On some generalization of the Möbius configuration

Krzysztof Petelczyc

Institute of Mathematics, University of Białystok,
Ciołkowskiego 1 M, 15-245 Białystok, Poland

Received 13 November 2014, accepted 28 January 2017, published online 19 February 2017

Abstract

The Möbius \((8_4)\) configuration is generalized in a purely combinatorial approach. We consider \((2n_n)\) configurations \(M_{(n,\varphi)}\) depending on a permutation \(\varphi\) in the symmetric group \(S_n\). Classes of non-isomorphic configurations of this type are determined. The parametric characterization of \(M_{(n,\varphi)}\) is given. The uniqueness of the decomposition of \(M_{(n,\varphi)}\) into two mutually inscribed \(n\)-simplices is discussed. The automorphisms of \(M_{(n,\varphi)}\) are characterized for \(n \geq 3\).

Keywords: Möbius configuration, \((8_4)\) configurations, Möbius pair, \(n\)-simplex.

Math. Subj. Class.: 51D20, 05B30, 51E30

1 Introduction

The Möbius \((8_4)\) configuration is a certain configuration in a projective 3-dimensional space consisting of two mutually inscribed and circumscribed tetrahedra (cf. [7]). Each vertex of one tetrahedron lies on a face plane of the other tetrahedron and vice versa. Configurations with parameters \((n_4)\) were studied in detail in [4], but this is not the case, since the Möbius \((8_4)\) configuration is not a point-line structure. An important role of the theorem connected with the Möbius configuration (which says, roughly speaking, that the Möbius configuration “closes”) in a projective 3-dimensional space was presented in [12]: it is equivalent to the commutativity of the ground division ring.

In this paper we deal with two \(n\)-simplices (simplices with \(n\) vertices, \(n \geq 3\)) instead of two tetrahedra (4-simplices). The way how an \(n\)-simplex is inscribed into another we define by a permutation \(\varphi\) in the group \(S_n\). The generalization of the Möbius configuration we obtain, is a \((2n_n)\)-configuration and it will be referred to as a Möbius pair of \(n\)-simplices.

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1 In geometry an \(n\)-simplex usually means a simplex having \(n + 1\) vertices. Our definition is slightly different.
or shortly a Möbius n-pair. Only a combinatorial scheme (an abstract incidence structure, see e.g. [2, 10]) of a Möbius n-pair is investigated and we do not discuss problems regarding embeddability into projective (or other) spaces. Although these problems have been partially solved in [5] (the case with \( \varphi = \text{id} \)), they are interesting and still open in general.

As we know from [6], in a projective space, up to an isomorphism there are five \((8_4)\) point-plane configurations with the property that at most two planes share two points, and dually at most two points are shared by two planes. These are precisely those configurations with two mutually circumscribed tetrahedra, and thus all of them are sometimes called the Möbius configurations. It is also known (cf. [10]), that these \((8_4)\) configurations correspond to conjugacy classes of the permutation group \(S_4\). We shall prove, that two Möbius n-pairs are isomorphic if and only if the permutations, that determine them, are conjugate. Another important impact of the permutation on the geometry of the Möbius n-pair is that the cycle structure of \(\varphi\) is associated with circuits in the incidence graph of the Möbius n-pair.

As we shall see, the decomposition of the points of the generalized Möbius configuration into two complementary and mutually inscribed simplices is, generally, a unique one. Exceptions appear “near” the classical case \(n = 4\). Three of five \((8_4)\) Möbius configurations contain at least two distinct pairs of complementary 4-simplices.

The next problem, which is considered in the paper, involves Möbius subpairs of a Möbius n-pair. We simply delete some number of points and blocks of one n-simplex and the same number of points and blocks of the second n-simplex with a hope to obtain a Möbius pair again. The conditions, under which we get a subpair in the Möbius n-pair, are determined.

In the last part we use most of the established properties to characterize the automorphism group of the Möbius n-pair for \(n \geq 3\).

## 2 Definitions, parameters and basic properties

By a configuration we mean any point-block structure \(\mathcal{M} = \langle S, \mathcal{L} \rangle\), where the blocks are subsets of the set of points, i.e. \(\mathcal{L} \subseteq 2^S\). The rank of a point is the number of blocks containing this point, and dually the size of a block is the number of points contained in this block. Let \(n\) be a natural number and \(X\) be a set. The family of all \(n\)-subsets of the set \(X\) will be denoted by \(\wp_n(X)\). Let \(n \geq 3\). We say that a configuration \(\mathcal{M}\) is an \(n\)-simplex iff \(|\mathcal{L}| = n\), there is a subset \(V \in \wp_n(S)\) such that for every \(V' \in \wp_{n-1}(V)\) there is a unique block \(L \in \mathcal{L}\) containing \(V'\), and the rank of each point \(s \in S \setminus V\) is less than \(n - 1\). Elements of \(V\) will be called vertices of the simplex, and blocks of the simplex are said to be its faces. We say that two configurations \(\mathcal{M}_1 = \langle S_1, \mathcal{L}_1 \rangle\), \(\mathcal{M}_2 = \langle S_2, \mathcal{L}_2 \rangle\) are isomorphic (and we write \(\mathcal{M}_1 \cong \mathcal{M}_2\)) iff there exists a bijective map \(f : S_1 \rightarrow S_2\) such that conditions \(k \in \mathcal{L}_1\) and \(f(k) \in \mathcal{L}_2\) are equivalent. In case \(\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}\) the map \(f\) will be called an automorphism of \(\mathcal{M}\).

Let us consider two sets \(A = \{a_1, \ldots, a_n\}\) and \(B = \{b_1, \ldots, b_n\}\) such that \(A \cap B = \emptyset\). Let \(\varphi \in S_n\) be a permutation of the set \(I = \{1, \ldots, n\}\). Now we introduce the following sets:

\[
\mathcal{L}_A := \{A' \cup \{b_i\} : A' \in \wp_{n-1}(A) \text{ and } a_i \notin A'\},
\]

\[
\mathcal{L}_B := \{B' \cup \{a_{\varphi(i)}\} : B' \in \wp_{n-1}(B) \text{ and } b_i \notin B'\}.
\]

The configuration

\[
\mathcal{M}_{(n, \varphi)} := (A \cup B, \mathcal{L}_A \cup \mathcal{L}_B),
\]

is called a Möbius \((n, \varphi)\)-pair.
will be called a M"obius n-pair. The M"obius configurations can be identified with the M"obius 4-pairs, which Levi graphs are Figures 1, 2, 3, 4, 5. All of them are also presented in [10]. In particular, \( \mathcal{M}_{(4,\text{id})} \) is the classical \((8_4)\) M"obius configuration.

Let \( \mathcal{M} \) be a M"obius n-pair. We write: \( A_i, B_i \) for blocks of \( \mathcal{M} \) not containing \( a_i, b_i \), respectively; \( a\)-points, \( b\)-points, \( A\)-blocks, \( B\)-blocks for points in \( A, B \), and blocks in \( \mathcal{L}_A, \mathcal{L}_B \), respectively. The configuration \( \mathcal{M} \) reflects the main abstract properties of the classical M"obius configuration.

1. The \( a \)-points yield a simplex in \( \mathcal{M} \): for any \((n-1)\)-subset \( A \setminus \{a_i\} \) of the \( a \)-points there is a unique block of \( \mathcal{M} \), which contains this subset (\( A_i \), a face of the simplex in question); the remaining points (\( b \)-points) yield another simplex.

2. The simplex with \( a \)-points and the simplex with \( b \)-points are mutually inscribed: on each face, \( A_i \), of the first simplex there is a unique vertex (\( b_i \)) of the second one; on each face, \( B_i \), of the second simplex there is a unique vertex (\( a_{\varphi(i)} \)) of the first simplex.

Thus, we can decompose \( \mathcal{M} \) into two complementary substructures \( S_A(\mathcal{M}) = \langle A, \mathcal{L}_A \rangle \) and \( S_B(\mathcal{M}) = \langle B, \mathcal{L}_B \rangle \), which we call simplices of \( \mathcal{M} \) (although, formally, a block of each of them is not a subset of its points; there is one extra point on each of its faces).

In the forthcoming part we will use the notion of the incidence graph (the Levi graph) \( G_\mathcal{M} \) associated with \( \mathcal{M} \). Recall that a Levi graph is a bipartite graph with partition induced by points vs. blocks (cf. [9, 10]). Two of its vertices \( x, y \) are said to be adjacent (which is written \( x \sim y \)) if \( x \) is a point, \( y \) is a block (or vice versa) and \( x \in y \) (or \( y \in x \)). Otherwise \( x \) is not adjacent to \( y \), which we write \( x \not\sim y \). The rank of a vertex is the number of vertices adjacent to it. A vertex of \( G_\mathcal{M} \) will be called point-vertex, block-vertex, \( a, b \)-vertex, \( A, B \)-vertex, or simply \( a_i, b_i, A_i, B_i \) as it corresponds to the point to the block of \( \mathcal{M} \). The Levi graph associated with \( S_A(\mathcal{M}), S_B(\mathcal{M}) \) will be denoted by \( G_{S_A(\mathcal{M})}, G_{S_B(\mathcal{M})} \), respectively.

