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Finite two-distance-transitive graphs of valency 6

Wei Jin *, Li Tan

School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, P.R.China
Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, P.R.China

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Abstract

A non-complete graph Γ is said to be $(G, 2)$-*distance-transitive* if, for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. This paper classifies the family of $(G, 2)$-distance-transitive graphs of valency 6 which are not $(G, 2)$-arc-transitive.

Keywords: 2-Distance-transitive graph, 2-arc-transitive graph, permutation group.

Math. Subj. Class.: 05E18, 05B25

1 Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. For the group theoretic terminology not defined here we refer the reader to [4, 8, 26]. Let $u, v \in V(\Gamma)$. Then the distance between u, v in Γ is denoted by $d_\Gamma(u, v)$. A non-complete graph Γ is said to be $(G, 2)$-*distance-transitive*, if for $i = 1, 2$ and for any two vertex pairs (u_1, v_1) and (u_2, v_2) with $d_\Gamma(u_1, v_1) = d_\Gamma(u_2, v_2) = i$, there exists $g \in G$ such that $(u_1, v_1)^g = (u_2, v_2)$. An *arc* is an ordered pair of adjacent vertices. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$. The graph Γ is said to be $(G, 2)$-*arc-transitive* if G is transitive on both the set of arcs and the set of 2-arcs.

The first remarkable result about $(G, 2)$-arc-transitive graphs comes from Tutte [20, 21], and since then, this family of graphs has been studied extensively, see [1, 12, 15, 16, 17, 23, 24]. By definition, every non-complete $(G, 2)$-arc-transitive graph is $(G, 2)$-distance-transitive. The converse is not necessarily true. If a $(G, 2)$-distance-transitive graph has

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E-mail addresses: jinwei@jxufe.edu.cn (Wei Jin), tltanli@126.com (Li Tan)

girth 3 (length of the shortest cycle is 3), then this graph is not $(G, 2)$-arc-transitive. Thus, the family of non-complete $(G, 2)$-arc-transitive graphs is properly contained in the family of $(G, 2)$-distance-transitive graphs. The graph in Figure 1 is the Kneser graph $KG_{6,2}$ which is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive of valency 6 for $G = \mathrm{Aut}(KG_{6,2})$. Therefore the following problem naturally arises: characterize the family of $(G, 2)$-distance-transitive graphs. At the moment, Corr, Schneider and the first author are investigating such graphs, and they classified the family of $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graphs of valency at most 5 in [6]. Hence 6 is the next smallest valency for $(G, 2)$-distance-transitive graphs to investigate. Our main theorem gives a classification of such graphs.

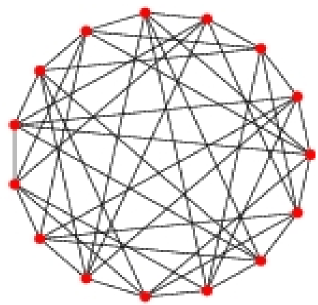

Figure 1: Kneser graph $KG_{6,2}$

Remark 1.1. Let Γ be a connected $(G, 2)$-distance-transitive graph. If Γ has girth at least 5, then for any two vertices u, v with $d_\Gamma(u, v) = 2$, there exists a unique 2-arc between u and v. Hence Γ is $(G, 2)$-distance-transitive implies that it is $(G, 2)$-arc-transitive. If Γ has girth 4, then Γ can be $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive. There are infinitely many such graphs. For instance, let Γ be the complement of the $(2 \times p^k)-$grid where p is a prime, and let $M = \mathbb{Z}_p^k : \mathbb{Z}_{p^k-1}$, $G = \mathbb{Z}_2 \times M$. Then Γ is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive of valency $p^k - 1$ and girth 4. There are also infinitely many $(G, 2)$-distance-transitive graphs of girth 4 that are $(G, 2)$-arc-transitive, for example the complete bipartite graphs $\mathrm{K}_{m,m}$. If Γ has girth 3, then since Γ is non-complete, it follows that G_u is not 2-transitive on $\Gamma(u)$, hence it is not $(G, 2)$-arc-transitive.

The *line graph* $L(\Gamma)$ of a graph Γ has the set of edges of Γ as its vertex set, and two edges are adjacent in $L(\Gamma)$ if and only if they have a common vertex in Γ. The line graph of a complete bipartite graph $\mathrm{K}_{m,n}$ is called an $(m \times n)-$grid. Let Γ be a connected graph. The *complement graph* $\overline{\Gamma}$ of Γ, is the graph with vertex $V(\Gamma)$, and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. The *Hamming graph* $\mathrm{H}(d, n)$ has vertex set $\mathbb{Z}_n^d = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$, and two vertices are adjacent if and only if they have exactly one different coordinate. We denote by $\mathrm{K}_{m[b]}$ the *complete multipartite graph* with m parts, and each part has b vertices where $m \geq 3, b \geq 2$. Let p be a prime such that $p \equiv 1 \pmod 4$. Then, the *Paley graph* $P(p)$ is the Cayley graph $\mathrm{Cay}(T, S)$ for the additive group $T = F_p^+$ with $S = \{w^2, w^4, \ldots, w^{p-1} = 1\}$ and $\Gamma_2(1) = \{w, w^3, \ldots, w^{p-2}\}$, where w is a primitive element of F_p, and $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$. In particular, Hamming graphs and Paley graphs are $(G, 2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, see [3, 13].

