One-point extensions in $n_3$ configurations

William L. Kocay *

Computer Science Department and St. Paul’s College,
University of Manitoba, Winnipeg, Manitoba, R3T 2N2 Canada

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Abstract

Given an $n_3$ configuration, a 1-point extension is a technique that constructs an $(n+1)_3$ configuration from it. It is proved that all $(n+1)_3$ configurations can be constructed from an $n_3$ configuration using a 1-point extension, except for the Fano, Pappus, and Desargues configurations, and a family of Fano-type configurations. A 3-point extension is also described. A 3-point extension of the Fano configuration produces the Desargues and anti-Pappian configurations.

The significance of the 1-point extension is that it can frequently be used to construct real and/or rational coordinatizations in the plane of an $(n+1)_3$ configuration, whenever it is geometric, and the corresponding $n_3$ configuration is also geometric.

Keywords: Fano configuration, Pappus, Desargues, $(n, 3)$-configuration.

Math. Subj. Class.: 51E20, 51E30

1 Projective Configurations

A projective configuration consists of a set $\Sigma$ of points and lines, and an incidence relation $\Pi$, such that two lines intersect in at most one point. We denote this by $(\Sigma, \Pi)$. For example, a triangle with points $A, B, C$ and lines $a, b, c$ can be represented by the pair $\{A, B, C, a, b, c\}, \{Ab, Ac, Ba, Bc, Ca, Cb\}$. A configuration $(\Sigma, \Pi)$ can also be viewed as a bipartite incidence graph of points versus lines. We will always assume that the incidence graph of a configuration is connected. Excellent references on configurations are the recent books by Grünbaum [7], and by Pisanski and Servatius [11].

An $n_3$-configuration is a projective configuration with $n$ points and $n$ lines such that every line is incident on 3 points, and every point is incident on 3 lines. There is a unique $7_3$-configuration, the Fano configuration, and a unique $8_3$-configuration, the Möbius-Kantor

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E-mail address: bkocay@cs.umanitoba.ca (William L. Kocay)

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configuration. In 1887, Martinetti [10] presented a method to construct the $(n+1)_3$ configurations from the $n_3$ configurations. This is described in [7, 6]. Boben [1, 2] has analysed and extended Martinetti’s construction significantly. Important related work has also been done by Carstens, Dinski and Steffen [4]. See also [12]. A recent paper [13] by Stokes studies extensions of configurations in a very general setting. The 1-point extension presented here can be related to Stokes’s construction, but does not follow directly from it.

An $n_3$ configuration which can be represented by a collection of points and straight lines in the real or rational plane, such that all incidences are respected, and no two points or two lines coincide, and no unwanted incidences occur, is termed a geometric $n_3$ configuration. In order to show that an $n_3$ configuration is geometric, the usual method is to assign suitable homogeneous coordinates to its points and lines. We call this a coordinatization of the configuration. Some $n_3$ configurations are not geometric configurations, although it is currently an unsolved problem to determine which $n_3$ configurations are geometric.

The purpose of this paper is to present a theorem, the 1-point extension theorem, which describes another method to construct an $(n+1)_3$-configuration from an $n_3$-configuration; and to characterize which configurations can be obtained in this way. The significance of this construction is that if the $n_3$ configuration is geometric, with a given coordinatization, then there is usually a simple method to extend the coordinatization to the $(n+1)_3$ configuration, that is, the $(n + 1)_3$ configuration will also be geometric. This is too long to include here, it will be the subject of another paper, currently in preparation [8].

In particular the following theorem is proved.

**Theorem 1.1.** Let $(\Sigma, \Pi)$ be an $(n+1)_3$-configuration. Then $(\Sigma, \Pi)$ can be constructed by a 1-point extension from an $n_3$-configuration if and only if $(\Sigma, \Pi)$ is not one of the following configurations:

a) the Fano configuration,

b) the Pappus configuration,

c) the Desargues configuration,

d) a Fano-type configuration (to be described).