**Remark 2.1.** Let \( \mathcal{M} \) be a M"obius n-pair. The Levi graph \( G_\mathcal{M} \) has the following properties:

(i) for \( X = A, B \), every point-vertex from \( G_{S_X(\mathcal{M})} \) is adjacent to all but one block-vertices from \( G_{S_X(\mathcal{M})} \), and vice versa,

(ii) for \( X, Y = A, B \) and \( X \neq Y \), every point-vertex from \( G_{S_X(\mathcal{M})} \) is adjacent to precisely one block-vertex from \( G_{S_Y(\mathcal{M})} \), and vice versa.

Immediately from the definition of \( \mathcal{M}_{(n,\varphi)} \), the number of its points coincides with the number of its blocks and equals \( 2n \), and the rank of every point coincides with the size of every block and equals \( n \). Thus the structures we investigate are \((2n_n)\)-configurations. A standard parametric question related to configurations is: what is the number of points that are contained in two distinct blocks, and dually: what is the number of blocks containing two distinct points.

**Proposition 2.2.** Let \( k, l \) be two different blocks of the structure \( \mathcal{M}_{(n,\varphi)} \). Then \( |k \cap l| \in \{0, 1, 2, n - 2\} \). If both \( k, l \) are \( A \)-blocks, or both \( k, l \) are \( B \)-blocks then \( |k \cap l| = n - 2 \). Otherwise, \( k = A_i \) and \( l = B_j \) for some \( i, j \in I \), and the following equivalences hold

(i) \(|A_i \cap B_j| = 0 \) iff \( \varphi(j) = i = j \).

(ii) \(|A_i \cap B_j| = 1 \) iff \( \varphi(j) = i \neq j \) or \( \varphi(j) \neq i = j \).
holds in \( M \) an deletion of some points and blocks. Informally, we say that these \( M \) Möbius pairs are with simplices distinct from \( i \). Then \( A_i \cap B_j = \{b_i, a_{\varphi(j)}\} \). Otherwise, for \( \varphi(j) = i \), we get \( A_i \cap B_j = \{b_i\} \). Let \( i = j \). If \( \varphi(i) \neq i \) we obtain \( A_i \cap B_i = \{a_{\varphi(i)}\} \). In case \( \varphi(i) = i \) it holds \( A_i \cap B_i = \emptyset \).

Each conjugacy class of \( S_n \) corresponds to exactly one decomposition of a permutation \( \varphi \in S_n \) into cycles, up to a permutation of the elements of \( I \). Now we describe how the cycle structure of \( \varphi \) is reflected in block paths of \( M_{(n,\varphi)} \).

**Fact 2.3.** A permutation \( \varphi \) contains a cycle of length \( k \leq n \) iff there is a closed path of length \( 2k \) consisting of blocks of \( M_{(n,\varphi)} \) such that every two consecutive blocks intersect in precisely one point of \( M_{(n,\varphi)} \).

**Proof.** Assume that \( \varphi \) contains the cycle \((i_1, i_2, \ldots, i_k)\). Then \( a_{i_{j+1}} \in A_{i_j} \cap B_{i_j} \) and \( b_{i_{j+1}} \in B_{i_j}, A_{i_{j+1}} \) for each \( j \leq k \). Thus, the closed path in question is the following: \( A_{i_1}, B_{i_1}, A_{i_2}, B_{i_2}, \ldots, A_{i_k}, B_{i_k} \).

Now assume that there exists a closed path \( l_1, l'_1, \ldots, l_k, l'_k \) of blocks of \( M_{(n,\varphi)} \) such that every two consecutive blocks intersect in a point. By Proposition 2.2(ii) every two consecutive blocks of the path are \( A_i \in L_A, B_j \in L_B \) with \( \varphi(j) = i \neq j \) or \( \varphi(j) \neq i = j \). Suppose \( \varphi(j) \neq i = j \) holds for the first two blocks of our path, namely \( l_1 = A_{i_1}, l'_1 = B_{i_1} \) and \( \varphi(i) \neq i \) for some \( i \in I \). To obtain \( |l'_1 \cap l_2| = 1 \) we must have \( l_2 = A_j \) with \( \varphi(i) = j \).

Thus the next two blocks are \( l_2 = A_{\varphi(i)}, l'_2 = B_{\varphi(i)} \) and \( \varphi(\varphi(i)) \neq \varphi(i) \). In general we obtain \( l_j = A_{\varphi^{j-1}(i)} \), \( l'_j = B_{\varphi^{j-1}(i)} \) and \( \varphi^{j-1}(i) \neq \varphi^{j-2}(i) \) for every \( j = 2, \ldots, k \). To close the path we need \( \varphi^k(i) = i \). Let us put \( i = i_0 \). Then the cycle \((i_0, i_1, \ldots, i_{k-1})\), where \( i_j = \varphi^j(i) \) for \( j = 0, \ldots, k - 1 \), is one of the cycles in the cycle decomposition of \( \varphi \). \( \square \)

As the configuration \( M_{(n,\varphi)} \) is symmetric, it makes sense to consider the dual configuration \( M_{(n,\varphi)}^\circ \).

**Fact 2.4.** The configuration \( M_{(n,\varphi)}^\circ \) is isomorphic to \( M_{(n,\varphi)} \).

**Proof.** It is easy to note that \( M_{(n,\varphi)}^\circ \cong M_{(n,\varphi^{-1})} \). Consider \( \alpha \in S_n \) such that \( \alpha(1) = 1 \) and \( \alpha(m) = n - m + 2 \) for \( m \in I \setminus \{1\} \). Let \( x \in \{a, b, A, B\} \), \( i \in I \). Then \( F : x_i \mapsto x_{\alpha(i)} \) is an isomorphism mapping \( M_{(n,\varphi^{-1})} \) onto \( M_{(n,\varphi)} \). \( \square \)

The problem of two isomorphic Möbius \( n \)-pairs will be considered in general in the last section of the paper. Another parametric characterization is now a simple consequence of Proposition 2.2 and Fact 2.4.

**Proposition 2.5.** Let \( x, y \) be two different points of \( M_{(n,\varphi)} \). There exist \( 0, 1, 2, \) or \( n - 2 \) blocks of \( M_{(n,\varphi)} \) containing \( x \) and \( y \).

## 3 Hidden Möbius pairs

The goal of this section is to characterize \( M = M_{(n,\varphi)} \) that can be transformed into Möbius pair with simplices distinct from \( S_A(M) \), \( S_B(M) \) by a decomposition of the points or by a deletion of some points and blocks. Informally, we say that these Möbius pairs are hidden in \( M \).
Figure 1: The Levi graph of $M_{(4,\text{id})}$ (isomorphic to the hypercube graph $Q_4$).

Figure 2: The Levi graph of $M_{(4,\varphi)}$ with $\varphi = (1234)$.

Figure 3: The Levi graph of $M_{(4,\varphi)}$ with $\varphi = (123)(4)$.

Figure 4: The Levi graph of $M_{(4,\varphi)}$ with $\varphi = (1)(2)(34)$.

Figure 5: The Levi graph of $M_{(4,\varphi)}$ with $\varphi = (12)(34)$. 
3.1 Möbius $n$-pairs with the special decompositions

Let us start with the following combinatorial observation:

**Remark 3.1.** The Möbius configuration $\mathcal{M} = \mathcal{M}_{(4,\text{id})}$ can be presented in 3 distinct ways as two mutually circumscribed simplices such that each of them is distinct from $S_A(\mathcal{M}), S_B(\mathcal{M})$.

One could say that there are four Möbius 4-pairs hidden in $\mathcal{M}_{(4,\text{id})}$. Let $n \geq 4, \mathcal{M} = \mathcal{M}_{(n,\varphi)}$, and assume that it is possible to decompose the points of $\mathcal{M}$ into two complementary and mutually inscribed simplices $S_1(\mathcal{M}), S_2(\mathcal{M})$ such that $S_t(\mathcal{M}) \neq S_X(\mathcal{M})$ for each $t = 1, 2, X = A, B$. Such a decomposition will be called a special decomposition.

**Lemma 3.2.** Let $S_1(\mathcal{M}), S_2(\mathcal{M})$ be two simplices, that arise from a special decomposition of $\mathcal{M}$.

(i) For each $i \in I$, and each $t = 1, 2$, it is impossible to have both $B_i, b_i$ in $S_t(\mathcal{M})$, or both $A_i, a_i$ in $S_t(\mathcal{M})$.

(ii) For each $t = 1, 2$, the blocks of $S_t(\mathcal{M})$ are two $B$-blocks and two $A$-blocks.

