$, $(1,1;2^{b-2}-1,2^{b-2}-1)$, $(1,1;2^{b-1}-1,2^{b-2}-1)$, $(1,1;2^{b-2}-1,2^{b-1}-1)$, $(1,1;2^{b-1}-1,2^{b-1}-1)$
$(3,3)$	$(0 \text{ or } 2,1;0,1)$, $(0 \text{ or } 2,1;2,1)$, $(0 \text{ or } 2,1;1,3)$, $(0 \text{ or } 2,1;3,3)$, $(1 \text{ or } 3,1;1,1)$, $(1 \text{ or } 3,1;3,1)$, $(1 \text{ or } 3,1;1,3)$, $(1 \text{ or } 3,1;3,3)$, $(1 \text{ or } 3,3;0,1)$, $(1 \text{ or } 3,3;1,1)$, $(1 \text{ or } 3,3;2,1)$, $(1 \text{ or } 3,3;3,1)$, $(1 \text{ or } 3,3;1,3)$, $(1 \text{ or } 3,3;3,3)$
$(3,4)$	$(0 \text{ or } 2,1;0,1)$, $(0 \text{ or } 2,1;4,1)$, $(0 \text{ or } 2,1;3,7)$, $(0 \text{ or } 2,1;7,7)$, $(1 \text{ or } 3,1;3,3)$, $(1 \text{ or } 3,1;7,3)$, $(1 \text{ or } 3,1;3,7)$, $(1 \text{ or } 3,1;7,7)$; $(1,3;x,1)$, $x=0,2,4,6$; $(3,3;s_2,t_2)$, $(s_2,t_2)\in\mathcal{P}(2^4)$
$(3,\geq 5)$	$(0 \text{ or } 2,1;0,1)$, $(0 \text{ or } 2,1;2^{b-2},1)$; $(0 \text{ or } 2,1;x,2^{b-1}-1)$, $x=2^{b-2}-1 \text{ or } 2^{b-1}-1$; $(1 \text{ or } 3,1;x,y)$, $x,y=2^{b-2}-1 \text{ or } 2^{b-1}-1$; $(1,3;i\cdot 2^{b-3},1)$, $i=0,1,2,3$; $(1,3;x,y)$, $x,y=2^{b-2}-1 \text{ or } 2^{b-1}-1$; $(3,3;i\cdot 2^{b-4},1)$, $i=0,1,\ldots,7$; $(3,3;x,y)$, $x,y=2^{b-2}-1 \text{ or } 2^{b-1}-1$
$(\geq 4,\geq a)$	$(0 \text{ or } 2^{a-2},1;x,y)$, $(x,y)=(0,1),(2^{b-2},1),(2^{b-2}-1,2^{b-1}-1) \text{ or } (2^{b-1}-1,2^{b-1}-1)$; $(2x,1;2^{b-2}-1,2^{b-1}-1)$, $(2x,1;2^{b-1}-1,2^{b-1}-1)$, $x=1,2,\ldots,2^{a-2}-1$ $(x\neq 2^{a-3})$; $(x,1 \text{ or } 2^{a-2}+1;y,z)$, $x=1,3,\ldots,2^{a-1}-1$, $y,z=2^{b-2}-1 \text{ or } 2^{b-1}-1$; $(2^{a-2}-1 \text{ or } 2^{a-1}-1, 2^{a-2}-1 \text{ or } 2^{a-1}-1;x,y)$, $x,y=2^{b-2}-1 \text{ or } 2^{b-1}-1$; $(2^{a-2}-1,2^{a-1}-1;i\cdot 2^{b-a},1)$, $i=0,1,\ldots,2^{a-1}-1$; Only when $a=b$: $(2^{a-2}-1 \text{ or } 2^{a-1}-1,2^{a-2}-1;x,1 \text{ or } 2^{b-2}+1)$, $x=1,3,\ldots,2^{b-1}-1$; Only when $a=b$: $(2^{a-2}-1,2^{a-1}-1;x,2^{b-2}+1)$, $x=1,3,\ldots,2^{b-1}-1$; Only when $a=b$ or $b=a+1$: $(2^{a-1}-1,2^{a-1}-1;x,1)$, $x=0,1,\ldots,2^{b-1}-1$; Only when $a=b$ or $b=a+1$: $(2^{a-1}-1,2^{a-1}-1;x,2^{b-2}+1)$, $x=1,3,\ldots,2^{b-1}-1$; Only when $b\geq a+2$: $(2^{a-1}-1,2^{a-1}-1;i\cdot 2^{b-a-1},1)$, $i=0,1,\ldots,2^a-1$