The *diameter* $\mathrm{diam}(\Gamma)$ of a graph Γ is the maximum distance occurring over all pairs of vertices. Let $u \in V(\Gamma)$ and $i = 1, 2, \ldots, \mathrm{diam}(\Gamma)$. We use $\Gamma_i(u)$ to denote the set of vertices at distance i with vertex u in Γ. Sometimes, $\Gamma_1(u)$ is also denoted by $\Gamma(u)$. Let Ω be a set of cardinality n. Then the *Kneser graph* $KG_{n,k}$ is the graph with vertex set all k-subsets of Ω, and two k-subsets are adjacent if and only if they are disjoint. The *triangular graph* $T(n)$ is the graph with vertex set all 2-subsets of Ω, and two 2-subsets are adjacent if and only if they share one common element. Thus $KG_{n,2} = \overline{T(n)}$. A subgraph X of Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we use $[U]$ to denote the subgraph of Γ induced by U.

Since complete graphs have diameter 1, they do not provide interesting examples. Our main theorem determines the family of non-complete $(G, 2)$-distance-transitive graphs of valency 6 which are not $(G, 2)$-arc-transitive.

Theorem 1.2. *Let Γ be a connected non-complete $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. Let $u \in V(\Gamma)$. Then one of the following holds.*

(1) Γ *has girth 4, and* $(\Gamma, G) = (\overline{(2 \times 7)-\mathsf{grid}}, S_2 \times M)$ *where M is a 2-transitive but not 3-transitive subgroup of S_7.*

(2) $[\Gamma(u)]$ *is connected, and Γ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $\mathrm{K}_{3[3]}$ or $\mathrm{K}_{4[2]}$.*

(3) $[\Gamma(u)]$ *is disconnected, and either*

(3.1) $[\Gamma(u)] \cong 2\mathrm{K}_3$, $\Gamma \cong \mathrm{H}(2, 4)$, *or* $|\Gamma_2(u)| = 18$ *and Γ is a line graph; or*

(3.2) $[\Gamma(u)] \cong 3\mathrm{K}_2$, $\Gamma \cong KG_{6,2}$, *or* $|\Gamma_2(u)| = 12$, *24.*

Remark 1.3. (1) There exist graphs Γ in Theorem 1.2 (3.1) such that $|\Gamma_2(u)| = 18$. For instance the generalized hexagon of order $(3, 1)$ and the generalized dodecagon of order $(3, 1)$. These two graphs are locally isomorphic to $2\mathrm{K}_3$ and $|\Gamma_2(u)| = 18$. By [3, p.223], they are $(G, 2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, since they are non-complete and have girth 3, they are not $(G, 2)$-arc-transitive.

(2) There exist graphs Γ in Theorem 1.2 (3.2) such that $|\Gamma_2(u)| = 12$ and also exist graphs such that $|\Gamma_2(u)| = 24$. For instance $\mathrm{H}(3, 3)$ has valency 6, $[\Gamma(u)] \cong 3\mathrm{K}_2$ and $|\Gamma_2(u)| = 12$; the halved foster graph has valency 6, $[\Gamma(u)] \cong 3\mathrm{K}_2$ and $|\Gamma_2(u)| = 24$. By [3, p.223], these two graphs are $(G, 2)$-distance-transitive for $G = \mathrm{Aut}(\Gamma)$, since they are non-complete and have girth 3, they are not $(G, 2)$-arc-transitive.

2 Proof of Theorem 1.2

In this section, we will prove our main theorem by a series of lemmas. All graphs are non-complete graphs.

A graph Γ is said to be *G-distance-transitive* if G is transitive on the ordered pairs of vertices at any given distance. The study of finite G-distance-transitive graphs goes back to Higman's paper [10] in which "groups of maximal diameter" were introduced. These are permutation groups G which act distance-transitively on some graph. Then G-distance-transitive graphs have been studied extensively and a classification is almost done, see [2, 9, 11, 18, 19, 22, 25]. By definition, every non-complete G-distance-transitive graph is $(G, 2)$-distance-transitive.

The following remark gives an useful observation.

Remark 2.1. Let Γ be a $(G, 2)$-distance-transitive graph. Let u, w be two vertices such that $d_\Gamma(u, w) = 2$.

Suppose that $|\Gamma_3(u) \cap \Gamma(w)| = 0$. Then since Γ is $(G, 2)$-distance-transitive, Γ has diameter 2 and so it is G-distance-transitive.

Suppose that $|\Gamma_3(u) \cap \Gamma(w)| = 1$. Let $(u_0, \ldots, u_i)$ be a path with $d_\Gamma(u_0, u_i) = i$ where $i = \mathrm{diam}(\Gamma)$. Then for each $j \leq \mathrm{diam}(\Gamma) - 2$, $|\Gamma_3(u_j) \cap \Gamma(u_{j+2})| = 1$. Note that, $\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2}) \subseteq \Gamma_3(u_j) \cap \Gamma(u_{j+2})$, and so $|\Gamma_{j+3}(u_0) \cap \Gamma(u_{j+2})| = 1$, hence Γ is also G-distance-transitive.