**Proof.** The proof involves only $S_1(\mathcal{M})$, since the reasoning for $S_2(\mathcal{M})$ will be the same.

(i) Assume that $S_1(\mathcal{M})$ contains both of $B_i, b_i$. Then also some $a_j$ is a point of $S_1(\mathcal{M})$ for $j \not\in I$. Consider the graph $G_M$. The vertices $B_i, b_i$ are not adjacent, so from Remark 2.1(i) $a_j \sim B_i$ and $j = \varphi(i)$. The unique block-vertex not adjacent to $a_j$ in $G_{S_1(\mathcal{M})}$ is $A_j$ or $B_s$ for some $s \neq \varphi^{-1}(j)$.

Let $A_j$ be this vertex, so from Remark 2.1(ii) $A_j \sim b_i$, and thus $j = i$. Consider in $G_{S_1(\mathcal{M})}$ another vertex $a_t$ or $b_t$ with $t \neq i$. Since $\varphi(i) = i \neq t$, a contradiction arises: $b_t \sim A_i$, and $a_t \sim B_i$ (see the scheme presented in Figure 6).

![Figure 6: The fragment of $G_{f(\mathcal{M})}$ containing $B_i, b_i$ and $A_i, a_i$.](image)

Assume that $s \neq \varphi^{-1}(j)$ and $B_s$ is a unique vertex not adjacent to $a_j$ in $G_{S_1(\mathcal{M})}$. We get $s \neq i$, as far as $b_i \sim B_s$. Let us take another vertex: $A_t$ or $B_t$. For $t \neq i$ there is no $B$-vertex adjacent to $a_j$, and $A_t \sim a_j, b_i$ if $t = i$. A vertex, which is not adjacent to $A_i$, is $a_i$ or $b_r$ with $r \neq i, s$. The vertex $a_i$ is not adjacent to $B_i$ since $\varphi(i) = j \neq i$, and thus $a_i$ cannot be the vertex in question. Consequently, this vertex is $b_r \sim B_i, B_s$. Following the
assumption \( n \geq 4 \), there exists another block in \( S_1(M) \), that is different from \( B_i, B_s, A_1 \). We have two \( b \)-points in \( S_1(M) \) so far, thus this block is a \( B \)-block. The \( B \)-vertex of \( G_M \), that is associated with this block, must be adjacent to \( a_\varphi(i) \). So this block is \( B_i \), which is already one of the blocks in \( S_1(M) \) (comp. with the scheme presented in Figure 7), a contradiction.

(ii) Let \( B_i \) be the unique \( B \)-block of \( S_1(M) \) for some \( i \in I \). Then the remaining blocks of \( S_1(M) \) are \( A \)-blocks. In view of Lemma 3.2(i), there are \( n - 1 \) \( b \)-vertices in \( G_{S_1(M)} \): every \( A \)-vertex is associated with the \( b \)-vertex, which is not adjacent to it. For \( n \geq 4 \) a contradiction with Remark 2.1(i) arises: every \( b \)-vertex is adjacent to precisely one of \( A \)-vertices, and thus it is not adjacent to at least two \( A \)-vertices in \( G_{S_1(M)} \).

Let \( S_1(M) \) contain at least three \( B \)-blocks. Without loss of generality, assume \( B_1, B_2, B_3 \) are blocks of \( S_1(M) \). From Lemma 3.2(i), \( b_1, b_2, b_3 \) are not in \( S_1(M) \). Thus, from Remark 2.1(i), \( S_1(M) \) contains \( a_{i_1}, a_{i_2}, a_{i_3} \) such that \( i_j \neq \varphi(j) \) for \( j = 1, 2, 3 \). Every block-vertex \( B_j \) must be adjacent to at least two of the point-vertices \( a_{i_j} \), with \( j' \neq j \). On the other hand, it is adjacent to at most one of them, what follows from Remark 2.1(ii) applied to \( G_M \). This contradiction actually completes the proof as other cases run dually. \( \Box \)

By Lemma 3.2 we prove a generalization of Remark 3.1.

**Proposition 3.3.** Let \( M = \mathcal{M}_{(n,\varphi)} \). The following conditions are equivalent

(i) there is a special decomposition of \( M \),

(ii) \( n = 4 \) and there is \( X \subset I \) such that \( |X| = 2 \) and \( \varphi(X) = X \).

**Proof.** (i) \( \Rightarrow \) (ii): From Lemma 3.2(ii) we get \( n = 4 \), and two \( B \)-vertices and two \( A \)-vertices in \( G_{S_1(M)} \). Let (e.g.) \( B_1, B_2 \) be the \( B \)-vertices of \( G_{S_1(M)} \). In view of Remark 2.1(i), there are vertices \( x, y \) in \( G_{S_1(M)} \) such that \( x \sim B_1, y \sim B_2 \) and \( x \sim B_2, y \sim B_1 \). By Lemma 3.2(i), \( x \neq b_2, y \neq b_2 \), and thus \( x = a_i, y = a_j \) where \( \varphi(1) = i, \varphi(2) = j \). Then two \( A \)-vertices in \( G_{S_1(M)} \) are \( A_s, A_t \) with \( s, t \neq i, j \). The remaining two point-vertices must be of the form \( b_{s'}, b_{t'} \) with \( s', t' \neq 1, 2 \), since they must be adjacent to both of \( B_1, B_2 \). On the other hand, \( b_{s'}, b_{t'} \) need to be adjacent to precisely one of \( A_s, A_t \), so \( \{s', t'\} = \{s, t\} \). Thus \( s, t \neq 1, 2, \{1, 2\} = \{i, j\} = \{\varphi(1), \varphi(2)\} \), and \( X = \{1, 2\} \) is the required set.

(ii) \( \Rightarrow \) (i): Assume, without loss of generality, \( X = \{1, 2\} \) and consider \( M = \mathcal{M}_{(4,\varphi)} \) with \( \varphi(X) = X \). Take blocks \( B_1, B_2, A_3, A_4 \) and points \( a_{\varphi(1)}, a_{\varphi(2)}, b_3, b_4 \) of \( M \), and consider \( G_M \). We have \( B_1 \sim a_{\varphi(2)}, B_2 \sim a_{\varphi(1)} \), and \( B_1, B_2 \sim b_3, b_4 \). Similarly \( A_3 \sim b_1, A_4 \sim b_3, A_3, A_4 \sim a_{\varphi(1)}, a_{\varphi(2)} \), since \( \varphi(1), \varphi(2) \in \{1, 2\} \). Thus the Levi graph we consider is a Levi graph of a 4-simplex. It is easy to verify that \( A_1, A_2, B_3, B_4 \) and \( b_1, b_2, a_3, a_4 \) form another 4-simplex. The two obtained simplices are mutually circumscribed. Indeed, \( B_1, b_2; B_2, b_1; A_3, a_4; A_4, a_3, A_1, a_{\varphi(1)} (or A_1, a_{\varphi(2)}); A_2, a_{\varphi(2)} (or A_2, a_{\varphi(1)}); B_3, b_4; B_4, b_3 \) are all pairs of adjacent vertices representing blocks (points) of the first simplex and points (blocks) of the second simplex in each pair. In other words, we have found a special decomposition of \( M \). \( \Box \)

Due to Proposition 3.3 there is a correspondence between the special decompositions of \( \mathcal{M}_{(n,\varphi)} \) and 2-subsets of \( I \) preserved by \( \varphi \). The correspondence is established up to complements, since the special decompositions arise only for \( n = 4 \), and thus if \( \varphi \) preserves a 2-subset of \( \{1, 2, 3, 4\} \) then it preserves its complement as well. So, directly from Proposition 3.3 we get:
Corollary 3.4. All (up to an isomorphism) Möbius n-pairs with a special decomposition are the following:

1. \( \mathcal{M}_{(4, \text{id})} \) with 3 distinct special decompositions associated with \( X = \{1, 2\}, \{1, 3\}, \{1, 4\} \).
2. \( \mathcal{M}_{(4,(13)(24))} \) with the special decomposition associated with \( X = \{1, 3\} \).
3. \( \mathcal{M}_{(4,(12)(3)(4))} \) with the special decomposition associated with \( X = \{1, 2\} \).

3.2 Subpairs of Möbius n-pairs

Let \( \mathcal{M} = \mathcal{M}_{(n, \varphi)} \), \( n \geq 4 \), \( k \geq 3 \), \( k < n \), and \( \mathcal{M}' \) be a Möbius k-pair obtained from \( \mathcal{M} \) by deleting \( 2(n-k) \) points and \( 2(n-k) \) blocks. We call \( \mathcal{M}' \) a k-subpair of \( \mathcal{M} \). The blocks of \( \mathcal{M}' \) are subblocks of \( \mathcal{M} \), that is every block of \( \mathcal{M}' \) arises as a block of \( \mathcal{M} \) with \( n-k \) points removed. The subblocks of the A-blocks, the B-blocks are called the A-subblocks, the B-subblocks, respectively. Let \( S_1(M') \), \( S_2(M') \) be two simplices of \( \mathcal{M}' \). For any \( t = 1, 2 \), \( X = A, B \) we write \( S_t(M') \prec S_X(M) \) if all the points and the blocks of \( S_t(M') \) are points and subblocks of \( S_X(M) \). Otherwise we write \( S_t(M') \nprec S_X(M) \). For \( Y \subset I \) by \( \varphi \upharpoonright_Y \) we mean the restriction of \( \varphi \) to the set \( Y \).

