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Independent sets on the Towers of Hanoi graphs*

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Abstract

The number of independent sets is equivalent to the partition function of the hard-core lattice gas model with nearest-neighbor exclusion and unit activity. In this article, we mainly study the number of independent sets $i(H_n)$ on the Tower of Hanoi graph H_n at stage n, and derive the recursion relations for the numbers of independent sets. Upper and lower bounds for the asymptotic growth constant μ on the Towers of Hanoi graphs are derived in terms of the numbers at a certain stage, where $\mu = \lim_{v \to \infty} \frac{\ln i(G)}{v(G)}$ and $v(G)$ is the number of vertices in a graph G. Furthermore, we also consider the number of independent sets on the Sierpiński graphs which contain the Towers of Hanoi graphs as a special case.

Keywords: Independent sets, the Tower of Hanoi graph, Sierpiński graph, recursion relation, asymptotic growth constant, asymptotic enumeration.

Math. Subj. Class.: 05C30, 05C69

1 Introduction

Counting sets satisfying a fixed property in graphs ranges among the classical tasks of combinatorics. There is a vast amount of literatures on this kind of combinatorial problems for various classes of graphs, especially for Sierpiński graphs, by different authors. We note that the set counting problems such as the number of independent sets and the number of matchings have been studied in the past [2, 4, 9, 10, 11, 26, 35, 36].

On one hand, all these graph invariants reflect the structure of a graph in some way, and therefore, some of them are even of interest in theoretical chemistry for the study of molecular graphs (see [32, 38]). For example, the number of independent sets is called

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Merrifield-Simmons index, the number of matchings is known as Hosoya index in chemistry. It was shown that both correlate well with physicochemical properties of the corresponding molecules (see [23, 30]).

On the other hand, the number of independent sets is equivalent to the partition function of the hard-core lattice gas model with nearest-neighbor exclusion and unit activity. The lattice gas with repulsive pair interaction is an important model in statistical mechanics [3, 13, 16, 33]. For the special case with hard-core nearest-neighbor exclusion such that each site can be occupied by at most one particle and no pair of adjacent sites can be simultaneously occupied, the partition function of the lattice gas coincides with the independence polynomial in combinatorics [14, 34]. This model is a problem of interest in mathematics [39, 15, 24]. The growth of the number of independent sets in the $m \times n$ grid graph is of interest in statistical physics (see [1]). It is known that the number of independent sets in the $m \times n$ grid graph grows with α^{mn}, where $\alpha = 1.503048082 \cdots$ is the so-called hard square entropy constant. The bound for this constant was successively improved by Weber [40], Engel [9] and Calkin and Wilf [4].

The number of independent sets and its bounds had been considered on various graphs [27, 29, 41]. It is of interest to consider independent sets on self-similar fractal lattices which have scaling invariance rather than translational invariance [35]. The recursion relations for the numbers of independent sets on the Sierpiński gasket were derived by Chang, Chen and Yan [6]. A special type of self-similar graph that has been of interest is the Hanoi graph, which has been extensively studied in several contexts [5, 7, 8, 12, 17, 18, 19, 20, 22, 25, 28, 31]. This graph, which is also known as the Tower of Hanoi graph, came from the well known Tower of Hanoi puzzle, as the graph is associated to the allowed moves in this puzzle. We shall derive the recursion relations for the numbers of independent sets on the Towers of Hanoi graphs. Upper and lower bounds for the asymptotic growth constant μ on the Tower of Hanoi graphs are derived in terms of the numbers at a certain stage, where $\mu = \lim_{v\to\infty} \frac{\ln i(G)}{v(G)}$, $i(G)$ and $v(G)$ are the number of independent sets and the number of vertices in a graph G, respectively. Furthermore, we also consider the Sierpiński graphs which include the Towers of Hanoi graphs as a special case.

2 Preliminaries

We recall some basic definitions about graphs. A graph $G = (V, E)$ with vertex set V and edge set E is always supposed to be undirected, without loops or multiple edges. Vertices x and y are adjacent if xy is an edge in E. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. An independent set is a subset of the vertices such that any two of them are not adjacent. When the number $i(G)$ of independent sets in G grows exponentially with $v(G)$ as $v(G) \to \infty$, let us define a constant μ describing this exponential growth:

$$\mu = \lim_{v(G)\to\infty} \frac{\ln i(G)}{v(G)}.$$

We will see that the limit exists for the Towers of Hanoi graphs and some other Sierpiński graphs considered in this paper.

There are many different approaches to construct self-similar graphs. A construction that is specifically geared to be used in the context of enumeration was described in [35], it is no restated and we will also make use of it here. Some examples can be seen in [37].