We begin with the idea of a 1-point extension in an $n_3$-configuration.

**Theorem 1.2. (1-Point Extension)** Let $(\Sigma, \Pi)$ be an $n_3$-configuration. Let $a_1, a_2, a_3$ be 3 distinct points in $\Sigma$, and let $\ell_1, \ell_2, \ell_3$ be 3 distinct lines in $\Sigma$ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$ and $a_3 \in \ell_3$, where $a_3 \notin \ell_1$. We can represent this in tabular form as

$$(\Sigma, \Pi) \begin{array}{cccc} \ell_1 & \ell_2 & \ell_3 & \cdots \\ a_1 & a_1 & a_2 & \cdots \\ \cdot & a_2 & a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}$$

where the dots indicate other points of the configuration. Let $\ell'$ be the third line containing $a_1$. Suppose further that if $\ell' \cap \ell_3 \neq \emptyset$, then $\ell' \cap \ell_3 = a_3$. Construct a new configuration $(\Sigma', \Pi')$ as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where $a_0$ is a new point and $\ell_0$ is a new line. $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3\} \cup \{a_1\ell_3, a_2\ell_0, a_3\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

$$(\Sigma', \Pi') \begin{array}{cccc} \ell_1 & \ell_2 & \ell_3 & \cdots \\ a_1 & a_1 & a_2 & \cdots \\ \cdot & a_2 & a_3 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}$$

where the dots indicate other points of the configuration. Let $\ell'$ be the third line containing $a_1$. Suppose further that if $\ell' \cap \ell_3 \neq \emptyset$, then $\ell' \cap \ell_3 = a_3$. Construct a new configuration $(\Sigma', \Pi')$ as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where $a_0$ is a new point and $\ell_0$ is a new line. $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, a_3\ell_3\} \cup \{a_1\ell_3, a_2\ell_0, a_3\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as
The Fano configuration can be represented by the following table.

Example 1.3. The Fano configuration can be represented by the following table.

<table>
<thead>
<tr>
<th>Fano</th>
<th>( \ell_1 )</th>
<th>( \ell_2 )</th>
<th>( \ell_3 )</th>
<th>( \ell_4 )</th>
<th>( \ell_5 )</th>
<th>( \ell_6 )</th>
<th>( \ell_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Choose \( \ell_1, \ell_2, \ell_3 \) as indicated, and choose \( a_1 = 2, a_2 = 3, a_3 = 6 \), and let \( a_0 = 8 \). Notice that the third line containing \( a_1 \) is \( \ell' = \ell_6 \), which intersects \( \ell_3 \) in \( a_3 = 6 \). Then by Theorem 1.2, the following table represents an 8₃-configuration, which is known to be unique.

<table>
<thead>
<tr>
<th>8₃-config</th>
<th>( \ell_0 )</th>
<th>( \ell_1 )</th>
<th>( \ell_2 )</th>
<th>( \ell_3 )</th>
<th>( \ell_4 )</th>
<th>( \ell_5 )</th>
<th>( \ell_6 )</th>
<th>( \ell_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

The 8₃-configuration can be viewed as a double cover of the cube [9]. It is possible to apply a 1-point extension to this configuration in two possible ways, resulting in two distinct 9₃-configurations. The third 9₃-configuration, known as the Pappus configuration, cannot be obtained in this way.

The 1-point extension theorem can be illustrated by the diagram of Figure 1. In \( (\Sigma, \Pi) \), we have a substructure consisting of 3 points \( a_1, a_2, a_3 \), and 3 lines, \( \ell_1, \ell_2, \ell_3 \), sequentially incident, forming a self-dual substructure contained in the \( n_3 \)-configuration. After the extension, we find that \( (\Sigma', \Pi') \) contains a triangle with vertices \( a_1, a_2, a_3 \) and sides \( \ell_2, \ell_3, \ell_0 \), where the third point on \( \ell_0 \) is \( a_3 \), and the third line through \( a_0 \) is \( \ell_1 \). This is again a self-dual substructure in the configuration.
Corollary 1.4. In $(\Sigma', \Pi')$, the third line through $a_1$ does not intersect $\ell_1$; the third point on $\ell_3$ is not collinear with $a_3$; and the third line through $a_2$ does not intersect $\ell_2$.