In order to determine all Möbius n-pairs with k-subpairs we need to prove some auxiliary facts.

Lemma 3.5. One of the following conditions holds

(i) \( S_1(M') \prec S_A(M) \) and \( S_2(M') \prec S_B(M) \),

(ii) \( S_2(M') \prec S_A(M) \) and \( S_1(M') \prec S_B(M) \),

(iii) \( S_1(M') \nprec S_A(M), S_B(M) \) and \( S_2(M') \nprec S_A(M), S_B(M) \).

Moreover, if \( \mathcal{M}' \) satisfies (iii) then there is a special decomposition of \( \mathcal{M}' \).

Proof. Let \( S_1(M') \prec S_A(M) \) and \( S_2(M') \nprec S_B(M) \). So there is an a-point or A-subblock in \( S_2(M') \). We consider only the case with an a-point, as the case with an A-subblock is symmetric. From Remark 2.1(ii) applied to \( G_M \), and Remark 2.1(i) applied to \( G_{M'} \), there are at most two B-subblocks in \( S_2(M') \). Since \( k \geq 3 \), there is at least one A-subblock in \( S_2(M') \). Note that a unique A-subblock, which does not contain an a-point of \( S_1(M') \), is a block of \( S_1(M') \). Thus all the points of \( S_1(M) \) are in an A-subblock of \( S_2(M') \). This yields a contradiction with Remark 2.1(ii). The proof for each of the remaining cases (i.e. \( S_2(M') \prec S_A(M) \) and \( S_1(M') \nprec S_B(M) \), \( S_1(M') \prec S_B(M) \) and \( S_2(M') \nprec S_A(M) \), or \( S_2(M') \prec S_B(M) \) and \( S_1(M') \nprec S_A(M) \) is analogous.

Let \( \mathcal{M}' \) satisfy (iii). The steps of the proof of Lemma 3.2 can be repeated for simplices of \( \mathcal{M}' \). As a result we get \( k = 4 \), and two A-subblocks and two B-subblocks in each of simplices of \( \mathcal{M}' \). Let \( Y \subset I \) be the set of subscripts of A-subblocks and B-subblocks in one of these simplices. From the reasoning analogous to the first part of the proof of Proposition 3.3 we get that \( Y \) is the set of all the subscripts used for labelling the points and the blocks of \( \mathcal{M}' \), and there is a two-element set \( X \subset Y \) such that \( \varphi \upharpoonright_Y (X) = X \). Therefore, in view of Proposition 3.3, there is a special decomposition of \( \mathcal{M}' \). □

Lemma 3.6. If the number of deleted B-blocks and the number of deleted A-blocks coincide (and equals \( n-k \)), then there is \( X \subset I \) such that \( |X| = n-k \) and \( \varphi(X) = X \).
Proof. Assume that $B_1, \ldots, B_{n-k}$ and $A_1, \ldots, A_{n-k}$ are removed blocks. Consider a vertex $a_{\varphi(i_j)}$ with $s = 1, \ldots, n - k$ of $G_{M'}$, and assume $a_{\varphi(i_j)}$ is in $G_{S_1(M')}$. The case with $a_{\varphi(i_j)}$ in $G_{S_2(M')}$ will be analogous. Note that $a_{\varphi(i_j)} \sim B_{i_j}$, and from Remark 2.1(ii) $B_{i_j}$ is a unique $B$-vertex adjacent to $a_{\varphi(i_j)}$. According to Lemma 3.5 two cases arise: (i) or (iii) holds for $M'$. Let $M'$ satisfy (i) of Lemma 3.5. Then there is a $B$-vertex in $G_{S_2(M')}$ adjacent to $a_{\varphi(i_j)}$, a contradiction. Consequently points $a_1, \ldots, a_{n-k}$ are removed. Likewise we consider the pairs $a_j, A_j$, $b_j, A_j$, and $b_i, B_i$. Each of these reasonings leads us to contradiction. Consequently points $a_j, A_j$, $b_j, A_j$, and $b_i, B_i$ are deleted as well. Hence

$$\{j_1, \ldots, j_{n-k}\} = \{\varphi(i_1), \ldots, \varphi(i_{n-k})\} \text{ and } \{i_1, \ldots, i_{n-k}\} = \{j_1, \ldots, j_{n-k}\}.$$ 

Finally we get $\{\varphi(i_1), \ldots, \varphi(i_{n-k})\} = \{i_1, \ldots, i_{n-k}\}$, and $X = \{i_1, \ldots, i_{n-k}\}$ is the set from our claim. \qed

Let us present a condition, which is sufficient and necessary to find a $k$-subpair in $M_{(n,\varphi)}$.

**Proposition 3.7.** Let $M = M_{(n,\varphi)}$. The following conditions are equivalent

(i) there is $M'$, which is a $k$-subpair of $M$,

(ii) there is $X \subset I$ such that $|X| = n - k$ and $\varphi(X) = X$.

Furthermore, if (ii) holds then $M' \cong M_{(k,\varphi \mid (I \setminus X))}$.

**Proof.** (i) $\Rightarrow$ (ii): By Lemma 3.5 $M'$ satisfies one of (i) – (iii) of Lemma 3.5. In cases (i) and (ii) of Lemma 3.5 the numbers of $A$-blocks and $B$-blocks deleted from $M$ coincide and are equal to $n - k$. The claim follows directly from Lemma 3.6. If case (iii) of Lemma 3.5 holds, then there is a special decomposition of $M'$, and we get our claim by Proposition 3.3.

(ii) $\Rightarrow$ (i): Without any loss of generality, let $X = \{1, \ldots, n - k\}$. Recall that the rank of every vertex in $G_{M}$ is $n$. Observe the Levi graph obtained from $G_{M}$ by removing the vertices $a_i, A_i$ and $b_i, B_i$ for every $i \in X$, and all edges passing through these vertices. We denote this Levi graph by $H$. Note that $a_{\varphi(i)}$ is not a vertex of $H$, since $\varphi(i) \not\in X$. Let $j \not\in X$ and take $A_j$. Clearly $A_j$ is a vertex of $H$. There are $n - k$ edges joining $A_j$ with all $a_i$ in $G_{M}$. Thus, the rank of $A_j$ in $H$ is $n - (n - k) = k$. Similarly we set ranks of the remaining vertices $a_j, b_j, B_j$ of $H$. All these ranks are $k$. From this and the construction of $H$ we get that $H$ is the Levi graph of two mutually circumscribed $k$-simplices, where the way they are inscribed one into another is induced by the action of $\varphi$ on the set $I \setminus X$. Therefore $H = G_{M'}$ for some $M'$, which is a $k$-subpair of $M$. \qed

4 Isomorphisms and automorphisms

4.1 Isomorphic Möbius $n$-pairs

Recall that the Möbius $(8,1)$ configurations (i.e. Möbius 4-pairs) correspond to conjugacy classes of the permutation group $S_4$. In this section we generalize this property to all Möbius $n$-pairs.

Let us start with a key lemma that gives an account on isomorphisms of configurations $M_{(n,\varphi)}$ with the unique decomposition into two $n$-simplices.
Lemma 4.1. Let $f$ be an isomorphism mapping $\mathcal{M}_{(n,\varphi)}$ onto $\mathcal{M}_{(n,\psi)}$. Assume that either $n = 4$ and both $\varphi, \psi \neq \text{id}$ contain no cycle of length 2, or $n \geq 5$. There is $\alpha \in S_n$ such that $f(B_i) = B_{\alpha(i)}$ for each $i \in I$, or $f(B_i) = A_{\alpha(i)}$ for each $i \in I$.

(i) If $f(B_i) = B_{\alpha(i)}$ then $f(b_i) = b_{\alpha(i)}$, $f(A_i) = A_{\alpha(i)}$, $f(a_i) = a_{\alpha(i)}$ for each $i \in I$.

(ii) If $f(B_i) = A_{\alpha(i)}$ then $f(b_i) = a_{\alpha(i)}$, $f(A_i) = B_{\psi^{-1}(\alpha(i))}$, $f(a_i) = b_{\psi^{-1}(\alpha(i))}$ for each $i \in I$.

Furthermore, $\alpha \varphi = \psi \alpha$ holds in both cases: (i) and (ii).