The Tower of Hanoi graph (or the Hanoi graph), invented in 1883 by the French math-

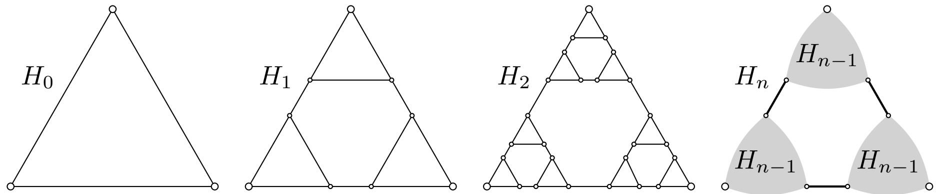

Figure 1: The Towers of Hanoi graphs H_0, H_1, H_2 and the construction of H_n.

ematician Edouard Lucas, has become a classic example in the analysis of algorithms and discrete mathematical structures. There exists an abundant literature on the properties of the Hanoi graph, which includes the study of shortest paths, average eccentricity, to name a few, see [21] and references therein. The Hanoi graph H_n is derived from the Tower of Hanoi puzzle with n discs. The vertices of the graph H_n in this sequence correspond to all possible configurations of the game Tower with $n + 1$ disks and three rods, whereas the edges describe transitions between configurations, see [17], and these graphs are finite Schreier graphs of the Hanoi tower group in [12]. Note that the Tower of Hanoi graph can be constructed by the following recursive-modular method. For $n = 0$, H_0 is the complete graph K_3 (also called a 3-clique or triangle). For $n \geq 1$, H_n is obtained from three copies of H_{n-1} joined by three new edges, each one connecting a pair of vertices from two different replicas of H_{n-1}, as show in Figure 1. From the construction rule, we can find that the number of vertices of H_n is 3^{n+1} while the number of edges is $\frac{3^{n+2}-3}{2}$.

3 The number of independent sets on H_n

In this section, we will derive the asymptotic growth constant for the number of independent sets on the Tower of Hanoi graph H_n in detail.

For the Tower of Hanoi graph H_n, i_n is its number of independent sets, f_n is its number of independent sets such that all three outmost vertices are not in the vertex subset, g_n is its number of independent sets such that only one specified vertex of three outmost vertices is in the vertex subset, h_n is its number of independent sets such that exact two specified vertices of the three outmost vertices are in the vertex subset, p_n is its number of independent sets such that all three outmost vertices are in the vertex subset. They are illustrated in Figures 2–5, where only the outmost vertices are shown and a solid circle is in the independent set and an open circle is not. Because of rotational symmetry, there are three possible g_n and three possible h_n such that

$$i_n = f_n + 3g_n + 3h_n + p_n$$

and $f_0 = g_0 = 1$, $h_0 = p_0 = 0$, $i_0 = f_0 + 3g_0 + 3h_0 + p_0 = 4$.

Lemma 3.1. *For any nonnegative integer n, we have*

$$\begin{aligned}
f_{n+1} =& f_n^3 + 6f_n^2 g_n + 3f_n^2 h_n + 9f_n g_n^2 + 6f_n g_n h_n + 2g_n^3,\\
g_{n+1} =& f_n^2 g_n + 2f_n^2 h_n + f_n^2 p_n + 4f_n g_n^2 + 8f_n g_n h_n + 2f_n g_n p_n + 2f_n h_n^2 + 3g_n^3\\
&+ 4g_n^2 h_n,\\
h_{n+1} =& f_n g_n^2 + 4f_n g_n h_n + 2f_n g_n p_n + 3f_n h_n^2 + 2f_n h_n p_n + 2g_n^3 + 7g_n^2 h_n + 2g_n^2 p_n\\
&+ 4g_n h_n^2,\\
p_{n+1} =& g_n^3 + 6g_n^2 h_n + 3g_n^2 p_n + 9g_n h_n^2 + 6g_n h_n p_n + 2h_n^3.
\end{aligned}$$

Proof. Note that the number f_{n+1} consists of (i) one configuration where all three H_n belong to the class that enumerated by f_n; (ii) six configurations where one of the H_n belongs to the class that enumerated by g_n and the other two belong to the class that enumerated by f_n; (iii) three configurations where one of the H_n belongs to the class that enumerated by h_n and the other two belong to the class that enumerated by f_n; (iv) nine configurations where one of the H_n belongs to the class that enumerated by f_n and the other two belong to the class that enumerated by g_n; (v) six configurations where all three H_n belong to the class that enumerated by f_n, g_n and h_n, respectively; (vi) two configurations where all three H_n belong to the class that enumerated by g_n as illustrated in Figure 2. And

$$f_{n+1} = f_n^3 + 6f_n^2 g_n + 3f_n^2 h_n + 9f_n g_n^2 + 6f_n g_n h_n + 2g_n^3$$

is verified by adding these configurations.

Similarly, the expressions of g_{n+1}, h_{n+1} and p_{n+1} can be obtained with appropriate configurations of its three H_n as illustrated in Figures 3–5. □

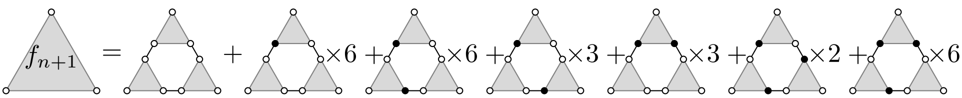

Figure 2: Illustration for the expression of f_{n+1}. The multiplication of three on the right-hand-side corresponds to the three possible orientations of H_{n+1}, the multiplication of two on the right-hand-side corresponds to reflection symmetry with respect to the central vertical axis and the multiplication of six on the right-hand-side corresponds to the six possible of considering both orientations and reflection symmetry.