Proof. If there were a line $\ell$ in $(\Sigma', \Pi')$ through $a_1$ which intersected $\ell_1$ in a point $u$, then in $(\Sigma, \Pi)$, $\ell$ would intersect $\ell_1$ in $u$ and $a_1$, which is impossible. If there were a point $x$ in $(\Sigma', \Pi')$ on $\ell_3$ collinear with $a_3$, then the line $\ell$ containing $a_3$ and $x$ would also be a line in $(\Sigma, \Pi)$, where it would intersect $\ell_3$ in two points. Finally, if there were a line $\ell$ in $(\Sigma', \Pi')$ through $a_2$ which intersected $\ell_2$ in a point $u$, then in $(\Sigma, \Pi)$, $\ell$ would intersect $\ell_2$ in $a_2$ and $u$, which is impossible. \hfill \Box

The purpose of this paper is to characterize the configurations that can be obtained using 1-point extensions. In practice, the 1-point extensions are very easy to find and apply, and can easily be done by computer. However, the characterization of which configurations can be obtained by them is very long and tedious. We shall refer to the Fano, Pappus, and Desargues configurations, illustrated in Figure 1.1.

![Figure 1: A 1-point extension with 3 points](image1)

Corollary 1.4 will be used frequently in the characterization. We state them here. We are concerned with an ordered triangle, denoted $\Delta(i, j, k)$, where $i, j$ and $k$ are the first, second, and third vertices, respectively, of the triangle. The line containing $i$ and $j$ is denoted $\ell_{ij}$, etc.

Definition 1.5. Let $(\Sigma, \Pi)$ be a configuration containing an ordered triangle $\Delta(i, j, k)$. We define the following 3 conditions:

A) The third line through $k$ intersects $\ell_{ij}$;
B) The third line through \( i \) intersects the third line through \( j \);

C) The third point on \( \ell_{ik} \) is collinear with the third point on \( \ell_{jk} \).

The definition is illustrated in Figure 3.

![Figure 3: Conditions A, B and C for triangle \( \Delta(i, j, k) \)](image)

**Theorem 1.6.** Let \((\Sigma', \Pi')\) be an \((n + 1)_3\)-configuration containing a triangle \( \Delta \). If conditions A, B and C do not apply to some ordering of the triangle, then \((\Sigma', \Pi')\) can be derived from an \(n_3\)-configuration by a 1-point extension.

**Proof.** Let the ordered triangle to which conditions A, B and C do not apply be \( \Delta(a_0, a_1, a_2) \), and let the sides of the triangle be \( \ell_0, \ell_2, \ell_3 \), where \( a_0 = \ell_0 \cap \ell_2 \), \( a_1 = \ell_2 \cap \ell_3 \), \( a_2 = \ell_3 \cap \ell_0 \). Let \( a_3 \) be the third point on \( \ell_0 \), and let \( \ell_1 \) be the third line through \( a_0 \). Observe that \( a_3 \not\in \ell_1 \). These incidences are characterized by the following table.

\[
\begin{array}{cccc}
(\Sigma, \Pi) & \ell_0 & \ell_1 & \ell_2 & \ell_3 \\
& a_2 & a_0 & a_1 & a_1 \\
& a_3 & a_0 & a_2 & \\
& a_0 & . & . & . \\
\end{array}
\]

We can then delete \( a_0 \) and \( \ell_0 \), and change the incidences to the following.

\[
\begin{array}{ccc}
(\Sigma', \Pi') & \ell_1 & \ell_2 & \ell_3 \\
& a_1 & a_1 & a_2 \\
& . & a_2 & a_3 \\
& . & . & . \\
\end{array}
\]