5 Classification of some groups

In this section, we aim to consider a presentation of the group $\langle x_\alpha, y_\beta\rangle$ for any $(\alpha,\beta) \in \mathrm{RET}_{m,n}$. And we give some sufficient conditions and necessary conditions for $\langle x_{\alpha_1}, y_{\beta_1}\rangle$ and $\langle x_{\alpha_2}, y_{\beta_2}\rangle$ to be isomorphic for any $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in \mathrm{RET}_{m,n}$. For any positive integers m and n, a group Γ such that

(i) $\Gamma = XY$ for some cyclic groups $X = \langle x\rangle$ of order n and $Y = \langle y\rangle$ of order m with $X \cap Y = \{1_\Gamma\}$ and

(ii) there exists an automorphism of Γ which sends x and y to x^{-1} and y^{-1}, respectively,

is isomorphic to $\langle x_\alpha, y_\beta\rangle$ for some $(\alpha,\beta) \in \mathrm{RET}_{m,n}$. For our convenience, call a group Γ satisfying the conditions (i) and (ii) in the above sentence a *reflexible product of two cyclic groups* of order m and n. Now to classify reflexible products of two cyclic groups of order m and n, it suffices to consider $\langle x_\alpha, y_\beta\rangle$, where $(\alpha,\beta) \in \mathrm{RET}_{m,n}$. Note that for any integers i, j and for any $(\alpha,\beta) \in \mathrm{RET}_{m,n}$,

$$y_\beta^i x_\alpha^j = x_\alpha^{\beta^i(j)} y_\beta^{\alpha^j(i)}.$$

For example, $y_\beta x_\alpha = x_\alpha^{\beta(1)} y_\beta^{\alpha(1)}$ and $y_\beta x_\alpha^2 = x_\alpha^{\beta(2)} y_\beta^{\alpha^2(1)}$.

For odd integers m and n, since $\mathrm{RET}_{m,n} = \{(\mathrm{id},\mathrm{id})\}$, there is a unique reflexible product of two cyclic groups of order m and n up to isomorphism, namely, an abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

Let

$$m = p_1^{a_1} p_2^{a_2} \cdots p_\ell^{a_\ell} p_{\ell+1}^{a_{\ell+1}} \cdots p_{\ell+f}^{a_{\ell+f}} \quad \text{(prime factorization)}$$

be odd and

$$n = 2^b p_1^{b_1} p_2^{b_2} \cdots p_\ell^{b_\ell} q_{\ell+1}^{b_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}} \quad \text{(prime factorization)}$$

be even. Let $\gcd(m,n) = p_1^{c_1} p_2^{c_2} \cdots p_\ell^{c_\ell}$ with $c_i \geq 1$ for any $i = 1,\ldots,\ell$. Now $|\,\mathrm{RET}_{m,n}\,| = 2^f(1+p_1^{c_1})\cdots(1+p_\ell^{c_\ell})$ by Theorem 3.2. Note that for any $(\alpha,\beta) \in \mathrm{RET}_{m,n}$ and for any integer k, $\alpha(k) = rk$, $\beta(2k) = 2k$, $\beta(2k+1) = 2k+1+2s$ for some integers $r \in [m]$ and $s \in [n]$ satisfying $r^2 \equiv 1 \pmod m$, $2s \equiv 0 \pmod{2^b q_{\ell+1}^{b_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}}}$ and for any $j = 1,2,\ldots,\ell$, $s \equiv 0 \pmod{p_j^{b_j}}$ if $r \equiv 1 \pmod{p_j^{a_j}}$; $s \equiv z \cdot p_j^{b_j - c_j} \pmod{p_j^{b_j}}$ for some integer z with $0 \leq z \leq p_j^{c_j} - 1$ if $r \equiv -1 \pmod{p_j^{a_j}}$. Let us denote such α and β by α_r and β_s. Considering commuting rule