We use $G_u^{[1]}$ to denote the kernel of the G_u-action on $\Gamma(u)$.

Lemma 2.2. *Let Γ be a $(G, 2)$-distance-transitive graph. Let $u, w \in V(\Gamma)$ be such that $d_\Gamma(u, w) = 2$. Let $g \in G_u^{[1]}$ be with order a prime p. Suppose that $|\Gamma_3(u) \cap \Gamma(w)| < p$. Then g is not trivial on $\Gamma_2(u)$.*

Proof. Suppose that g is trivial on $\Gamma_2(u)$. Let $w_i \in \Gamma_2(u)$. Since $g \in G_u^{[1]}$ and g is trivial on $\Gamma_2(u)$, g fixes all the vertices in $(\Gamma(u) \cup \Gamma_2(u)) \cap \Gamma(w_i)$ and $g \in G_{w_i}$. In particular, g fixes $\Gamma_3(u) \cap \Gamma(w_i)$ setwise.

Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_3(u) \cap \Gamma(w)| < p$, $|\Gamma_3(u) \cap \Gamma(w_i)| < p$. Since the order of g is prime p and g fixes $\Gamma_3(u) \cap \Gamma(w_i)$ setwise, it follows that g fixes all the vertices in $\Gamma_3(u) \cap \Gamma(w_i)$. Thus $g \in G_{w_i}^{[1]}$. Since w_i is any vertex of $\Gamma_2(u)$, g fixes all the vertices of $\Gamma_3(u)$. For any $v \in \Gamma(u)$, $\Gamma_2(v) \subseteq \Gamma(u) \cup \Gamma_2(u) \cup \Gamma_3(u)$. Thus $g \in G_v^{[1]}$ and fixes all the vertices of $\Gamma_2(v)$.

Since Γ is $(G, 2)$-distance-transitive, for any $z \in \Gamma_2(v)$, $|\Gamma_3(v) \cap \Gamma(z)| < p$. Since g fixes all the vertices in $(\Gamma(v) \cup \Gamma_2(v)) \cap \Gamma(z)$, g fixes all the vertices in $\Gamma_3(v) \cap \Gamma(z)$. Thus $g \in G_z^{[1]}$. In particular, g fixes all the vertices of $\Gamma_4(u)$. Since Γ is connected, by induction, g fixes all the vertices of Γ, so $g = 1$, which is a contradiction. Thus g is not trivial on $\Gamma_2(u)$. □

Lemma 2.3. *Let Γ be a $(G, 2)$-distance-transitive graph of valency 6. Let $u, w \in V(\Gamma)$ be such that $d_\Gamma(u, w) = 2$. If Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 3$, then Γ is $(G, 2)$-arc-transitive.*

Proof. Suppose that Γ has girth 4 and $|\Gamma(u) \cap \Gamma(w)| = 3$. Let (u, v, w) be a 2-arc. Then $d_\Gamma(u, w) = 2$ and $|\Gamma_2(u) \cap \Gamma(v)| = 5$. Since Γ is $(G, 2)$-distance-transitive, there are 30 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma(u) \cap \Gamma(w)| = 3$ and $|\Gamma(u) \cap \Gamma(w)| \cdot |\Gamma_2(u)| = 30$, it follows that $|\Gamma_2(u)| = 10$. Again since Γ is $(G, 2)$-distance-transitive, G_u is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so both $|\Gamma(u)|$ and $|\Gamma_2(u)|$ divide $|G_u|$, hence 30 divides $|G_u|$. Thus 5 divides $|G_{u,v}|$, so $G_{u,v}$ has an element g of order 5. Therefore either $\langle g \rangle$ is regular on $\Gamma(u) \setminus \{v\}$ or is trivial on $\Gamma(u) \setminus \{v\}$. If $\langle g \rangle$ is regular on $\Gamma(u) \setminus \{v\}$, then $G_{u,v}$ is transitive on $\Gamma(u) \setminus \{v\}$, so G_u is 2-transitive on $\Gamma(u)$. Thus Γ is $(G, 2)$-arc-transitive.

Now suppose that g is trivial on $\Gamma(u) \setminus \{v\}$. Then $g \in G_u^{[1]}$. Since $|\Gamma(u) \cap \Gamma(w)| = 3$, it follows that $|\Gamma_3(u) \cap \Gamma(w)| \leq 3 < 5$. Thus by Lemma 2.2, g is not trivial on $\Gamma_2(u)$. Hence $\langle g \rangle$ has orbits of size 5 on $\Gamma_2(u)$. Since g fixes $\Gamma_2(u) \cap \Gamma(v_i)$ setwise and $|\Gamma_2(u) \cap \Gamma(v_i)| = 5$, it follows that $\langle g \rangle$ is transitive on $\Gamma_2(u) \cap \Gamma(v_i)$. Thus G_{u,v_i} is transitive on $\Gamma_2(u) \cap \Gamma(v_i)$, so Γ is $(G, 2)$-arc-transitive. □

Lemma 2.4. ([6]) *Let $\Gamma \cong \mathrm{K}_{m,m}$ with $m \geq 2$. Then Γ is $(G, 2)$-distance-transitive if and only if it is $(G, 2)$-arc-transitive.*

A permutation group G on a set Ω is said to be 2-*homogeneous*, if G is transitive on the set of 2-subsets of Ω.