Call the result \((\Sigma', \Pi')\). If \( \ell \) is the third line through \( a_2 \) in \((\Sigma, \Pi)\), then since condition A does not apply, we know that in \((\Sigma', \Pi')\), \( \ell \) and \( \ell_2 \) intersect in just one point. If \( \ell \) is the third line through \( a_1 \) in \((\Sigma, \Pi)\), then since condition B does not apply, we know that in \((\Sigma', \Pi')\), \( \ell \) and \( \ell_1 \) intersect in just one point, \( a_1 \). Since \( \ell \cap \ell_3 = a_1 \) in \((\Sigma, \Pi)\), it follows that in \((\Sigma', \Pi')\), if \( \ell \) and \( \ell_3 \) intersect, they intersect in \( a_3 \).

If \( \ell \) is any line other than \( \ell_0 \) through \( a_3 \) in \((\Sigma, \Pi)\), then since condition C does not apply, we know that in \((\Sigma', \Pi')\), \( \ell \) and \( \ell_3 \) intersect in just one point. The result is an \(n_3\)-configuration to which Theorem 1.2 applies.

\( \square \)
Given an ordered triangle $\Delta(i, j, k)$, the dual is an ordered triangle whose sides are lines which can be denoted $i', j', k'$. The dual of condition $A$ is that the third point on $k'$ is collinear with $i' \cap j'$. But this is just condition $A$ again applied to the triangle $\Delta(i' \cap k', j' \cap k', i' \cap j')$. So condition $A$ is self-dual. The dual of condition $B$ is that the third point on $i'$ is collinear with the third point on $j'$. This is just condition $C$ applied to the triangle $\Delta(i' \cap k', j' \cap k', i' \cap j')$. So $B$ and $C$ are dual conditions.

Theorem 1.6 is the main tool which we will use to characterize the extensions. We will find all configurations such that at least one of conditions $A$, $B$, and $C$ apply to every ordering of every triangle. We will also need longer cycles than triangles.

2 The General Extension Theorem

Before beginning the characterization of the $n_3$-configurations that can be obtained by 1-point extensions, we generalize Theorem 1.2 to $m$ points and $m$ lines, sequentially incident.

Theorem 2.1. (General 1-Point Extension) Let $(\Sigma, \Pi)$ be an $n_3$-configuration. Let $a_1, a_2, \ldots, a_m$ be $m$ distinct points in $\Sigma$, where $3 \leq m \leq n$, and let $\ell_1, \ell_2, \ldots, \ell_m$ be $m$ distinct lines in $\Sigma$ such that $a_1 = \ell_1 \cap \ell_2$, $a_2 = \ell_2 \cap \ell_3$, \ldots, $a_{m-1} = \ell_{m-1} \cap \ell_m$, and $a_m \in \ell_m$.

Suppose that $a_{m-1}, a_m \notin \ell_1, \ell_2$, and that $a_i \notin \ell_{i+3}$, where $i = 1, 2, \ldots, m-3$. We can represent this in tabular form as

\[
(\Sigma, \Pi) \begin{array}{cccccc}
\ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\
 a_1 & a_2 & a_3 & \ldots & a_{m-2} & a_{m-1} \\
 \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
 a_m & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\]

where the dots indicate other points of the configuration. Let $\ell_i'$ be the third line containing $a_i$, where $1 \leq i \leq m-2$. Suppose further that if $\ell_i' \cap \ell_{i+2} \neq \emptyset$, then $\ell_i' \cap \ell_{i+2} = a_{i+2}$. Construct a new configuration $(\Sigma', \Pi')$ as follows. $\Sigma' = \Sigma \cup \{a_0, \ell_0\}$ where $a_0$ is a new point and $\ell_0$ is a new line. $\Pi' = \Pi - \{a_1\ell_1, a_2\ell_2, \ldots, a_m\ell_m\} \cup \{a_1\ell_3, a_2\ell_4, \ldots, a_{m-2}\ell_m, a_{m-1}\ell_0, a_m\ell_0, a_0\ell_0, a_0\ell_1, a_0\ell_2\}$. We can represent this in tabular form as

\[
(\Sigma', \Pi') \begin{array}{cccccc}
\ell_0 & \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_{m-1} & \ell_m \\
 a_{m-1} & a_0 & a_0 & a_1 & \ldots & a_{m-3} & a_{m-2} \\
 a_m & \cdot & a_1 & a_2 & \ldots & a_{m-2} & a_{m-1} \\
 a_0 & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot
\end{array}
\]