$$y_\beta^i x_\alpha^j = x_\alpha^{\beta^i(j)} y_\beta^{\alpha^j(i)},$$

one can check that the centralizer of $\langle x_{\alpha_r}, y_{\beta_s}\rangle$ is

$$\{x_{\alpha_r}^{2i} y_{\beta_s}^j : i \in \left[\frac{n}{2}\right], j(r-1) \equiv 0 \pmod m\} = \langle x_{\alpha_r}^2, y_{\beta_s}^k\rangle,$$

where k is the smallest positive integer j satisfying $j(r-1) \equiv 0 \pmod m$. This implies that for any $(\alpha_{r_1},\beta_{s_1}),(\alpha_{r_2},\beta_{s_2}) \in \mathrm{RET}_{m,n}$, if two groups $\langle x_{\alpha_{r_1}}, y_{\beta_{s_1}}\rangle$ and $\langle x_{\alpha_{r_2}}, y_{\beta_{s_2}}\rangle$ are isomorphic, then $r_1 = r_2$. Note that

$$y_{\beta_s} x_{\alpha_r} = x_{\alpha_r}^{\beta_s(1)} y_{\beta_s}^{\alpha_r(1)} = x_{\alpha_r}^{2s+1} y_{\beta_s}^r \quad \text{and}$$
$$y_{\beta_s} x_{\alpha_r}^2 = x_{\alpha_r}^{\beta_s(2)} y_{\beta_s}^{\alpha_r^2(1)} = x_{\alpha_r}^2 y_{\beta_s}.$$

In fact, the above two equations determine the whole commuting rules. For any $u \in [m]$ and $v \in [n]$, if v is even, then $y^u_{\beta_s} x^v_{\alpha_r} = x^v_{\alpha_r} y^u_{\beta_s}$, and if v is odd, then

$$\begin{aligned} y^u_{\beta_s} x^v_{\alpha_r} &= x^{v-1}_{\alpha_r} y^u_{\beta_s} x_{\alpha_r} = x^{v-1}_{\alpha_r} y^{u-1}_{\beta_s} x^{2s+1}_{\alpha_r} y^r_{\beta_s} \\ &= x^{v-1+2s}_{\alpha_r} y^{u-1}_{\beta_s} x_{\alpha_r} y^r_{\beta_s} = x^{v-1+2s}_{\alpha_r} y^{u-2}_{\beta_s} x^{2s+1}_{\alpha_r} y^{2r}_{\beta_s} \\ &= x^{v-1+4s}_{\alpha_r} y^{u-2}_{\beta_s} x_{\alpha_r} y^{2r}_{\beta_s} = \cdots = x^{v+2us}_{\alpha_r} y^{ur}_{\beta_s}. \end{aligned}$$

For any $v \in [n]$ with $\gcd(v, n) = 1$,

$$y_{\beta_s} x^v_{\alpha_r} = x^{\beta_s(v)}_{\alpha_r} y^{\alpha^v_r(1)}_{\beta_s} = x^{v+2s}_{\alpha_r} y^r_{\beta_s} = x^{v(2v^{-1}s+1)}_{\alpha_r} y^r_{\beta_s}$$

because v is odd, where v^{-1} is an integer satisfying $vv^{-1} \equiv 1 \pmod{n}$. For any $s_1, s_2 \in [\frac{n}{2}]$ with $\gcd(s_1, n) = \gcd(s_2, n)$, one can choose $v \in [n]$ satisfying that $\gcd(v, n) = 1$ and $v^{-1}s_1 \equiv s_2 \pmod{n}$. Therefore for any $(\alpha_{r_1}, \beta_{s_1}), (\alpha_{r_2}, \beta_{s_2}) \in \mathrm{RET}_{m,n}$, if $r_1 = r_2$ and $\gcd(s_1, n) = \gcd(s_2, n)$ then $\langle x_{\alpha_{r_1}}, y_{\beta_{s_1}} \rangle$ is isomorphic to $\langle x_{\alpha_{r_2}}, y_{\beta_{s_2}} \rangle$. This means that the number of non-isomorphic reflexible product of two cyclic groups of order m and n is at most $2^f(2 + c_1) \cdots (2 + c_\ell)$. So any reflexible product of two cyclic groups of order m and n is isomorphic to