Lemma 2.5. ([8, Theorem 9.4B]) *Let G be a 2-homogeneous permutation group which is not 2-transitive of degree n. Then $n = p^e \equiv 3 \pmod 4$ where p is a prime.*

Lemma 2.6. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. If Γ has girth 4, then $(\Gamma, G) = (\overline{(2 \times 7)-\text{grid}}, S_2 \times M)$ where M is a 2-transitive but not 3-transitive subgroup of S_7.*

Proof. Suppose that Γ has girth 4. Let (u, v, w) be a 2-arc. Then $d_\Gamma(u, w) = 2$, $|\Gamma_2(u) \cap \Gamma(v)| = 5$ and $|\Gamma(u) \cap \Gamma(w)| \geq 2$. Further there are 30 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since Γ is $(G, 2)$-distance-transitive, $|\Gamma(u) \cap \Gamma(w)|$ divides 30. Since $2 \leq |\Gamma(u) \cap \Gamma(w)| \leq 6$, we have $|\Gamma(u) \cap \Gamma(w)| = 2$, 3, 5 or 6.

Suppose first that $|\Gamma(u) \cap \Gamma(w)| = 2$. Then since Γ has girth 4, each 2-arc of Γ lies in a unique 4-cycle. Thus, there is a 1-1 mapping between the unordered vertex pairs in $\Gamma(u)$ and vertices in $\Gamma_2(u)$. Since G_u is transitive on $\Gamma_2(u)$, it follows that G_u is transitive on the set of unordered vertex pairs in $\Gamma(u)$. Hence $G_u^{\Gamma(u)}$ is 2-homogeneous on $\Gamma(u)$. Further, since Γ is not $(G, 2)$-arc-transitive, $G_u^{\Gamma(u)}$ is not 2-transitive on $\Gamma(u)$. Thus by Lemma 2.5, the valency of Γ is $p^e \equiv 3 \pmod 4$ where p is a prime, contradicting the fact that Γ has valency 6.

Next, if $|\Gamma(u) \cap \Gamma(w)| = 3$, then by Lemma 2.3, Γ is $(G, 2)$-arc-transitive, which is a contradiction.

Thirdly, suppose that $|\Gamma(u) \cap \Gamma(w)| = 5$. Then $|\Gamma_3(u) \cap \Gamma(w)| \leq 1$. It follows from Remark 2.1 that Γ is G-distance-transitive. By inspecting the graphs in [3, p. 222-223], Γ is isomorphic to $\overline{(2 \times 7)-\text{grid}}$. Noting that $\overline{(2 \times 7)-\text{grid}}$ is $(\text{Aut}(\Gamma), 2)$-arc-transitive. Thus $S_2 < G < \text{Aut}(\Gamma) \cong S_2 \times S_7$. Let $G = S_2 \times M$ where $M < S_7$. Then $G_u = M_u$. Since Γ is $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive, M_u is transitive but not 2-transitive on $\Gamma(u)$. Thus M is a 2-transitive but not 3-transitive subgroup of S_7.

Finally, if $|\Gamma(u) \cap \Gamma(w)| = 6$, then $\Gamma \cong \mathrm{K}_{6,6}$, and by Lemma 2.4, Γ is $(G, 2)$-distance-transitive implies that it is $(G, 2)$-arc-transitive, which is a contradiction. □

In a non-complete graph Γ, a 2-*geodesic* of Γ is a 2-arc (u_0, u_1, u_2) such that $d_\Gamma(u_0, u_2) = 2$. The graph Γ is said to be $(G, 2)$-*geodesic-transitive*, if G is transitive on both the set of arcs and the set of 2-geodesics. Hence, a non-complete G-arc-transitive graph is $(G, 2)$-geodesic-transitive if, for any arc (u, v), $G_{u,v}$ is transitive on $\Gamma_2(u) \cap \Gamma(v)$. By definition, every $(G, 2)$-geodesic-transitive graph is $(G, 2)$-distance-transitive.

Suppose that Γ is a G-distance-transitive graph of valency k and diameter d. Then the cells of the distance partition with respect to vertex u are orbits of G_u, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i-1}(u)$, say c_i. Similarly, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i+1}(u)$, say b_i. The notation $(k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d)$ is called the *intersection array* of Γ.

Lemma 2.7. *Let Γ be a $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. Let $u \in V(\Gamma)$. If $[\Gamma(u)]$ is connected, then Γ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $\mathrm{K}_{3[3]}$ or $\mathrm{K}_{4[2]}$.*

Proof. Suppose that $[\Gamma(u)]$ is connected. Let (u, v, w) be a 2-arc such that $d_\Gamma(u, w) = 2$. Since Γ is $(G, 2)$-distance-transitive, G_u is transitive on $\Gamma(u)$, so $[\Gamma(u)]$ is a vertex-transitive graph. Let k be the valency of $[\Gamma(u)]$. Since $[\Gamma(u)]$ is connected and $|\Gamma(u)| = 6$, it follows that $k = 2, 3, 4, 5$. Let $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

If $k = 5$, then $[\Gamma(u)] \cong \mathrm{K}_6$, and so $\Gamma \cong \mathrm{K}_7$, contradicting the fact that Γ is non-complete.