Here the dots represent exactly the same points as in the previous table. Then $(\Sigma', \Pi')$ is an $(n+1)_3$-configuration.

Proof. The only incidences in which $(\Sigma', \Pi')$ and $(\Sigma, \Pi)$ differ are those involving $\ell_0, \ell_1, \ell_2, \ldots, \ell_m$. It is easy to verify from the tables that each of $a_1, a_2, \ldots, a_m$ occurs in exactly 3 lines in both $(\Sigma', \Pi')$ and $(\Sigma, \Pi)$, and that $a_0$ also occurs in exactly 3 lines. We must still verify that any two lines of $(\Sigma', \Pi')$ intersect in at most one point. Notice that $\ell_0$ intersects $\ell_1$ and $\ell_2$ in exactly one point, since $a_{m-1}, a_m \notin \ell_1, \ell_2$. It does not intersect $\ell_3, \ldots, \ell_{m-1}$, and it intersects $\ell_m$ in exactly one point.

Let $\ell \neq \ell_1, \ell_2, \ldots, \ell_m$ be a line of $(\Sigma, \Pi)$. If $\ell$ intersects $\ell_1$ in $(\Sigma, \Pi)$, then in $(\Sigma', \Pi')$, it intersects $\ell_1$ in either 0 or 1 point. If $\ell$ intersects $\ell_2$ in $(\Sigma, \Pi)$, then in $(\Sigma', \Pi')$, it intersects
Corollary 2.2. Let \((\Sigma, \Pi)\) and \((\Sigma', \Pi')\) be as in Theorem 2.1, so that \(C = (a_0, \ell_2, a_1, \ell_3, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0)\) is an ordered cycle in \((\Sigma', \Pi')\). Then in \((\Sigma', \Pi')\):

i) the third points of \(\ell_m\) and \(\ell_0\) are not collinear;

ii) the third point on \(\ell_i\) is not contained in the third line through \(a_i\), for \(i = 2, \ldots, m - 1\);

iii) the third lines through \(a_0\) and \(a_1\) do not intersect.

\(\ell_2\) in either 0 or 1 point. Suppose that \(\ell\) intersects \(\ell_3\) in \((\Sigma, \Pi)\). If \(\ell = \ell_1\), then \(\ell \cap \ell_3 = a_3\) in \((\Sigma, \Pi)\) according to the condition of the theorem concerning \(\ell_1\). It follows that \(\ell \cap \ell_3 = a_1\) in \((\Sigma', \Pi')\). If \(\ell \neq \ell_1\), then \(\ell\) intersects \(\ell_3\) in either 0 or 1 point in \((\Sigma', \Pi')\). An identical argument holds if \(\ell\) intersects one of \(\ell_4, \ldots, \ell_m\) in \((\Sigma, \Pi)\).

Suppose that \(\ell\) does not intersect \(\ell_1\) in \((\Sigma, \Pi)\). Then it also does not intersect \(\ell_1\) in \((\Sigma', \Pi')\). Similarly, if \(\ell\) does not intersect \(\ell_2\) in \((\Sigma, \Pi)\), then it also does not intersect \(\ell_2\) in \((\Sigma', \Pi')\). Suppose that \(\ell\) does not intersect \(\ell_3\) in \((\Sigma, \Pi)\). Then in \((\Sigma', \Pi')\), it may intersect \(\ell_3\) only in \(a_1\). A similar argument holds if \(\ell\) does not intersect \(\ell_4, \ldots, \ell_m\).