$$\langle x, y \mid x^n = y^m = 1,\ yx = x^{2s+1}y^r,\ yx^2 = x^2y \rangle$$

for some $r \in [m]$ and $s \in [n]$ satisfying $r^2 \equiv 1 \pmod{m}$, $2s \equiv 0 \pmod{2^b q^{b_{\ell+1}}_{\ell+1} \cdots q^{b_{\ell+g}}_{\ell+g}}$ and for any $j = 1, 2, \ldots, \ell$, $s \equiv 0 \pmod{p^{b_j}_j}$ if $r \equiv 1 \pmod{p^{a_j}_j}$; $s \equiv p^{b_j - c_j + z}_j \pmod{p^{b_j}_j}$ for some integer $z = 0, 1, \ldots, c_j$ if $r \equiv -1 \pmod{p^{a_j}_j}$.

Conversely, assume that for some $(\alpha_{r_1}, \beta_{s_1}), (\alpha_{r_2}, \beta_{s_2}) \in \mathrm{RET}_{m,n}$, $\langle x_{\alpha_{r_1}}, y_{\beta_{s_1}} \rangle$ is isomorphic to $\langle x_{\alpha_{r_2}}, y_{\beta_{s_2}} \rangle$. Let $\psi\colon \langle x_{\alpha_{r_1}}, y_{\beta_{s_1}} \rangle \to \langle x_{\alpha_{r_2}}, y_{\beta_{s_2}} \rangle$ be an isomorphism such that $\psi(x^u_{\alpha_{r_1}}) = x_{\alpha_{r_2}}$ and $\psi(y^v_{\beta_{s_1}}) = y_{\beta_{s_2}}$.

For the remaining case, let

$$\begin{aligned} m &= 2^a p^{a_1}_1 p^{a_2}_2 \cdots p^{a_\ell}_\ell p^{a_{\ell+1}}_{\ell+1} \cdots p^{a_{\ell+f}}_{\ell+f} \quad \text{and} \\ n &= 2^b p^{b_1}_1 p^{b_2}_2 \cdots p^{b_\ell}_\ell q^{a_{\ell+1}}_{\ell+1} \cdots q^{b_{\ell+g}}_{\ell+g} \quad \text{(prime decompositions)} \end{aligned}$$

with $\gcd(m, n) = 2^c p^{c_1}_1 p^{c_2}_2 \cdots p^{c_\ell}_\ell$, where $1 \le a \le b$ and $c_i \ge 1$ for any $i = 1, \ldots, \ell$. For any $(\alpha, \beta) \in \mathrm{RET}_{m,n}$ and for any integer k,

$$\begin{aligned} \alpha(2k) &= 2kt_1, \\ \alpha(2k+1) &= 2kt_1 + 2s_1 + 1, \\ \beta(2k) &= 2kt_2 \quad \text{and} \\ \beta(2k+1) &= 2kt_2 + 2s_2 + 1 \end{aligned}$$

for some $(s_1, t_1; s_2, t_2) \in \mathcal{Q}(m, n)$. Let α and β be such permutations. Note that

$$\begin{aligned} y_\beta x_\alpha &= x^{\beta(1)}_\alpha y^{\alpha(1)}_\beta = x^{2s_2+1}_\alpha y^{2s_1+1}_\beta, \\ y_\beta x^2_\alpha &= x^{\beta(2)}_\alpha y^{\alpha^2(1)}_\beta = x^{2t_2}_\alpha y^{2s_1(t_1+1)+1}_\beta, \\ y^2_\beta x_\alpha &= x^{\beta^2(1)}_\alpha y^{\alpha(2)}_\beta = x^{2s_2(t_2+1)+1}_\alpha y^{2t_1}_\beta \quad \text{and} \\ y^2_\beta x^2_\alpha &= x^{\beta^2(2)}_\alpha y^{\alpha^2(2)}_\beta = x^2_\alpha y^2_\beta. \end{aligned}$$