Suppose that $k = 4$. Then $|\Gamma(u) \cap \Gamma(v_1)| = 4$, say $\Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4, v_5\}$. Since $|\Gamma(u) \cap \Gamma(v_6)| = 4$ and v_1, v_6 are non-adjacent, it follows that $\Gamma(u) \cap \Gamma(v_6) = \{v_2, v_3, v_4, v_5\}$. Thus $[\Gamma(u)]$ has diameter 2, and $\{v_1, v_6\}$ is a block. Since $[\Gamma(u)]$ is vertex-transitive, $[\Gamma(u)] \cong \mathrm{K}_{3[2]}$, and by [3, p.5] or [5], $\Gamma \cong \mathrm{K}_{4[2]}$.

Suppose that $k = 3$. Then $|\Gamma(u) \cap \Gamma(v_1)| = 3$, say $\Gamma(u) \cap \Gamma(v_1) = \{v_2, v_3, v_4\}$. Assume first that $[\Gamma(u)]$ does not have triangles. Then every vertex of $\{v_2, v_3, v_4\}$ is adjacent to both v_5 and v_6. Thus $[\Gamma(u)] \cong \mathrm{K}_{3,3}$. Then by [3, p.5] or [5], $\Gamma \cong \mathrm{K}_{3[3]}$. Next, assume that $[\Gamma(u)]$ has a triangle. Since $[\Gamma(u)]$ is vertex-transitive, every vertex of $\Gamma(u)$ lies in a triangle. Let (v_1, v_2, v_3) be a triangle. Since $[\Gamma(u)]$ is connected, v_4 is adjacent to neither v_2 nor v_3. Thus v_4 is adjacent to both v_5 and v_6. Since v_4 lies in a triangle and $\{v_5, v_6\} \subset \Gamma_2(v_1)$, it follows that v_5, v_6 are adjacent. Further, v_2 is adjacent to one of $\{v_5, v_6\}$, say v_5, and v_3 is adjacent to the remaining vertex v_6. Thus $[\Gamma(u)]$ is isomorphic to the 3-prism, (v_1, v_2, v_3) and (v_4, v_5, v_6) are the two triangles, and $\{v_1, v_4\}$, $\{v_2, v_5\}$ and $\{v_3, v_6\}$ are edges. Since $k = 3$, it follows that $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$. Set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2\}$. Then $\Gamma(v_1) = \{u, v_2, v_3, v_4, w_1, w_2\}$. Since $[\Gamma(v_1)]$ is isomorphic to the 3-prism, it follows that v_4 is adjacent to both w_1 and w_2, v_2 is adjacent to one of $\{w_1, w_2\}$, say w_1, and v_3 is adjacent to w_2. Thus $\Gamma(v_4) = \{u, v_1, v_5, v_6, w_1, w_2\}$. Since $[\Gamma(v_4)]$ is isomorphic to the 3-prism, it follows that w_1 is adjacent to one of $\{v_5, v_6\}$, say v_5. Thus $\{v_1, v_2, v_4, v_5\} \subseteq \Gamma(u) \cap \Gamma(w_1)$. Since $w_2 \in \Gamma(w_1)$, it follows that $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$. Thus by Remark 2.1, Γ is G-distance-transitive.

Since $\{v_1, v_2, v_4, v_5\} \subseteq \Gamma(u) \cap \Gamma(w_1)$ and $\{w_1\} \subseteq \Gamma_2(u) \cap \Gamma(w_1)$, it follows that $|\Gamma(u) \cap \Gamma(w_1)| = 4$ or 5. Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 2$, there are 12 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Thus $|\Gamma(u) \cap \Gamma(w_1)|$ divides 12, so $|\Gamma(u) \cap \Gamma(w_1)| = 4$. Hence $|\Gamma_2(u)| = 3$. Since G_u is transitive on $\Gamma_2(u)$, $[\Gamma_2(u)]$ is a vertex-transitive regular graph. Since w_1, w_2 are adjacent, $[\Gamma_2(u)] \cong C_3$. Therefore, $|\Gamma_3(u) \cap \Gamma(w_1)| = 0$, Γ has diameter 2 and has 10 vertices. In particular, the intersection array of Γ is $(6, 2; 1, 4)$. By inspecting the graphs in [3, p.222-223], Γ is $T(5)$ (also known as the Johnson graph $J(5, 2)$).