In the following, we will estimate the value $\mu = \lim_{v\to\infty} \frac{\ln i(H_n)}{v(H_n)}$ of the asymptotic growth constant for the Tower of Hanoi graph H_n. The values of f_n, g_n, h_n, p_n for small n are listed in Table 1 by Lemma 3.1, and grow exponentially. For the Tower of Hanoi graph H_n, define the ratios

$$\alpha_n = \frac{g_n}{f_n}, \quad \beta_n = \frac{h_n}{g_n}, \quad \gamma_n = \frac{p_n}{h_n}$$

where n is a positive integer. Their values for small n are listed in Table 2. From the initial values of f_n, g_n, h_n, p_n, it is easy to see that $f_n > g_n > h_n > p_n$ for all positive integer n by induction. Alternatively, these inequalities can be obtained by an injection. For instance, if one of the independent sets enumerated by g_n is given, one can remove

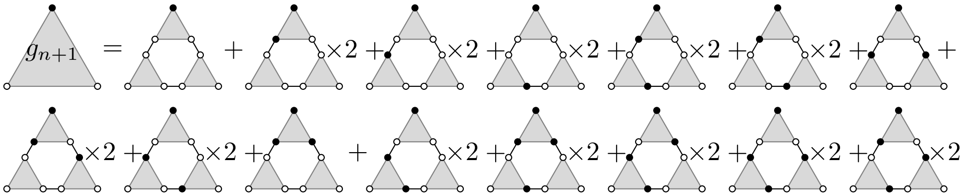

Figure 3: Illustration for the expression of g_{n+1}. The multiplication of two on the right-hand-side are corresponds to the reflection symmetry with respect to the central vertical axis.

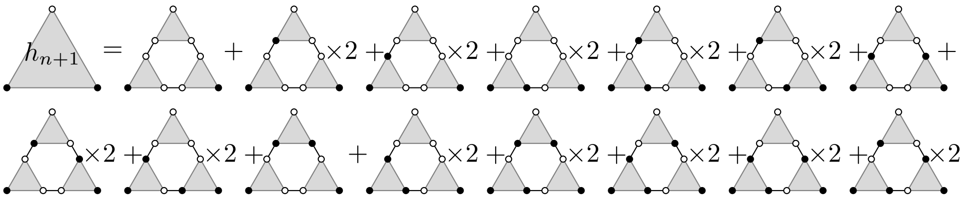

Figure 4: Illustration for the expression of h_{n+1}. The multiplication of two on the right-hand-side are corresponds to the reflection symmetry with respect to the central vertical axis.

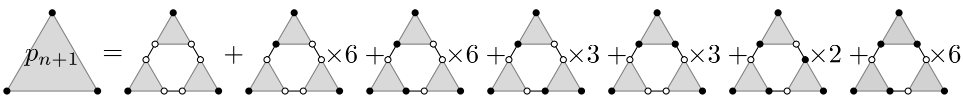

Figure 5: Illustration for the expression of p_{n+1}. The multiplication of three on the right-hand-side corresponds to the three possible orientations of H_{n+1}, the multiplication of two on the right-hand-side corresponds to reflection symmetry with respect to the central vertical axis and the multiplication of six on the right-hand-side corresponds to the six possible of considering both orientations and reflection symmetry.

Table 1: The first few values of f_n, g_n, h_n, p_n and i_n on H_n.

n	0	1	2	3
f_n	1	18	38284	342408411795232
g_n	1	8	15840	141595222762112
h_n	0	3	6546	58553484583728
p_n	0	1	2702	24213460330512
i_n	4	52	108144	967067994163264

the corner vertex to obtain another independent set that are enumerated by f_n such that $f_n > g_n$ is established. Similarly, other two inequalities can be established. It follows that $\alpha_n, \beta_n, \gamma_n \in (0,1)$.

Table 2: The first few values of $\alpha_n, \beta_n, \gamma_n$ on H_n.

n	1	2	3
α_n	0.444444444444444	0.413749869397137	0.413527290465016
β_n	0.375	0.413257575757575	0.413527260606109
γ_n	0.333333333333333	0.412771157959058	0.413527230747269

Lemma 3.2. *For any positive integer n, the ratios satisfy*

$$\alpha_n > \beta_n > \gamma_n.$$

When n increases, the ratio α_n decreases monotonically while γ_n increases monotonically. The three ratios in the large n limit are equal to each other

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n.$$

Proof. By the definition of $\alpha_n, \beta_n, \gamma_n$, we have

$$\alpha_{n+1} = \alpha_n \frac{B_n}{A_n},\ \beta_{n+1} = \alpha_n \frac{C_n}{B_n},\ \gamma_{n+1} = \alpha_n \frac{D_n}{C_n}$$

for a positive integer n, where

$A_n = 1 + 6\alpha_n + 3\alpha_n\beta_n + 9\alpha_n^2 + 6\alpha_n^2\beta_n + 2\alpha_n^3$,

$B_n = 1 + 2\beta_n + \beta_n\gamma_n + 4\alpha_n + 8\alpha_n\beta_n + 2\alpha_n\beta_n\gamma_n + 2\alpha_n\beta_n^2 + 3\alpha_n^2 + 4\alpha_n^2\beta_n$,

$C_n = 1 + 4\beta_n + 2\beta_n\gamma_n + 3\beta_n^2 + 2\beta_n^2\gamma_n + 2\alpha_n + 7\alpha_n\beta_n + 2\alpha_n\beta_n\gamma_n + 4\alpha_n\beta_n^2$,

$D_n = 1 + 6\beta_n + 3\beta_n\gamma_n + 9\beta_n^2 + 6\beta_n^2\gamma_n + 2\beta_n^3$.