Proof. Let $\mathcal{M}_1 := \mathcal{M}_{(n,\varphi)}$ and $\mathcal{M}_2 := \mathcal{M}_{(n,\psi)}$. Let $i, j \in I$ and $B_i$ be an arbitrary $B$-block of $\mathcal{M}_1$. Clearly, either $f(B_i) = B_j$ for some $B$-block $B_j$ of $\mathcal{M}_2$, or $f(B_i) = A_j$ for some $A$-block $A_j$ of $\mathcal{M}_2$.

Assume that $f(B_i) = B_j$. In view of Corollary 3.4, both $\mathcal{M}_1$, $\mathcal{M}_2$ are Möbius $n$-pairs without the special decompositions. Thus all the $B$-blocks of $\mathcal{M}_1$ are mapped onto $B$-blocks of $\mathcal{M}_2$. We introduce a map $\alpha \in S_n$ associated with $f$ by the formula

$$\alpha : i \mapsto j \text{ iff } f(B_i) = B_j,$$

for all $i, j \in I$. Then $f(B_i) = B_{\alpha(i)}$. Let us analyze graphs $G_{\mathcal{M}_1}$ and $G_{\mathcal{M}_2}$: $f(b_i) = b_{\alpha(i)}$ as $b_i$, $b_{\alpha(i)}$ are unique $b$-vertices not adjacent to $B_i$, $B_{\alpha(i)}$ respectively in graphs $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$; $f(A_i) = A_{\alpha(i)}$ as $A_i$, $A_{\alpha(i)}$ are unique $A$-vertices adjacent to $b_i$, $b_{\alpha(i)}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$; $f(a_i) = a_{\alpha(i)}$ as $a_i$, $a_{\alpha(i)}$ are unique $a$-vertices not adjacent to $A_i$, $A_{\alpha(i)}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$. On the other hand, $f(a_{\varphi(i)}) = a_{\psi(\alpha(i))}$ as $a_{\varphi(i)}$, $a_{\psi(\alpha(i))}$ are unique $a$-vertices adjacent to $B_i$, $B_{\alpha(i)}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$. So $a_{\alpha(\varphi(i))} = a_{\psi(\alpha(i))}$ and thus $\alpha \varphi = \psi \alpha$.

In case $f(B_i) = A_j$ the map $\alpha \in S_n$ is determined by the condition

$$\alpha : i \mapsto j \text{ iff } f(B_i) = A_j,$$

for all $i, j \in I$. Then we proceed in a similar way as in the former case, namely: $f(b_i) = a_{\alpha(i)}$ as $b_i$, $a_{\alpha(i)}$ are a unique $b$-vertex and $a$-vertex not adjacent to $B_i$, $A_{\alpha(i)}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$; $f(A_i) = B_{\psi^{-1}(\alpha(i))}$ as $A_i$, $B_{\psi^{-1}(\alpha(i))}$ are a unique $A$-vertex and $B$-vertex adjacent to $b_i$, $a_{\alpha(i)}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$; $f(a_i) = b_{\psi^{-1}(\alpha(i))}$ as $a_i$, $b_{\psi^{-1}(\alpha(i))}$ are a unique $a$-vertex and $b$-vertex not adjacent to $A_i$, $B_{\psi^{-1}(\alpha(i))}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$. But also $f(a_{\varphi(i)}) = b_{\alpha(i)}$ as $a_{\varphi(i)}$, $b_{\alpha(i)}$ are a unique $a$-vertex and $b$-vertex adjacent to $B_i$, $A_{\alpha(i)}$ respectively in $G_{\mathcal{M}_1}$, $G_{\mathcal{M}_2}$. Hence $b_{\psi^{-1}(\alpha(\varphi(i)))} = b_{\alpha(i)}$, and consequently $\alpha \varphi = \psi \alpha$. \qed

We are ready to characterize two isomorphic Möbius $n$-pairs.

Theorem 4.2. Let $n \geq 4$ and $\varphi, \psi, \alpha \in S_n$. The following conditions are equivalent:

(i) $\varphi^\alpha = \psi$,

(ii) $\mathcal{M}_{(n,\varphi)} \cong \mathcal{M}_{(n,\psi)}$.

Proof. Let $\mathcal{M}_1 = \mathcal{M}_{(n,\varphi)}$ and $\mathcal{M}_2 = \mathcal{M}_{(n,\psi)}$.

(i) $\Rightarrow$ (ii): Let $i \in I$, $a_i, b_i$ be points and $A_i, B_i$ be blocks of $\mathcal{M}_1$. Consider a map $f$ associated to the permutation $\alpha$ given by the formula

$$f(x_i) = x_{\alpha(i)} \text{ for } x \in \{a, b\},$$

for all $i, j \in I$. Then $f(B_i) = B_{\alpha(i)}$ for each $i \in I$, or $f(B_i) = A_{\alpha(i)}$ for each $i \in I$.
which maps the points of \( M_1 \) onto the points of \( M_2 \). Then \( f(A_i) = A_{\alpha(i)} \) and \( f(B_i) = B_{\alpha(i)} \), as the conditions \( a_i \notin A_i, b_i \notin B_i \) uniquely determine blocks \( A_i, B_i \), respectively. Clearly, conditions \( b_i \in A_i \) and \( b_{\alpha(i)} \in A_{\alpha(i)} \) are equivalent. Note that \( a_{\alpha(\varphi(i))} \in B_{\alpha(i)} \) is equivalent to \( a_{\psi(\alpha(i))} \in B_{\alpha(i)} \) as well, since \( \alpha \varphi = \psi \alpha \). Thus \( f \) is the required isomorphism.

(ii) \( \Rightarrow \) (i): We restrict ourselves to \( n \geq 5 \) since for \( n = 4 \) this fact is well known, as it was mentioned at the beginning of this section. Let \( f \) be an isomorphism mapping \( M_1 \) onto \( M_2 \). By Lemma 4.1, there is \( \alpha \in S_n \) associated with \( f \) such that \( \alpha \varphi = \psi \alpha \).

According to Theorem 4.2, the number of non-isomorphic configurations \( \mathcal{M}_{(n, \varphi)} \) is equal to the number of partitions \( p(n) \) of a positive integer \( n \). There is the generating function, recursive formula, asymptotic formula, and direct formula for \( p(n) \) (cf. [1]). The increase of \( n \) implies quick growth of \( p(n) \): \( p(5) = 7, p(6) = 11, \ldots, p(100) = 190569292, \ldots, p(1000) = 24061467864032622473692149727991 \).

### 4.2 The automorphism group structure of a Möbius \( n \)-pair

For \( n = 3 \) the structure \( \mathcal{M}_{(3, \varphi)} \) consists of two mutually inscribed triangles. From [8] the automorphism group of \( \mathcal{M}_{(3, \varphi)} \) is isomorphic to \( S_3 \rtimes C_2 \). From the original paper of Möbius [7] the automorphism group of \( \mathcal{M}_{(4, \text{id})} \) has order 192. The Möbius configuration is also a particular case of the Cox configuration. Recall the definition of the Cox configuration (comp. [3]). Let \( X \) be a set with \( n \) elements. The incidence structure

\[
(Cx)_X = (Cx)_n = \left\{ \{\varphi_{2k+1}(X) : 0 \leq k \leq n\}, \{\varphi_{2k}(X) : 0 \leq k \leq n\}, \subset \cup \supset \right\}
\]

is the \((2^{n-1})_n\) configuration, which is called the Cox configuration. Since the automorphism group of \((Cx)_n\) is established in [11] and \( \mathcal{M}_{(4, \text{id})} = (Cx)_4 \) (see Figure 8), we get the following:

**Fact 4.3.** The automorphism group of \( \mathcal{M}_{(4, \text{id})} \) is isomorphic to \( S_4 \rtimes C_2^3 \).

It follows from Theorem 4.2 that the centralizer of \( \varphi \) in \( S_n \) consists of automorphisms of \( \mathcal{M}_{(n, \varphi)} \) for any \( n \). Nevertheless, we will give a detailed characterization of automorphism group of \( \mathcal{M}_{(4, \text{id})} \) with \( \varphi \neq \text{id} \), and of \( \mathcal{M}_{(n, \varphi)} \) with \( n \geq 5 \).