If $k = 2$, then $[\Gamma(u)] \cong C_6$. Let $(v_1, \ldots, v_6)$ be a 6-cycle. Then $|\Gamma_2(u) \cap \Gamma(v_1)| = 3$, and set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3\}$. Then $\Gamma(v_1) = \{u, v_2, v_5, w_1, w_2, w_3\}$. Since $[\Gamma(v_1)] \cong C_6$ and (v_2, u, v_6) is a 2-arc, it follows that v_2 is adjacent to one of $\{w_1, w_2, w_3\}$, say w_1; v_6 is adjacent to one of $\{w_2, w_3\}$, say w_3; and w_2 is adjacent to both w_1 and w_3. In particular, v_2 is not adjacent to any of $\{w_2, w_3\}$, and v_6 is not adjacent to any of $\{w_1, w_2\}$. Since $|\Gamma_2(u) \cap \Gamma(v_2)| = 3$, there exist w_4, w_5 in $\Gamma_2(u)$ that are adjacent to v_2, and so $\Gamma(v_2) = \{u, v_1, v_3, w_1, w_4, w_5\}$. Noting that $[\Gamma(v_2)] \cong C_6$ and (w_1, v_1, u, v_3) is a 3-arc, so v_3 is adjacent to one of $\{w_4, w_5\}$, say w_5, w_1 is adjacent to w_4, and w_4, w_5 are adjacent. Thus, $\{v_1, v_2, w_2, w_4\} \subseteq (\Gamma(u) \cup \Gamma_2(u)) \cap \Gamma(w_1)$. Hence $2 \leq |\Gamma(u) \cap \Gamma(w_1)| \leq 4$ and $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$. Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma(u) \cap \Gamma(w_1)|$ divides 18, $|\Gamma(u) \cap \Gamma(w_1)| = 2$ or 3.

Suppose that $|\Gamma(u) \cap \Gamma(w_1)| = 2$. Then $|\Gamma_2(u)| = 9$. Since $|\Gamma_2(u) \cap \Gamma(w_1)| \geq 2$, $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 2$. If $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$, then by Remark 2.1, Γ is G-distance-transitive. Inspecting the graphs in [3, p. 222-223], such a Γ does not exist. Hence $|\Gamma_3(u) \cap \Gamma(w_1)| = 2$. Since Γ is $(G, 2)$-distance-transitive, both $|\Gamma(u)|$ and $|\Gamma_2(u)|$ divide $|G_u|$, hence 18 divides $|G_u|$. Thus 3 divides $|G_{u,v}|$. Therefore $G_{u,v}$ has an element g of order 3. Since $|\Gamma(u) \setminus \{v\}| = 5$, it follows that g is trivial on $\Gamma(u) \setminus \{v\}$, so $g \in G_u^{[1]}$. Hence g fixes $\Gamma_2(u) \cap \Gamma(v_i)$ setwise. By Lemma 2.2, g is not trivial on $\Gamma_2(u)$. Hence $\langle g \rangle$ has orbits of size 3 on $\Gamma_2(u)$. Since g fixes $\Gamma_2(u)\cap\Gamma(v_i)$ setwise and $|\Gamma_2(u)\cap\Gamma(v_i)| = 3$, it follows that $\langle g\rangle$ is transitive on $\Gamma_2(u)\cap\Gamma(v_i)$. Thus G_{u,v_i} is transitive on $\Gamma_2(u)\cap\Gamma(v_i)$. Therefore Γ is $(G,2)$-geodesic-transitive. Then by [7, Corollary 1.4], Γ is either the Octahedron or the Icosahedron. However, these two graphs do not have valency 6, which is a contradiction.

Finally, suppose that $|\Gamma(u)\cap\Gamma(w_1)| = 3$. Since there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$, and $|\Gamma_2(u)|\cdot|\Gamma(u)\cap\Gamma(w_1)| = 18$, $|\Gamma_2(u)| = 6$. Since $|\Gamma_2(u)\cap\Gamma(w_1)| \geq 2$, $|\Gamma_3(u)\cap\Gamma(w_1)| \leq 1$. Thus by Remark 2.1, Γ is G-distance-transitive. Inspecting the graphs in [3, p. 222-223], Γ is the Paley graph $P(13)$. □

Lemma 2.8. *Let Γ be a $(G,2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. If $[\Gamma(u)]\cong 2\mathrm{K}_3$, then $|\Gamma_2(u)| = 9$ or 18.*

Proof. Suppose that $[\Gamma(u)]\cong 2\mathrm{K}_3$. Then each arc lies in a unique K_4. Let $\Gamma(u) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$ such that (v_1,v_2,v_3) and (v_4,v_5,v_6) are two triangles. Then for each v_i, $|\Gamma_2(u)\cap\Gamma(v_i)| = 3$. Since $[\Gamma(v_1)]\cong 2\mathrm{K}_3$, it follows that $\Gamma_2(u)\cap\Gamma(v_i)\cap\Gamma(v_j) = \emptyset$ for $i,j\in\{1,2,3\}$. Thus $|\Gamma_2(u)|\geq 9$.

On the other hand, since Γ is $(G,2)$-distance-transitive and $|\Gamma_2(u)\cap\Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Thus $|\Gamma_2(u)|$ divides 18, and so $|\Gamma_2(u)| = 9$ or 18. □

If further $|\Gamma_2(u)| = 9$, then such a graph is unique.