In the following, we show that $\frac{1}{3} \le \gamma_n < \beta_n < \alpha_n \le \frac{4}{9}$ by induction on n. It is true for $n = 1, 2, 3, 4$ from Table 2. Suppose that $\frac{1}{3} \le \gamma_n < \beta_n < \alpha_n \le \frac{4}{9}$ for $n \ge 4$.

Let $\varepsilon_n = \alpha_n - \gamma_n$. Then $\varepsilon_n > \alpha_n - \beta_n$, $\varepsilon_n > \beta_n - \gamma_n$ and $\varepsilon_n \in (0, \frac{1}{9})$. Now,

$$\begin{aligned}\alpha_n - \alpha_{n+1} =& \alpha_n - \alpha_n\frac{B_n}{A_n} = \frac{\alpha_n(A_n - B_n)}{A_n}\\ =& \frac{\alpha_n}{A_n}[(2 + 6\alpha_n + 4\alpha_n\beta_n + 2\alpha_n^2 + \beta_n)(\alpha_n - \beta_n)\\ & + (2\alpha_n\beta_n + \beta_n)(\beta_n - \gamma_n)] > 0,\\ \alpha_{n+1} - \beta_{n+1} =& \frac{\alpha_n(B_n^2 - A_nC_n)}{A_nB_n} > 0,\end{aligned}$$

where

$$\begin{aligned}B_n^2 - A_nC_n =& (10\alpha_n^2\beta_n + 5\alpha_n^2 + \alpha_n\beta_n + 4\alpha_n + \beta_n^2 + 1)(\alpha_n - \beta_n)^2 + (4\alpha_n^2\beta_n^2 + 2\alpha_n\beta_n^2\\ & + 6\alpha_n\beta_n + 2\beta_n)(\alpha_n - \beta_n)(\alpha_n - \gamma_n) + (4\alpha_n^3\beta_n + 10\alpha_n^2\beta_n + 2\alpha_n\beta_n^2\\ & + 2\alpha_n\beta_n + \beta_n^2)(\beta_n - \gamma_n)(\alpha_n - \beta_n) + (2\alpha_n\beta_n^2 + \beta_n^2)(\alpha_n - \gamma_n)(\beta_n - \gamma_n)\\ & + (4\alpha_n^2\beta_n^2 + 2\alpha_n\beta_n^2)(\beta_n - \gamma_n)^2 > 0,\end{aligned}$$

$$\begin{aligned}A_nB_n =&10\alpha_n + 2\beta_n + 23\alpha_n\beta_n + \beta_n\gamma_n + 8\alpha_n\beta_n^2 + 88\alpha_n^2\beta_n + 133\alpha_n^3\beta_n + 70\alpha_n^4\beta_n\\ &+ 8\alpha_n^5\beta_n + 36\alpha_n^2 + 56\alpha_n^3 + 35\alpha_n^4 + 6\alpha_n^5 + 48\alpha_n^2\beta_n^2 + 6\alpha_n^2\beta_n^3 + 78\alpha_n^3\beta_n^2\\ &+ 12\alpha_n^3\beta_n^3 + 12\alpha_n^3\beta_n^2\gamma_n + 28\alpha_n^4\beta_n^2 + 12\alpha_n^2\beta_n^2\gamma_n + 8\alpha_n\beta_n\gamma_n + 3\alpha_n\beta_n^2\gamma_n\\ &+ 21\alpha_n^2\beta_n\gamma_n + 20\alpha_n^3\beta_n\gamma_n + 4\alpha_n^4\beta_n\gamma_n + 1\\ >&4\alpha_n^4\beta_n + 8\alpha_n^3\beta_n^2 + 20\alpha_n^3\beta_n + 5\alpha_n^3 + 8\alpha_n^2\beta_n^2 + 9\alpha_n^2\beta_n + 4\alpha_n^2 + 3\alpha_n\beta_n^2\\ &+ 2\alpha_n\beta_n + \alpha_n.\end{aligned}$$

Then

$$\begin{aligned}\alpha_{n+1} - \beta_{n+1} =&\frac{\alpha_n(B_n^2 - A_nC_n)}{A_nB_n}\\ <&\frac{\varepsilon_n^2}{A_nB_n}[4\alpha_n^4\beta_n + 8\alpha_n^3\beta_n^2 + 20\alpha_n^3\beta_n + 5\alpha_n^3 + 8\alpha_n^2\beta_n^2 + 9\alpha_n^2\beta_n + 4\alpha_n^2\\ &+ 3\alpha_n\beta_n^2 + 2\alpha_n\beta_n + \alpha_n]\\ <&\varepsilon_n^2,\end{aligned}$$

since $\varepsilon_n > \alpha_n - \beta_n$ and $\varepsilon_n > \beta_n - \gamma_n$.