In fact, the above four equations determine the whole commuting rules as follows. For any $i \in [m]$ and $j \in [n]$,

$$\begin{aligned}
y_\beta^{2i}x_\alpha^{2j} &= x_\alpha^{2j}y_\beta^{2i}\\
y_\beta^{2i}x_\alpha^{2j+1} &= x_\alpha^{2j}y_\beta^{2i}x_\alpha = x_\alpha^{2j}y_\beta^{2(i-1)}x_\alpha^{2s_2(t_2+1)+1}y_\beta^{2t_1}\\
&= x_\alpha^{2j+2s_2(t_2+1)}y_\beta^{2(i-1)}x_\alpha y_\beta^{2t_1} = \cdots = x_\alpha^{2j+2is_2(t_2+1)+1}y_\beta^{2it_1}\\
y_\beta^{2i+1}x_\alpha^{2j} &= y_\beta x_\alpha^{2j}y_\beta^{2i} = x_\alpha^{2t_2}y_\beta^{2s_1(t_1+1)+1}x_\alpha^{2(j-1)}y_\beta^{2i}\\
&= x_\alpha^{2t_2}y_\beta x_\alpha^{2(j-1)}y_\beta^{2i+2s_1(t_1+1)} = \cdots = x_\alpha^{2jt_2}y_\beta^{2i+2js_1(t_1+1)+1}\\
y_\beta^{2i+1}x_\alpha^{2j+1} &= y_\beta^{2i}y_\beta x_\alpha x_\alpha^{2j} = y_\beta^{2i}x_\alpha^{2s_2+1}y_\beta^{2s_1+1}x_\alpha^{2j} = x_\alpha^{2s_2}y_\beta^{2i}x_\alpha y_\beta x_\alpha^{2j}y_\beta^{2s_1}\\
&= x_\alpha^{2s_2}(x_\alpha^{2is_2(t_2+1)+1}y_\beta^{2it_1})(x_\alpha^{2jt_2}y_\beta^{2js_1(t_1+1)+1})y_\beta^{2s_1}\\
&= x_\alpha^{2jt_2+2is_2(t_2+1)+2s_2+1}y_\beta^{2it_1+2js_1(t_1+1)+2s_1+1}.
\end{aligned}$$

So any reflexible product of two cyclic groups of order m and n is isomorphic to

$$\langle x, y \mid x^n = y^m = 1,\ yx = x^{2s_2+1}y^{2s_1+1},\ yx^2 = x^{2t_2}y^{2s_1(t_1+1)+1},\\ y^2x = x^{2s_2(t_2+1)+1}y^{2t_1},\ y^2x^2 = x^2y^2\rangle$$

for some $(s_1, t_1; s_2, t_2) \in \mathcal{Q}(m, n)$. In summary, we have the following theorem.

Theorem 5.1. *For any positive integers m and n, let Γ be a group such that $\Gamma = XY$ for some cyclic groups $X = \langle x\rangle$ of order n and $Y = \langle y\rangle$ of order m with $X \cap Y = \{1_\Gamma\}$ and there exists an automorphism of Γ which sends x and y to x^{-1} and y^{-1}, respectively.*

(1) If both m and n are odd, Γ is isomorphic to the abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

(2) Let

$$m = p_1^{a_1}\cdots p_\ell^{a_\ell}p_{\ell+1}^{a_{\ell+1}}\cdots p_{\ell+f}^{a_{\ell+f}} \quad \textit{(prime factorization)}$$

be odd and let

$$n = 2^b p_1^{b_1}\cdots p_\ell^{b_\ell}q_{\ell+1}^{b_{\ell+1}}\cdots q_{\ell+g}^{b_{\ell+g}} \quad \textit{(prime factorization)}$$

be even with $\gcd(m, n) = p_1^{c_1}\cdots p_\ell^{c_\ell}$, where $c_i \geq 1$ for any $i = 1, \ldots, \ell$. Then Γ is isomorphic to

$$\langle x, y \mid x^n = y^m = 1,\ yx = x^{2s+1}y^r,\ yx^2 = x^2y\rangle$$

for some $r \in [m]$ and $s \in [\frac{n}{2}]$ satisfying

$$r^2 \equiv 1 \pmod{m}, \qquad 2s \equiv 0 \pmod{2^b q_{\ell+1}^{b_{\ell+1}}\cdots q_{\ell+g}^{b_{\ell+g}}},$$

and for any $j = 1, 2, \ldots, \ell$, $s \equiv 0 \pmod{p_j^{b_j}}$ if

$$r \equiv 1 \pmod{p_j^{a_j}}, \qquad s \equiv p_j^{b_j-c_j+z} \pmod{p_j^{b_j}}$$

for some $z = 0, 1, \ldots, c_j$ if $r \equiv -1 \pmod{p_j^{a_j}}$.