Lemma 2.9. *Let Γ be a $(G,2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. Suppose that $[\Gamma(u)]\cong 2\mathrm{K}_3$ and $|\Gamma_2(u)| = 9$. Then $\Gamma\cong \mathrm{H}(2,4)$*

Proof. Since $[\Gamma(u)]\cong 2\mathrm{K}_3$, each arc lies in a unique K_4. Let $\Gamma(u) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$. Let (v_1,v_2,v_3) and (v_4,v_5,v_6) be the two triangles of $[\Gamma(u)]$. Then for each v_i, $|\Gamma_2(u)\cap\Gamma(v_i)| = 3$. Since $[\Gamma(v_1)]\cong 2\mathrm{K}_3$, it follows that $\Gamma_2(u)\cap\Gamma(v_i)\cap\Gamma(v_j) = \emptyset$ for $i\neq j\in\{1,2,3\}$. Since $|\Gamma_2(u)| = 9$, $\Gamma_2(u) = (\Gamma_2(u)\cap\Gamma(v_1))\cup(\Gamma_2(u)\cap\Gamma(v_2))\cup(\Gamma_2(u)\cap\Gamma(v_3))$. Set $\Gamma_2(u)\cap\Gamma(v_1) = \{w_1,w_2,w_3\}$, $\Gamma_2(u)\cap\Gamma(v_2) = \{w_4,w_5,w_6\}$, and $\Gamma_2(u)\cap\Gamma(v_3) = \{w_7,w_8,w_9\}$. Since $[\Gamma(v_1)]\cong[\Gamma(v_2)]\cong[\Gamma(v_3)]\cong 2\mathrm{K}_3$, it follows that (w_1,w_2,w_3), (w_4,w_5,w_6) and (w_7,w_8,w_9) are three triangles.

Since Γ is $(G,2)$-distance-transitive and $|\Gamma_2(u)\cap\Gamma(v_1)| = 3$, there are 18 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma_2(u)| = 9$, it follows that for each w_i, $|\Gamma(u)\cap\Gamma(w_i)| = 2$. By the previous argument, w_1 is not adjacent to any of $\{v_2,v_3\}$, so w_1 is adjacent to one of $\{v_4,v_5,v_6\}$, say v_4. Then $\Gamma(u)\cap\Gamma(w_1) = \{v_1,v_4\}$. As each arc lies in a unique K_4 and (v_1,w_1,w_2,w_3) is a K_4, it follows that v_4 is not adjacent to any of $\{w_2,w_3\}$. Since $|\Gamma_2(u)\cap\Gamma(v_4)| = 3$ and $|\Gamma(v_i)\cap\Gamma(v_4)| = 2$ for $i = 1,2,3$, v_4 is adjacent to one of $\{w_4,w_5,w_6\}$, say w_4, and is adjacent to one of $\{w_7,w_8,w_9\}$, say w_7. Then $\Gamma(v_4) = \{u,v_5,v_6,w_1,w_4,w_7\}$. Since $[\Gamma(v_4)]\cong 2\mathrm{K}_3$ and (u,v_5,v_6) is a triangle, it follows that (w_1,w_4,w_7) is a triangle. Thus, $\Gamma(w_1) = \{v_1,v_4,w_2,w_3,w_4,w_7\}$, and so $\Gamma_3(u)\cap\Gamma(w_1) = \emptyset$. Since Γ is $(G,2)$-distance-transitive, it follows that Γ is G-distance-transitive with diameter 2 and has 16 vertices. Thus by inspecting the graphs in [3, p. 222-223], $\Gamma\cong\mathrm{H}(2,4)$. □

Lemma 2.10. *Let Γ be a $(G,2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. If $[\Gamma(u)]\cong 3\mathrm{K}_2$, then $|\Gamma_2(u)| = 8, 12,$ or 24.*

Proof. Suppose that $[\Gamma(u)]\cong 3\mathrm{K}_2$. Then each arc lies in a unique triangle. Let $\Gamma(u) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$ be such that (v_1,v_2), (v_3,v_4), and (v_5,v_6) are three arcs. Then for each v_i, $|\Gamma_2(u) \cap \Gamma(v_i)| = 4$. Since $[\Gamma(v_1)] \cong 3K_2$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$. Thus $|\Gamma_2(u)| \geq 8$.

Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 24 edges between $\Gamma(u)$ and $\Gamma_2(u)$. Since $|\Gamma_2(u)|$ divides 24, it follows that $|\Gamma_2(u)| = 8$, 12, or 24. □

If further $|\Gamma_2(u)| = 8$, then Γ is known.

Lemma 2.11. *Let Γ be a $(G, 2)$-distance-transitive graph of valency 6. Let u be a vertex of Γ. Suppose that $[\Gamma(u)] \cong 3K_2$ and $|\Gamma_2(u)| = 8$. Then $\Gamma \cong KG_{6,2}$*

Proof. Since Γ is symmetric and $[\Gamma(u)] \cong 3K_2$, each arc lies in a unique triangle. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Let (v_1, v_2), (v_3, v_4) and (v_5, v_6) be three arcs. Then for each v_i, $|\Gamma_2(u) \cap \Gamma(v_i)| = 4$. Since $[\Gamma(v_1)] \cong 3K_2$, it follows that $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \emptyset$. Since $|\Gamma_2(u)| = 8$, $\Gamma_2(u) = (\Gamma_2(u) \cap \Gamma(v_1)) \cup (\Gamma_2(u) \cap \Gamma(v_2))$. Set $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3, w_4\}$, and $\Gamma_2(u) \cap \Gamma(v_2) = \{w_5, w_6, w_7, w_8\}$. Since $[\Gamma(v_1)] \cong [\Gamma(v_2)] \cong 3K_2$, it follows that (w_1, w_2), (w_3, w_4), (w_5, w_6) and (w_7, w_8) are arcs.