Similarly, we have $\beta_{n+1} - \gamma_{n+1} = \frac{\alpha_n(C_n^2 - B_nD_n)}{B_nC_n} > 0$, where

$$\begin{aligned}C_n^2 - B_nD_n =&[(10\beta_n^3 + 4\beta_n^2\gamma_n^2 + 4\beta_n^2 + 4\beta_n + 1)(\alpha_n - \beta_n) + (2\beta_n^3 + 9\beta_n^2\\ &+ 2\beta_n)(\alpha_n - \gamma_n) + (2\alpha_n\beta_n^2 + \alpha_n\beta_n)(\beta_n - \gamma_n)](\alpha_n - \beta_n)\\ &+ [(4\alpha_n\beta_n^3 + 10\beta_n^3)(\alpha_n - \beta_n) + (4\alpha_n\beta_n^3 + 2\beta_n^3)(\alpha_n - \gamma_n)\\ &+ (2\alpha_n\beta_n^2 + \beta_n^2)(\beta_n - \gamma_n)](\beta_n - \gamma_n) > 0,\end{aligned}$$

$$\begin{aligned}B_nC_n =&16\alpha_n^3\beta_n^3 + 8\alpha_n^3\beta_n^2\gamma_n + 40\alpha_n^3\beta_n^2 + 6\alpha_n^3\beta_n\gamma_n + 29\alpha_n^3\beta_n + 6\alpha_n^3 + 8\alpha_n^2\beta_n^4\\ &+ 20\alpha_n^2\beta_n^3\gamma_n + 58\alpha_n^2\beta_n^3 + 4\alpha_n^2\beta_n^2\gamma_n^2 + 44\alpha_n^2\beta_n^2\gamma_n + 101\alpha_n^2\beta_n^2 + 18\alpha_n^2\beta_n\gamma_n\\ &+ 60\alpha_n^2\beta_n + 11\alpha_n^2 + 4\alpha_n\beta_n^4\gamma_n + 6\alpha_n\beta_n^4 + 4\alpha_n\beta_n^3\gamma_n^2 + 30\alpha_n\beta_n^3\gamma_n + 40\alpha_n\beta_n^3\\ &+ 6\alpha_n\beta_n^2\gamma_n^2 + 43\alpha_n\beta_n^2\gamma_n + 64\alpha_n\beta_n^2 + 14\alpha_n\beta_n\gamma_n + 35\alpha_n\beta_n + 6\alpha_n + 2\beta_n^3\gamma_n^2\\ &+ 7\beta_n^3\gamma_n + 6\beta_n^3 + 2\beta_n^2\gamma_n^2 + 10\beta_n^2\gamma_n + 11\beta_n^2 + 3\beta_n\gamma_n + 6\beta_n + 1.\end{aligned}$$

Thus, we have

$$\begin{aligned}\beta_{n+1} - \gamma_{n+1} =&\frac{\alpha_n(C_n^2 - B_nD_n)}{B_nC_n}\\ <&\frac{\varepsilon_n^2}{B_nC_n}[8\alpha_n^2\beta_n^3 + 4\alpha_n^2\beta_n^2 + \alpha_n^2\beta_n + 24\alpha_n\beta_n^3 + 4\alpha_n\beta_n^2\gamma_n^2 + 14\alpha_n\beta_n^2\\ &+ 6\alpha_n\beta_n + \alpha_n] < \varepsilon_n^2.\end{aligned}$$

And

$$\begin{aligned}\gamma_{n+1}-\gamma_n =&\frac{1}{C_n}(\alpha_n D_n-\gamma_n C_n)\\ =&\frac{1}{C_n}[(1+4\beta_n+2\beta_n^2+2\beta_n^2\gamma_n)(\alpha_n-\gamma_n)+(2\alpha_n+7\alpha_n\beta_n+2\beta_n\gamma_n\\ &+2\alpha_n\beta_n\gamma_n+2\alpha_n\beta_n^2)(\beta_n-\gamma_n)+3\beta_n\gamma_n(\alpha_n-\beta_n)]>0.\end{aligned}$$

So, we have (i) $\alpha_n-\alpha_{n+1}>0$, (ii) $0<\alpha_{n+1}-\beta_{n+1}<\varepsilon_n^2$, (iii) $0<\beta_{n+1}-\gamma_{n+1}<\varepsilon_n^2$ and (iv) $\gamma_{n+1}-\gamma_n>0$.

From (ii) and (iii), we can obtain that $\varepsilon_{n+1}=\alpha_{n+1}-\gamma_{n+1}<2\varepsilon_n^2<\frac{2}{81}$ for all positive integer n by induction. It follows that for any positive integer $m\le n$,

$$\varepsilon_n<2\varepsilon_{n-1}^2<2[2\varepsilon_{n-2}^2]^2<\cdots<\frac{1}{2}[2\varepsilon_m]^{2^{n-m}}.$$

Since $\varepsilon_m\in(0,\frac{1}{9})$ for any positive integer m, we have that the values of $\alpha_n,\beta_n,\gamma_n$ are close to each other when n becomes large. □

Numerically, we can find

$$\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\gamma_n=0.41352727694875959999\cdots$$

From the lemmas above, we get the bounds for the number of independent sets.