Let \( M = \mathcal{M}_{(n, \varphi)} \) and \( 1 \leq \nu_1 < \ldots < \nu_r \) be the lengths of the cycles which are contained in the cycle decomposition of \( \varphi \in S_n \). Assume that there are \( m_t \) cycles of length \( \nu_t \), so \( n = \sum_{t=1}^{r} m_t \nu_t \). In other words

\[
\varphi = \underbrace{\varphi_{\nu_1} \varphi_{\nu_2} \ldots \varphi_{m_1} \varphi_{\nu_2} \varphi_{\nu_3} \ldots \varphi_{m_2} \varphi_{\nu_3} \ldots \varphi_{m_r} \varphi_{\nu_r} \ldots \varphi_{m_r}}_{\text{cyclic}}
\]

where \( \varphi_{k}^{\nu_k} \) is a cycle of length \( \nu_k \) for \( k \leq m_t, t \leq r \). In view of Theorem 4.2 we can assume, that each cycle consists of consecutive natural numbers. If we set \( \mu_k^t := \sum_{i=1}^{t-1} m_i \nu_i + (k-1)\nu_t + 1 \) then

\[
\varphi_{k}^{\nu_k} : \mu_k^t \mapsto \mu_k^t + 1 \mapsto \mu_k^t + 2 \mapsto \ldots \mapsto \mu_k^t + (\nu_t - 1) \mapsto \mu_k^t,
\]

and the effective domain of \( \varphi_{k}^{\nu_k} \) is the set \( X_{k}^{\nu_k} := \{\mu_k^t, \mu_k^t + 1, \ldots, \mu_k^t + (\nu_t - 1)\} \subseteq I \). Taking all the domains of all the cycles we obtain the family of pairwise disjoint sets \( X_{k}^{\nu_k}, X_{k+1}^{\nu_k}, \ldots, X_{m_t}^{\nu_k}, \ldots, X_{k}^{\nu_r}, X_{k+1}^{\nu_r}, \ldots, X_{m_t}^{\nu_r} \) that yields a covering of \( I \). Thus for any cycle \( \varphi_{k}^{\nu_k} \) we have \( \varphi_{k}^{\nu_k}(X_{k}^{\nu_k}) = X_{k}^{\nu_k} \) and \( \varphi_{k}^{\nu_k} \upharpoonright I \setminus X_{k}^{\nu_k} = \text{id} \).
that

\[ (g) \]

In like manner we define the map 

\[ \text{Set} \]

\[ i \]

\[ \text{onto} \]

\[ S \]

\[ f \]

\[ \text{Lemma 4.4.} \]

The points and the blocks of \( M \) can be identified with the sequences \( (t, k, i, \varepsilon) \) such that \( t \leq r, k \leq m_t, i = 0, \ldots, \nu_t - 1, \) and \( \varepsilon \in \{1, 2, -1, -2\} \) according to the formula:

\[
(t, k, i, \varepsilon) = \begin{cases} 
  a_i + \mu_k^t & \text{for } \varepsilon = 1, \\
  b_i + \mu_k^t & \text{for } \varepsilon = -1, \\
  A_i + \mu_k^t & \text{for } \varepsilon = 2, \\
  B_i + \mu_k^t & \text{for } \varepsilon = -2.
\end{cases} \tag{4.1}
\]

Let \( \nu_t = (v_1^t, \ldots, v_{m_t}^t) \in C_{m_t}^{m_t}, \alpha_t \in S_{m_t}, \) and \( \nu = (v_1, \ldots, v_r) \in \times_{t=1}^r C_{m_t}^{m_t}, \) \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \times_{t=1}^r S_{m_t}. \) With the pair \( (\nu, \alpha) \) we associate the map \( f_{(\nu, \alpha)} \) as follows:

\[
f_{(\nu, \alpha)}((t, k, i, \varepsilon)) = (t, \alpha_t(k), i + v_k^t \mod \nu_t, \varepsilon). \tag{4.2}
\]

In like manner we define the map \( g_{(\nu, \alpha)} \) by:

\[
g_{(\nu, \alpha)}((t, k, i, \varepsilon)) = \begin{cases} 
  (t, \alpha_t(k), i + v_k^t - 1 \mod \nu_t, -\varepsilon) & \text{for } \varepsilon = 1, 2, \\
  (t, \alpha_t(k), i + v_k^t \mod \nu_t, -\varepsilon) & \text{for } \varepsilon = -1, -2.
\end{cases} \tag{4.3}
\]

Lemma 4.4. The map \( f_{(\nu, \alpha)} \) is an automorphism of \( M, \) which preserves each of simplices \( S_A, S_B. \)

Proof. It follows directly from (4.2), that \( f_{(\nu, \alpha)} \) maps \( S_A \) onto \( S_A, \) and \( f_{(\nu, \alpha)} \) maps \( S_B \) onto \( S_B. \) Let \( i \in X^{r\times}_{k} \) and \( j \in I. \)

Assume that \( b_j \in B_i. \) By (4.1), \( B_i = (t, k, i_0, -2) \) for some \( i_0 \in \{0, \ldots, \nu_t - 1\}, \) and \( b_j = (t', k', j_0, -1) \) for some \( t' \leq r, k' \leq m_{t'}, j_0 \in \{0, \ldots, \nu_{t'} - 1\}. \) Then \( f(B_i) = (t, \alpha_t(k), i_0 + v_{\alpha_t(k)}^t \mod \nu_t, -2) \) and \( f(b_j) = (t', \alpha_{t'}(k'), j_0 + v_{\alpha_{t'}(k')}^{t'} \mod \nu_{t'}, -2). \) Set \( i' = (i_0 + v_{\alpha_t(k)}^t \mod \nu_t) + \mu^t_{\alpha_t(k)} \) and \( j' = (j_0 + v_{\alpha_{t'}(k')}^{t'} \mod \nu_{t'}) + \mu^{t'}_{\alpha_{t'}(k')}, \) so

Figure 8: The Möbius configuration as \( (\text{Cx})_4. \)
f(B_i) = B'_i and f(b_j) = B'_j. Recall that b_j ∈ B_i iff j ≠ i. If j’ ≠ i’ then: firstly t’ = t, next α_t(k’) = α_t(k) and thus k’ = k, and finally j_0 = i_0. It means that j = i, which yields a contradiction. Hence f(b_j) ∉ f(B_i).

Let a_j ∈ B_i. Then j = φ(i). We have a_φ(i) = (t, k, i_0 + 1 mod ν_t, 1), so f(a_φ(i)) = (t, k, i_0 + 1 + ν_t k mod ν_t, 1) = a_φ(i'). Therefore f(a_φ(i)) ∈ f(B_i).

The incidence (membership) relation is preserved by f(v,α) in case a_j ∈ A_i and in case b_j ∈ A_i as well, that can be easily proved by similar reasoning.

Let \( v_t = (v_1, \ldots, v_t) \) for all \( t ≤ r \), and \( v = (v_1, \ldots, v_r) \). Let us put \( g_0 := g(0, id) \).

**Lemma 4.5.** The map \( g_0 \) is an automorphism of \( \mathcal{M} \), which interchanges simplices \( S_A, S_B \).

**Proof.** Immediately from (4.2), \( g_0 \) maps \( S_A \) onto \( S_B \), and \( S_B \) onto \( S_A \). We restrict our proof to the incidence relation involving the \( B \)-blocks, as the case with the \( A \)-blocks runs similarly. Let \( i ∈ X_k^\nu \). From (4.1) \( B_i \) is represented by the sequence \( (t, k, i_0, -2) \) for some \( i_0 ∈ \{0, \ldots, \nu - 1\} \). The points that belongs to \( B_i \) are \( b_j \) with \( j ∈ I \setminus \{i\} \) and \( a_φ(i) \). Clearly, \( g_0(b_j) = a_j ∈ A_i = g_0(B_i) \). We have \( a_φ(i) = (t, k, i_0 + 1 \text{ mod } \nu_t, 1) \) and thus \( g_0(a_φ(i)) = (t, k, i_0, -1) = b_i \). Then finally \( g_0(a_φ(i)) ∈ g_0(B_i) \).

Since \( g(v,α) = g_0 f(v,α) \), from Lemma 4.4 and Lemma 4.5 we infer that:

**Corollary 4.6.** The map \( g(v,α) \) is an automorphism of \( \mathcal{M} \), which interchanges simplices \( S_A, S_B \).

We write \( \mathcal{M}_k^\nu \) for the set of all the points and the blocks of \( \mathcal{M} \) labelled by the elements of the set \( X_k^\nu \), and \( \mathcal{M}^\nu = \{ \mathcal{M}_k^\nu : k ≤ m_\nu \} \).

**Lemma 4.7.** Let \( f \) be an automorphism of \( \mathcal{M} \), which

1. maps the \( B \)-blocks onto the \( B \)-blocks, or
2. maps the \( B \)-blocks onto the \( A \)-blocks.

There is \( v ∈ ×_{t=1}^r C_{\nu t}^m \) and \( α ∈ ×_{t=1}^r S_{m_t} \) such that

(i) \( f = f(v,α) \) in case (1), or

(ii) \( f = g(v,α) \) in case (2).

In particular, for each \( k ≤ m_\nu \) there is \( k' ≤ m_\nu \) such that \( f(\mathcal{M}_k^\nu) = \mathcal{M}_k^\nu \).