Since Γ is $(G, 2)$-distance-transitive and $|\Gamma_2(u) \cap \Gamma(v_1)| = 4$, there are 24 edges between $\Gamma(u)$ and $\Gamma_2(u)$. As $|\Gamma_2(u)| = 8$, it follows that for each w_i, $|\Gamma(u) \cap \Gamma(w_i)| = 3$. By the previous argument, w_1 is not adjacent to v_2. Noting that $\Gamma_2(u) \cap \Gamma(v_i) \cap \Gamma(v_j) = \emptyset$ for $(i, j) = (1, 2), (3, 4), (5, 6)$. Thus w_1 is adjacent to one of $\{v_3, v_4\}$, say v_3, and is also adjacent to one of $\{v_5, v_6\}$, say v_5. Then $\Gamma(u) \cap \Gamma(w_1) = \{v_1, v_3, v_5\}$. Since each arc lies in a unique triangle and (v_1, w_1, w_2) is a triangle, it follows that v_3 is not adjacent to w_2. By $|\Gamma_2(u) \cap \Gamma(v_3)| = 4$ and $|\Gamma(v_i) \cap \Gamma(v_3)| = 3$ for $i = 1, 2$, v_3 is adjacent to one of $\{w_3, w_4\}$, say w_3, and is also adjacent to two vertices of $\{w_5, w_6, w_7, w_8\}$, say w_5, w_7.

Then $\Gamma(v_3) = \{u, v_4, w_1, w_3, w_5, w_7\}$. Since $[\Gamma(v_3)] \cong 3K_2$ and (u, v_4) is an arc, it follows that (w_1, w_5) and (w_3, w_7) are two arcs. Thus, $\{v_1, v_3, v_5\} \cup \{w_2, w_5\} \subseteq \Gamma(w_1)$, and so $|\Gamma_3(u) \cap \Gamma(w_1)| \leq 1$. Since Γ is $(G, 2)$-distance-transitive, it follows from Remark 2.1 that Γ is G-distance-transitive. One part of the intersection array of Γ is $(6, 4, \ldots; 1, 3, \ldots)$. By inspecting the graphs in [3, p.221], $\Gamma \cong KG_{6,2}$. □

Lemma 2.12. *Let Γ be an arc-transitive graph and let u be a vertex of Γ. Suppose that $\Gamma(u) = U \cup W$, where $|U| = |W| = n$ and $U \cap W = \emptyset$. Assume further that $[U] \cong [W] \cong K_n$. Let $v_1 \in U$. If $|\Gamma(u) \cap \Gamma(v_1) \cap W| \leq n - 2$, then Γ is a line graph.*

Proof. Suppose that $|\Gamma(u) \cap \Gamma(v_1) \cap W| \leq n - 2$. Then $[U]$ and $[W]$ are the only two n-cliques of $\Gamma(u)$. It follows from [14, Proposition 2.1] that Γ is a line graph. □

Proof of Theorem 1.2. Let Γ be a connected non-complete $(G, 2)$-distance-transitive but not $(G, 2)$-arc-transitive graph of valency 6. If Γ has girth at least 5, then for any two vertices with distance 2, there exists a unique 2-arc between these two vertices. Thus Γ is $(G, 2)$-arc-transitive, which is a contradiction. Hence Γ has girth 3 or 4. If Γ has girth 4, then it follows from Lemma 2.6 that $(\Gamma, G) = (\overline{(2 \times 7)\text{-grid}}, S_2 \times M)$ where M is a 2-transitive but not 3-transitive subgroup of S_7, so that (1) holds.

Suppose that Γ has girth 3. Let (u, v, w) be a 2-arc such that $d_\Gamma(u, w) = 2$. If $[\Gamma(u)]$ is connected, then by Lemma 2.7, Γ is isomorphic to one of: $T(5)$, Paley graph $P(13)$, $K_{3[3]}$ or $K_{4[2]}$, (2) holds. If $[\Gamma(u)]$ is disconnected, then G_u has blocks in $\Gamma(u)$, and each block has cardinality 2 or 3. If each block has cardinality 3, then $[\Gamma(u)] \cong 2K_3$; if each block has cardinality 2, then $[\Gamma(u)] \cong 3K_2$. Suppose that $[\Gamma(u)] \cong 2K_3$. Then by Lemma 2.8, $|\Gamma_2(u)| = 9$ or 18. If $|\Gamma_2(u)| = 9$, then by Lemma 2.9, $\Gamma \cong H(2, 4)$. If $|\Gamma_2(u)| = 18$, then by Lemma 2.12, Γ is a line graph, (3.1) holds.

Finally, if $[\Gamma(u)] \cong 3\mathrm{K}_2$, then by Lemma 2.10, $|\Gamma_2(u)| = 8, 12$, or 24. In particular, if $|\Gamma_2(u)| = 8$, then by Lemma 2.11, $\Gamma \cong KG_{6,2}$, so that (3.2) holds. □

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