Theorem 3.3. *For any positive integer $m\le n$,*

$$f_m^{3^{n-m}}(1+2\gamma_m)^{\frac{3(3^{n-m}-1)}{2}}(1+\gamma_n)^3<i_n<f_m^{3^{n-m}}(1+2\alpha_m)^{\frac{3(3^{n-m}-1)}{2}}(1+\alpha_n)^3$$

Proof. By Lemmas 3.1 and 3.2 and the definition of $\alpha_n,\beta_n,\gamma_n$, we have

$$\begin{aligned}f_n =&f_{n-1}^3(1+6\alpha_{n-1}+3\alpha_{n-1}\beta_{n-1}+9\alpha_{n-1}^2+6\alpha_{n-1}^2\beta_{n-1}+2\alpha_{n-1}^3)\\ <&f_{n-1}^3(1+6\alpha_{n-1}+12\alpha_{n-1}^2+8\alpha_{n-1}^3)\\ =&[f_{n-1}(1+2\alpha_{n-1})]^3<[f_{n-2}^3(1+2\alpha_{n-2})^3]^3(1+2\alpha_{n-1})^3\\ <&f_{n-2}^{3^2}(1+2\alpha_{n-2})^{3^2+3^1}\\ <&\cdots<f_m^{3^{n-m}}(1+2\alpha_m)^{\frac{3(3^{n-m}-1)}{2}}.\end{aligned}$$

And

$$\begin{aligned}i_n =&f_n+3g_n+3h_n+p_n=f_n(1+3\alpha_n+3\alpha_n\beta_n+\alpha_n\beta_n\gamma_n)\\ <&f_n(1+3\alpha_n+3\alpha_n^2+\alpha_n^3)=f_n(1+\alpha_n)^3<f_m^{3^{n-m}}(1+2\alpha_m)^{\frac{3(3^{n-m}-1)}{2}}(1+\alpha_n)^3.\end{aligned}$$

Similarly, the lower bound for i_n can be derived. □

Theorem 3.4. *The asymptotic growth constant for the number of independent sets in H_n is bounded by*

$$\frac{\ln f_m}{3^{m+1}}+\frac{\ln(1+2\gamma_m)}{2\times 3^m}\le\mu\le\frac{\ln f_m}{3^{m+1}}+\frac{\ln(1+2\alpha_m)}{2\times 3^m}$$

where m is a positive integer.

Proof. Note that the number of vertices of H_n is $v(H_n) = 3^{n+1}$, by Theorem 3.3, we have

$$\frac{\ln i_n}{v(H_n)} < \frac{\ln f_m}{3^{m+1}} + \frac{\ln(1+2\alpha_m)}{2\times 3^m} - \frac{\ln(1+2\alpha_m)}{2\times 3^n} + \frac{\ln(1+\alpha_n)}{3^n}$$

and

$$\frac{\ln i_n}{v(H_n)} > \frac{\ln f_m}{3^{m+1}} + \frac{\ln(1+2\gamma_m)}{2\times 3^m} - \frac{\ln(1+2\gamma_m)}{2\times 3^n} + \frac{\ln(1+\gamma_n)}{3^n}$$

So, the bounds for $\mu = \lim_{v(H_n)\to\infty} \frac{\ln i_n}{v(H_n)}$ follow. □

As m increases, the difference between the upper and lower bounder in Theorem 3.4 becomes small and the convergence is rapid. Numerically, the asymptotic growth constant for the number of independent sets of the Tower of Hanoi graph H_n in the large n limit is $\mu = 0.42433435855938823\cdots$. In fact, the numerical value of μ can be obtained with more than a hundred significant figures accurate when m is equal to seven.

4 The number of independent sets on graphs $S_{k,n}$

The Sierpiński graphs $S_{k,n}$ were introduced by Klavžar and Milutinović in 1997 in [25]. The graph $S_{k,0}$ is simply the complete graph on k vertices, $S_{k,n}$ is constructed from $S_{k,n-1}$ by copying k times $S_{k,n-1}$ and adding exactly one edge between each pair of copies. For the construction, one can easily derive that the total number of vertices and edges in $S_{k,n}$ are $v(S_{k,n}) = k^{n+1}$ and $e(S_{k,n}) = \frac{1}{2}(k^{n+2} - k)$, respectively. In particularly, we can see those graphs are exactly the graphs of the Tower of Hanoi problem for $k = 3$ and another case as shown in Figure 6 for $k = 4$.

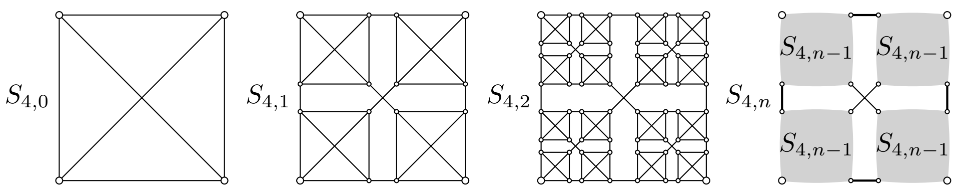

Figure 6: The graphs $S_{4,0}$, $S_{4,1}$, $S_{4,2}$ and the construction of $S_{4,n}$.