(3) Let

$$m = 2^a p_1^{a_1} \cdots p_\ell^{a_\ell} p_{\ell+1}^{a_{\ell+1}} \cdots p_{\ell+f}^{a_{\ell+f}} \quad \text{and}$$
$$n = 2^b p_1^{b_1} \cdots p_\ell^{b_\ell} q_{\ell+1}^{a_{\ell+1}} \cdots q_{\ell+g}^{b_{\ell+g}} \quad \text{(prime factorization)}$$

with $\gcd(m,n) = 2^c p_1^{c_1} p_2^{c_2} \cdots p_\ell^{c_\ell}$, *where* $1 \le a \le b$ *and* $c_i \ge 1$ *for any* $i = 1, \ldots, \ell$. *Now* Γ *is isomorphic to*

$$\langle x, y \mid x^n = y^m = 1,\ yx = x^{2s_2+1} y^{2s_1+1},\ yx^2 = x^{2t_2} y^{2s_1(t_1+1)+1},\ y^2x = x^{2s_2(t_2+1)+1} y^{2t_1},\ y^2x^2 = x^2y^2 \rangle$$

for some $(s_1, t_1; s_2, t_2) \in \mathcal{Q}(m,n)$.

For any positive integers m and n and for any $(\alpha, \beta), (\alpha', \beta') \in \mathrm{RET}_{m,n}$, we do not know a necessary and sufficient condition for $\langle x_\alpha, y_\beta \rangle \simeq \langle x_{\alpha'}, y_{\beta'} \rangle$. So we propose the following problem.

Problem 5.2. For any positive integers m and n and for any $(\alpha, \beta), (\alpha', \beta') \in \mathrm{RET}_{m,n}$, find a necessary and sufficient condition for $\langle x_\alpha, y_\beta \rangle \simeq \langle x_{\alpha'}, y_{\beta'} \rangle$. Consequently calculate the number of reflexible products of two cyclic groups of order m and n up to isomorphism.

References

[1] S.-F. Du, G. Jones, J. H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. I: Metacyclic case, *European J. Combin.* **28** (2007), 1595–1609, doi:10.1016/j.ejc.2006.08.012.

[2] S.-F. Du, G. Jones, J. H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. II: The non-metacyclic case, *European J. Combin.* **31** (2010), 1946–1956, doi:10.1016/j.ejc.2010.01.009.

[3] J. E. Graver and M. E. Watkins, Locally finite, planar, edge-transitive graphs, *Mem. Amer. Math. Soc.* **126** (1997), no. 601, doi:10.1090/memo/0601.

[4] G. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with a unique regular embedding, *J. Comb. Theory Ser. B* **98** (2008), 241–248, doi:10.1016/j.jctb.2006.07.004.

[5] G. A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, *Proc. Lond. Math. Soc.* **101** (2010), 427–453, doi:10.1112/plms/pdp061.

[6] G. A. Jones, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is an odd prime power, *European J. Combin.* **28** (2007), 1863–1875, doi:10.1016/j.ejc.2005.07.021.

[7] J. H. Kwak and Y. S. Kwon, Regular orientable embeddings of complete bipartite graphs, *J. Graph Theory* **50** (2005), 105–122, doi:10.1002/jgt.20097.

[8] J. H. Kwak and Y. S. Kwon, Classification of nonorientable regular embeddings of complete bipartite graphs, *J. Comb. Theory Ser. B* **101** (2011), 191–205, doi:10.1016/j.jctb.2011.03.003.

[9] Y. S. Kwon, Classification of reflexible edge-transitive embeddings of $K_{m,n}$ for odd m, n, *East Asian Math. J.* **25** (2009), 533–541, https://ynmath.jams.or.kr/jams/download/KCI_FI001404071.pdf.

[10] R. Nedela, M. Škoviera and A. Zlatoš, Regular embeddings of complete bipartite graphs, *Discrete Math.* **258** (2002), 379–381, doi:10.1016/s0012-365x(02)00539-3.