**Proof.** (i): Let \( i ∈ X_k^\nu \). Assume that \( f(B_i) = B_j \) for some \( j ∈ I \). According to (4.1) there is \( i_0 ∈ \{0, \ldots, \nu_t - 1\} \) such that \( B_i = (t, k, i_0, -2) \), and \( j_0 ∈ \{0, \ldots, \nu - 1\} \) such that \( B_j = (t', k', j_0, -2) \) for some \( t' ≤ r \), \( k' ≤ m_\nu \). Then, by Lemma 4.1(ii) we get \( f((t, k, i_0, ε)) = (t', k', j_0, ε) \) for each value of \( ε \). The unique \( B \)-block containing \( a_1 = (t, k, i_0, 1) \) is \( B_{2-1(i)} = (t, k, i_0 - 1 \text{ mod } \nu_t, -2) \), and a unique \( B \)-block containing \( a_1 = B_{2-1(j)} = (t', k', j_0 - 1 \text{ mod } \nu_\nu, -2) \). Hence, \( f \) maps \( (t, k, i_0 - 1 \text{ mod } \nu_t, ε) \) onto \( (t', k', j_0 - 1 \text{ mod } \nu_\nu, ε) \) and \( f \) maps \( (t, k, i_0 - 1 \text{ mod } \nu_t, ε) \) onto \( (t', k', j_0 - 1 \text{ mod } \nu_\nu, ε) \) generally. By induction we get

\[ f: (t, k, i_0 - u \text{ mod } \nu_t, ε) → (t', k', j_0 - u \text{ mod } \nu_\nu, ε) \] for all \( u = 0, \ldots, \nu_t - 1 \).
This characterizes the action of \( f \) on \( \mathcal{M}'_v \), in particular, \( f(\mathcal{M}'_v) \subseteq \mathcal{M}'_{v'} \). Conversely, \( f^{-1} \) maps \( B_j \) onto \( B_i \). By the reasoning, analogous to this, which has been already done, we come to \( f^{-1}(\mathcal{M}'_v) \subseteq \mathcal{M}'_v \). Consequently, \( f(\mathcal{M}'_v) = \mathcal{M}'_{v'} \), and therefore \( t' = t \) since \( f \) is a bijection. It provides that \( f \) preserves the set \( \mathcal{M}'_v \). We define the map \( \alpha \in S_{m_t} \) associated with \( f \upharpoonright \mathcal{M}'_v \) by the formula

\[
\alpha: k \mapsto k' \text{ iff } f(\mathcal{M}'_v) = \mathcal{M}'_{v'}
\]

for all \( k, k' \leq m_t \). Set \( v^0_k = j_0 - i_0 \mod \nu_t \). Finally, the formula for \( f \) is the following:

\[
f: (t, k, i, \varepsilon) \mapsto (t, \alpha(k), i + v^0_k \mod \nu_t, \varepsilon) \text{ for all } i = 0, \ldots, \nu_t - 1.
\]

(ii): Based on Lemma 4.5, \( g_0f \) is an automorphism of \( \mathcal{M} \), which maps the \( B \)-blocks onto the \( B \)-blocks. Then, from Lemma 4.7(i), \( g_0f = f(v, \alpha) \) for some \( v \in \times_{t=1}^{\nu_t} S_{m_t} \) and \( \alpha \in \times_{t=1}^{t} S_{m_t} \), and thus \( f = g^{-1}_0f(v, \alpha) \). Note that \( g^{-1}_0 = g(1, \text{id}) \). Consequently, \( f = g(1, \text{id})f(v, \alpha) = g_0f(v, \alpha) = g(v+1, \alpha) \). What is more, \( f \) preserves the set \( \mathcal{M}'_v \), that follows directly from (4.3).

Now we characterize automorphisms of \( \mathcal{M}(n, \varphi) \), which can be uniquely decomposed into two mutually inscribed \( n \)-simplices.

**Theorem 4.8.** Let \( \mathcal{M} = \mathcal{M}(n, \varphi) \) and \( 1 \leq \nu_1 < \ldots < \nu_r \) be the lengths of the cycles in the cycle decomposition of \( \varphi \in S_n \). Assume that either \( n = 4 \) and \( \varphi \neq \text{id} \) contains no cycle of length 2, or \( n \geq 5 \). Then \( \text{Aut}(\mathcal{M}) \cong \bigoplus_{i=1}^{r} \left( C_{2\nu_i} \rtimes S_{m_i} \right) \).

**Proof.** Let \( F \) be an automorphism of \( \mathcal{M} \). By Proposition 3.3, there is no special decomposition of \( \mathcal{M} \). Thus, \( F \) either interchanges \( A(\mathcal{M}) \) with \( B(\mathcal{M}) \) or preserves each of them. According to Lemma 4.7 there is \( v_0 \in \times_{t=1}^{\nu_t} C_{m_t} \) and \( \alpha_0 \in \times_{t=1}^{t} S_{m_t} \) such that \( F = f(v_0, \alpha_0) \) or \( F = g(v_0, \alpha_0) = g_0f(v_0, \alpha_0) \). Furthermore, every \( f(v, \alpha), g(v, \alpha) \) with \( v \in \times_{t=1}^{\nu_t} C_{m_t} \) and \( \alpha \in \times_{t=1}^{t} S_{m_t} \) is a bijection. It provides that \( f \) is a bijection. It provides that \( \alpha \) is a bijection. It provides that \( f \) preserves the set \( \mathcal{M}'_v \), we can restrict the proof to the one fixed set \( \mathcal{M}'_v \). Thus, we assume that \( i = 0, \ldots, \nu_t - 1, k \leq m_t \). For the simplicity of the notation, we will write \( (\alpha(k), i + v_\alpha(k), \varepsilon) \) instead of \( (t, \alpha(t(k)), i + v_\alpha(t(k)) \mod \nu_t, \varepsilon) \). Moreover, we identify \( f(v, \alpha) \) with \( f(v, \alpha) \upharpoonright \mathcal{M}'_v \) and \( g(v, \alpha) \upharpoonright \mathcal{M}'_v \), so we assume \( v \in C_{m_t} \), \( \alpha \in S_{m_t} \). Let \( w \in C_{m_t} \), \( \beta \in S_{m_t} \) and note that

\[
f_{(w, \beta)}f(v, \alpha)((k, i, \varepsilon)) = f_{(w, \beta)}((\alpha(k), i + v_k, \varepsilon)) = (\beta(\alpha(k), i + v_k + w_\alpha(k), \varepsilon)) \]

Let \( \phi^\alpha: S_{m_t} \to \text{Aut}(C_{m_t}) \) be the map defined by

\[
\phi^\alpha: (v_1, \ldots, v_{m_t}) \mapsto (v_{\alpha(1)}, \ldots, v_{\alpha(m_t)}).
\]

Then the formula for the composition of \( f(v, \alpha) \) and \( f_{(w, \beta)} \) is

\[
f_{(w, \beta)}f(v, \alpha) = f(v + \phi^\beta(w), \beta \alpha).
\]

It is not difficult to check that \( g_0 \) and \( f(v, \alpha) \) commute. Note also that

\[
g_0^z = \begin{cases} f(-\frac{z}{2}, \text{id}) & \text{if } z \text{ is even}, \\ g(-\frac{z-1}{2}, \text{id}) & \text{if } z \text{ is odd}. \end{cases}
\]
Let \( k' \leq m \). We introduce the family of maps
\[
g_{0_k}((k', i, \varepsilon)) = \begin{cases} g_0((k', i, \varepsilon)) & \text{if } k' = k, \\ (k', i, \varepsilon) & \text{otherwise.} \end{cases}
\]

Then the following equalities hold
\[
\begin{align*}
\{g_{0_k}^z : z = 0, \ldots, 2\nu_t - 1 \text{ and } z \text{ is even}\} &= \{f_{(v, \text{id})} : v_{k'} = 0 \text{ for } k' \neq k\}, \\
\{g_{0_k}^z : z = 0, \ldots, 2\nu_t - 1 \text{ and } z \text{ is odd}\} &= \{g_{(v, \text{id})} : v_{k'} = 0 \text{ for } k' \neq k\}.
\end{align*}
\]