The method given in the previous section can be applied to enumeration the number of independent sets on this Sierpiński graphs with $k \geq 4$, but the items of the recursion relations will become larger and larger with the increase of k.

To seek the number of independent sets on $S_{4,n}$, we use the following definitions: (i) Define $f_{4,n}$ as the number of independent sets such that all four outmost vertices are not in the vertex sets. (ii) Define $g_{4,n}$ as the number of independent sets such that only one certain outmost vertex are in the vertex sets. (iii) Define $h_{4,n}$ as the number of independent sets such that exactly two certain outmost vertex are in the vertex sets. (iv) Define $p_{4,n}$ as the number of independent sets such that exactly three certain outmost vertex are in the vertex sets. (v) Define $q_{4,n}$ as the number of independent sets such that all four outmost vertex are in the vertex sets.

Table 3: The first few values of $f_{4,n}$, $g_{4,n}$, $h_{4,n}$, $p_{4,n}$, $q_{4,n}$ and $i_{4,n}$ on $S_{4,n}$.

n	1	2	3
$f_{4,n}$	163	13064274739	497661511371512614009322138806617451967507
$g_{4,n}$	52	3951119257	150487045809089786329485928937399858428184
$h_{4,n}$	15	1194624638	45505530112368879421817904248654649805971
$p_{4,n}$	4	361093492	13760342318790991781550553074012255470504
$q_{4,n}$	1	109115158	4160967243331065589513567798163834387921
$i_{4,n}$	478	37589988721	1431845211800580068573889060142357640786006

Table 4: The first few values of $\alpha_{4,n}$, $\beta_{4,n}$, $\gamma_{4,n}$ and $\delta_{4,n}$ on $S_{4,n}$.

n	1	2	3
$\alpha_{4,n}$	0.319018404907975	0.302436938592921	0.302388355077651
$\beta_{4,n}$	0.288461538461538	0.302350944199809	0.302388354211550
$\gamma_{4,n}$	0.266666666666666	0.302265230863252	0.302388353345449
$\delta_{4,n}$	0.25	0.302179796693760	0.302388352479348

The quantities $f_{4,n}$, $g_{4,n}$, $h_{4,n}$, $p_{4,n}$, $q_{4,n}$ of $S_{4,n}$ are lengthy and given in the appendix. Some values of $f_{4,n}$, $g_{4,n}$, $h_{4,n}$, $p_{4,n}$, $q_{4,n}$, $i_{4,n}$ are listed in Table 3. These numbers grow exponentially, and have no integer factorizations. There are four equivalent $g_{4,n}$, six equivalent $h_{4,n}$, and four equivalent p_n. By definition, we have

$$i_{4,n} = f_{4,n} + 4g_{4,n} + 6h_{4,n} + 4p_{4,n} + q_{4,n}.$$

The initial values at stage zero are $f_{4,0} = g_{4,0} = 1$, $h_{4,0} = p_{4,0} = q_{4,0} = 0$ and $i_{4,0} = 5$.

Define ratios $\alpha_{4,n} = g_{4,n}/f_{4,n}$, $\beta_{4,n} = h_{4,n}/g_{4,n}$,$\gamma_{4,n} = p_{4,n}/h_{4,n}$, $\delta_{4,n} = q_{4,n}/p_{4,n}$. As n increases, we find $\alpha_{4,n}$ decrease monotonically while $\beta_{4,n}$, $\gamma_{4,n}$ and $\delta_{4,n}$ increase monotonically with the relation $\alpha_{4,n} > \beta_{4,n} > \gamma_{4,n} > \delta_{4,n}$. The values of these ratios for small n are listed in Table 4. Numerically, we can find

$$\lim_{n\to\infty} \alpha_{4,n} = \lim_{n\to\infty} \beta_{4,n} = \lim_{n\to\infty} \gamma_{4,n} = \lim_{n\to\infty} \delta_{4,n} = 0.302388354588805297767\cdots$$

By a similar argument as the Tower of Hanoi graph H_n in the last section, the asymptotic growth constant for the number of independent sets on $S_{4,n}$ is bounded by

$$\frac{\ln f_{4,m}}{4^{m+1}} + \frac{\ln(1+2\delta_{4,m})}{2\times 4^m} \le \mu_4 \le \frac{\ln f_{4,m}}{4^{m+1}} + \frac{\ln(1+2\alpha_{4,m})}{2\times 4^m}$$

where $\mu_4 = \lim_{v(S_{4,n})\to\infty} \frac{\ln i_{4,n}}{v(S_{4,n})}$ and m is a positive integer.

Then, we can obtain the asymptotic growth constant for the number of independent sets on the Sierpińsk graph $S_{4,n}$ in the large n limit is $\mu = 0.378737140730676994823835\cdots$.

We can also continue verify a similarly bound for the asymptotic growth constant on $S_{5,n}$, in order to avoid verbosity, we are not to describe here. However, the recursion relations of the number of independent sets for general k are difficult to obtain. From what has been discussed above, we have the following conjecture for the Sierpiński graphs $S_{k,n}$ with positive integers k and m.

Conjecture 4.1. *The asymptotic growth constant for the number of independent sets on the Sierpińsk graph $S_{4,n}$ is bounded by*

$$\frac{\ln f_{k,m}}{k^{m+1}} + \frac{\ln(1+2\phi_{k,m})}{2\times k^m} \le \mu_k \le \frac{\ln f_{k,m}}{k^{m+1}} + \frac{\ln(1+2\alpha_{k,m})}{2\times k^m}$$

where the ratios are defined as $\alpha_{k,n} = g_{k,n}/f_{k,n}$, $\phi_{k,n} = w_{k,n}/y_{k,n}$, $f_{k,n}$ is the number of independent sets such that all k outmost vertices are not in the vertex subset, $g_{k,n}$ is the number of independent sets such that one certain outmost vertex is in the vertex subset, $y_{k,n}$ is number of independent sets such that all but one certain outmost vertex are in the vertex subset, and $w_{k,n}$ is the number of independent sets such that all k outmost vertices are in the vertex subset.

Appendix: Recursion relation for $S_{4,n}$

We give the recursive relation for the Siepiński graph $S_{4,n}$ here. Since the subscript is $k = 4$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_{4,n+1}$ and similar notations for other quantities. For any non-negative integer n, we have

$$f_{n+1} = f_n^4+12f_n^3g_n+12f_n^3h_n+48f_n^2g_n^2+4f_n^3p_n+84f_n^2g_nh_n+72f_ng_n^3+24f_n^2g_np_n+30f_n^2h_n^2+156f_ng_n^2h_n+30g_n^4+12f_n^2h_np_n+36f_ng_n^2p_n+84f_ng_nh_n^2+60g_n^3h_n+24f_ng_nh_np_n+8g_n^3p_n+8f_nh_n^3+24g_n^2h_n^2,$$

$$g_{n+1} = f_n^3g_n+3f_n^3h_n+9f_n^2g_n^2+3f_n^3p_n+33f_n^2g_nh_n+24f_ng_n^3+f_n^3q_n+24f_n^2g_np_n+21f_n^2h_n^2+96f_ng_n^2h_n+18g_n^4+6f_n^2g_nq_n+21f_n^2h_np_n+51f_ng_n^2p_n+93f_ng_nh_n^2+69g_n^3h_n+3f_n^2h_nq_n+9f_ng_n^2q_n+3f_n^2p_n^2+66f_ng_nh_np_n+24g_n^3p_n+21f_nh_n^3+66g_n^2h_n^2+6f_ng_nh_nq_n+2g_n^3q_n+6f_ng_np_n^2+12f_nh_n^2p_n+24g_n^2h_np_n+14g_nh_n^3,$$

$$h_{n+1} = f_n^2g_n^2+6f_n^2g_nh_n+6f_ng_n^3+6f_n^2g_np_n+8f_n^2h_n^2+38f_ng_n^2h_n+8g_n^4+2f_n^2g_nq_n+14f_n^2h_np_n+30f_ng_n^2p_n+64f_ng_nh_n^2+50g_n^3h_n+4f_n^2h_nq_n+8f_ng_n^2q_n+5f_n^2p_n^2+80f_ng_nh_np_n+30g_n^3p_n+26f_nh_n^3+87g_n^2h_n^2+2f_n^2p_nq_n+16f_ng_nh_nq_n+6g_n^3q_n+18f_ng_np_n^2+34f_nh_n^2p_n+72g_n^2h_np_n+44g_nh_n^3+4f_ng_np_nq_n+4f_nh_n^2q_n+8g_n^2h_nq_n+8f_nh_np_n^2+8g_n^2p_n^2+28g_nh_n^2p_n+4h_n^4,$$

$$p_{n+1} = f_ng_n^3+9f_ng_n^2h_n+3g_n^4+9f_ng_n^2p_n+24f_ng_nh_n^2+27g_n^3h_n+3f_ng_n^2q_n+42f_ng_nh_np_n+22g_n^3p_n+18f_nh_n^3+75g_n^2h_n^2+12f_ng_nh_nq_n+6g_n^3q_n+15f_ng_np_n^2+39f_nh_n^2p_n+99g_n^2h_np_n+69g_nh_n^3+6f_ng_np_nq_n+9f_nh_n^2q_n+21g_n^2h_nq_n+21f_nh_np_n^2+24g_n^2p_n^2+96g_nh_n^2p_n+15h_n^4+6f_nh_np_nq_n+6g_n^2p_nq_n+12g_nh_n^2q_n+2f_np_n^3+24g_nh_np_n^2+14h_n^3p_n,$$

$$q_{n+1} = g_n^4+12g_n^3h_n+12g_n^3p_n+48g_n^2h_n^2+4g_n^3q_n+84g_n^2h_np_n+72g_nh_n^3+24g_n^2h_nq_n+30g_n^2p_n^2+156g_nh_n^2p_n+30h_n^4+12g_n^2p_nq_n+36g_nh_n^2q_n+84g_nh_np_n^2+60h_n^3p_n+24g_nh_np_nq_n+8h_n^3q_n+8g_np_n^3+24h_n^2p_n^2.$$

There are always $729 = 3^6$ terms in these equations.

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