Therefore, for each \( v \in C_{\nu_t} \) we have \( f_{(v, \text{id})} = g_{0_1}^z g_{02}^z \cdots g_{0_m}^z \), where all numbers \( z_k = 0, \ldots, 2\nu_t - 1 \) are even. Likewise \( g_{(v, \text{id})} = g_{01}^z g_{02}^z \cdots g_{0_m}^z \), where all numbers \( z_k \) are odd. Hence, for each \( F \in \text{Aut}(\mathcal{M}) \) there is \( v \in C_{\nu_t} \) and \( \alpha \in S_{m_t} \) such that \( F = f_{(v, \text{id})} f_{(0, \alpha)} \) or \( F = g_{(v, \text{id})} f_{(0, \alpha)} \). To complete the proof it suffices to determine the remaining compositions:
\[
\begin{align*}
f_{(v, \text{id})} f_{(w, \text{id})} &= f_{(w+v, \text{id})}, \\
f_{(0, \alpha)} f_{(0, \beta)} &= f_{(0, \beta \alpha)}, \\
f_{(v, \text{id})} f_{(0, \alpha)} &= f_{(v, \phi_\alpha(v), \alpha)}, \\
g_{(v, \text{id})} f_{(0, \alpha)} &= g_{0} f_{(v, \text{id})} f_{(0, \alpha)} = g_{(\phi_{0}(v), \alpha)}, \\
f_{(v, \text{id})} g_{(w, \text{id})} &= f_{(v, \text{id})} g_{(w, \text{id})} = g_{0} f_{(v, \text{id})} f_{(w, \text{id})} = g_{(w+v, \text{id})}, \\
g_{(v, \text{id})} g_{(w, \text{id})} &= g_{0} f_{(v, \text{id})} g_{(w, \text{id})} = g_{0}^2 f_{(w+v, \text{id})} = f_{(-1, \text{id})} f_{(w+v, \text{id})} = f_{(w+v-1, \text{id})}.
\end{align*}
\]

The Möbius \( n \)-pairs, which automorphism groups are not characterized by Theorem 4.8, admit a special decomposition. We say that an automorphism \( f \) of a Möbius \( n \)-pair \( \mathcal{M} \) yields a special decomposition of \( \mathcal{M} \) if \( f \) maps the pair \( \{S_A, S_B\} \) onto a distinct pair of mutually inscribed simplices.

**Theorem 4.9.** The automorphism group of \( \mathcal{M}_{(4, \varphi)} \) is isomorphic to
\[
\begin{align*}
\text{(i) } (C_4 \oplus S_2) \rtimes C_2 & \text{ if } \varphi \in S_4 \text{ contains precisely one cycle of length } 2, \\
\text{(ii) } (C_4^2 \times S_2) \rtimes C_2 & \text{ if } \varphi \in S_4 \text{ contains two cycles of length } 2.
\end{align*}
\]

**Proof.** In view of Theorem 4.2, without loss of generality we can consider \( \mathcal{M}_1 = \mathcal{M}_{(4, \varphi_1)} \) with \( \varphi_1 = (1)(2)(34) \) in case (i), and \( \mathcal{M}_2 = \mathcal{M}_{(4, \varphi_2)} \) with \( \varphi_2 = (12)(34) \) in case (ii) (comp. Figures 4, 5). Let \( F_s \in \text{Aut}(\mathcal{M}_s) \) for \( s = 1, 2 \). By Corollary 3.4, there is the special decomposition of each of \( \mathcal{M}_s \). Thus, \( F_s \) maps the pair \( \{S_A, S_B\} \) onto \( \{S_A, S_B\} \) or \( F_s \) yields the special decomposition of \( \mathcal{M}_s \). In case \( F_s \) maps the pair \( \{S_A, S_B\} \) onto \( \{S_A, S_B\} \), by Lemma 4.7, there is \( v_0 \in \{0\} \times \{0\} \times C_2, \alpha_0 \in S_2 \times \{\text{id}\} \) for \( \mathcal{M}_1 \), or \( v_0 \in C_2 \times C_2, \alpha_0 \in S_2 \) for \( \mathcal{M}_2 \) such that \( F_s = f_{(v_0, \alpha_0)} \) or \( F_s = g_{(v_0, \alpha_0)} \), respectively for \( s = 1, 2 \). By Lemma 4.4 and Corollary 4.6 all maps \( F_s f_{(v, \alpha)}, F_s g_{(v, \alpha)} \), where \( v \in \{0\} \times \{0\} \times C_2 \) and \( \alpha \in S_2 \times \{\text{id}\} \) if \( s = 1 \), or \( v \in C_2 \times C_2, \alpha \in S_2 \) if \( s = 2 \), are automorphisms of \( \mathcal{M}_s \) preserving the pair \( \{S_A, S_B\} \). Based on the proof of Theorem 4.8, these maps form the group \( C_4 \oplus S_2 \) if \( s = 1 \), and the group \( C_4^2 \times S_2 \) if \( s = 2 \).
Consider the following two transformations:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{f}(x)$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$a_4$</td>
<td>$a_3$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$b_3$</td>
<td>$b_4$</td>
</tr>
<tr>
<td>$\tilde{f}(x)$</td>
<td>$a_2$</td>
<td>$a_1$</td>
<td>$b_3$</td>
<td>$b_4$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$a_3$</td>
<td>$a_4$</td>
</tr>
</tbody>
</table>

The map $\tilde{f}$ is an automorphism, which yields a special decomposition of $M_1$; and $\hat{f}$ is an automorphism, which yields a special decomposition of $M_2$. Assume that $F_s$ yields a special decomposition of $M_s$. Then $F_1 = \hat{f}F_1'$ and $F_2 = \tilde{f}F_2'$, where $F_s'$ is the automorphism of $M_s$ given by (4.2) or (4.3).

Let us set the commutativity rules in the automorphism group of $M_s$. By (4.1), the points of $M_1$, $M_2$ correspond to the sequences $(t, i, k, \varepsilon)$ with $\varepsilon = 1, -1$. Using the convention introduced at the beginning of this paragraph we get $t = 1, 2$, $\nu_1 = 1$, $\nu_2 = 2$, $m_1 = 2$, $m_2 = 1$ and $X_1^1 = \{1\}$, $X_2^1 = \{2\}$, $X_1^2 = \{3, 4\}$ for $M_1$; $t = 1$, $\nu_1 = 2$, $m_1 = 2$, and $X_1^2 = \{1, 2\}$, $X_2^2 = \{3, 4\}$ for $M_2$. To avoid any misunderstanding, in case $M_2$ we will write $Y_1^2, Y_2^2$ instead of $X_1^2, X_2^2$ respectively. Then $\tilde{f}$ maps the points of $M_1$ by the formula:

$$\tilde{f}((t, k, i, \varepsilon)) = \begin{cases} (t, k, i, -\varepsilon) & \text{for } i + \mu^t_k \in X_1^1, X_2^1, \\ (t, k, i + 1 \mod 2, \varepsilon) & \text{for } \varepsilon = 1, i + \mu^t_k \in X_1^2, \\ (t, k, i, \varepsilon) & \text{for } \varepsilon = -1, i + \mu^t_k \in X_2^1. \end{cases} \quad (4.4)$$

The map $\hat{f}$ can be defined on points of $M_2$ as:

$$\hat{f}((t, k, i, \varepsilon)) = \begin{cases} (t, k, i + 1 \mod 2, \varepsilon) & \text{for } \varepsilon = 1, i + \mu^t_k \in Y_1^2, \\ (t, k, i, \varepsilon) & \text{for } \varepsilon = -1, i + \mu^t_k \in Y_1^2, \\ (t, k, i, -\varepsilon) & \text{for } i + \mu^t_k \in Y_2^2. \end{cases} \quad (4.5)$$

Note, that $\tilde{f}^2 = \hat{f}^2 = \text{id}$. Hence the cyclic group generated by $\tilde{f}$ and the cyclic group generated by $\hat{f}$ both coincide with $C_2$. All the formulas for compositions of $\tilde{f}$ with $g_0$, and $\hat{f}$ with $f = f_{(v, \alpha)}$ can be calculated using (4.4) and (4.5) (it is rather technical and thus omitted) and then we get

$$\tilde{f}f = f\hat{f} \quad \text{and} \quad \tilde{f}g_0 = \hat{f}g_{0, \text{id}} = g_{(\tau_{(0,0,1)}(0), \text{id})}\tilde{f},$$

where $\tau_{(0,0,1)}(v) = \tau_{(0,0,1)}((v_1^1, v_2^1, v_1^2)) = (v_1^1, v_2^1, v_2^1 + 1)$. Analogous calculation can be done for $\hat{f}$. If we set $\tau_{(1,1)}(v) = \tau_{(1,1)}((v_1^1, v_2^1)) = (v_1^1 + 1, v_2^1 + 1)$, then

$$\hat{f}g_0 = \hat{f}g_{(0, \text{id})} = g_{(\tau_{(1,1)}(0), \text{id})}\hat{f},$$

$$\tilde{f}f_{(v, \alpha)} = f_{(v, \alpha)}\hat{f} \quad \text{if only} \quad \alpha = \text{id},$$

$$\tilde{f}f_{(v, \alpha)} = g_0f_{(v, \alpha)}\hat{f} = g_{(v, \alpha)}\hat{f} \quad \text{provided that} \quad i + \mu^t_k \in Y_1^2 \text{ and } \alpha = (12),$$

$$\tilde{f}f_{(v, \alpha)} = g_0^{-1}f_{(v, \alpha)}\hat{f} = g_{(\tau_{(1,1)}(v), \alpha)}\hat{f} \quad \text{as long as} \quad i + \mu^t_k \in Y_2^2 \text{ and } \alpha = (12).$$

□
References