Finally, let \(\ell_i\) and \(\ell_j\), where \(1 \leq i < j \leq m\), be two lines of \((\Sigma, \Pi)\). If \(j = i + 1\), then \(\ell_i\) and \(\ell_j\) intersect in one point in both \((\Sigma, \Pi)\) and \((\Sigma', \Pi')\). Suppose that \(j = i + 2\). If \(\ell_i \cap \ell_j = \emptyset\) in \((\Sigma, \Pi)\), then it is also \(\emptyset\) in \((\Sigma', \Pi')\). Now \(\ell_i \cap \ell_j \neq a_{i-1}\) in \((\Sigma, \Pi)\) (when \(i > 1\)), because of the hypothesis that \(a_k \notin \ell_{k+3}\). Also, \(\ell_i \cap \ell_j \neq a_i\), because \(\ell_{i+1}\) contains \(a_i\) and \(a_{i+1}\). It follows that \(|\ell_i \cap \ell_j|\) is the same in \((\Sigma, \Pi)\) and \((\Sigma', \Pi')\) when \(j = i + 2\). Suppose now that \(j \geq i + 3\). It is easy to see that \(|\ell_i \cap \ell_j| \leq 1\) in \((\Sigma', \Pi')\). This completes the proof of the theorem. \(\square\)

Theorem 2.1 is illustrated in Figure 4, with \(m = 4\). This general form of Theorem 2.1 is stated separately from Theorem 1.2, because the form with \(m = 3\) is simpler, and because we shall mostly only require Theorems 1.2 and 1.6 when characterizing extensions.

![Figure 4: A 1-point extension with 4 points](image_url)
Proof. The third point of \(\ell_0\) is \(a_m\). If there were a line \(\ell\) in \((\Sigma', \Pi')\) containing \(a_m\) and the third point of \(\ell_m\), then in \((\Sigma, \Pi), \ell\) and \(\ell_m\) would intersect in two points, which is impossible.

Let \(\ell\) be the third line through \(a_i\) in \((\Sigma', \Pi')\), for some \(i = 2, \ldots, m - 1\), and let \(u\) be the third point on \(\ell_i\). Suppose that \(u \not\in \ell\). In \((\Sigma', \Pi')\), \(a_i\) is contained in \(\ell_{i+1}\) and \(\ell_{i+2}\), but in \((\Sigma, \Pi), a_i\) is contained in \(\ell_i\) and \(\ell_{i+1}\). We then find that in \((\Sigma, \Pi), \ell \cap \ell_i = \{u, a_i\}\), which is impossible.

The third line through \(a_0\) is \(\ell_1\). Let \(\ell\) be the third line through \(a_1\). If \(\ell \cap \ell_1 = u\) in \((\Sigma', \Pi')\), then in \((\Sigma, \Pi), \ell_i \cap \ell_i = \{u, a_1\}\), which is impossible. □

Observe that a triangle is a set of three distinct points and lines that are cyclically incident. Similarly, we define a quadrangle to be a set of four distinct points and lines that are cyclically incident. We will also need conditions similar to \(A, B, C\) for quadrangles. An ordered quadrangle with vertices \(i, j, k, m\) is denoted \(\Box(i, j, k, m)\). In analogy with Definition 1.5 and Corollary 2.2, we make the following definition for a quadrangle.

Definition 2.3. Let \((\Sigma, \Pi)\) be a configuration containing an ordered quadrangle \(\Box(i, j, k, m)\). We define the following 4 conditions:

D) The third point on \(\ell_{im}\) is collinear with the third point on \(\ell_{km}\);

E) The third line through \(m\) intersects \(\ell_{jk}\);

F) The third line through \(k\) intersects \(\ell_{ij}\);

G) The third line through \(j\) intersects the third line through \(i\).

These conditions are illustrated in Figure 5.

![Figure 5: Conditions D, E, F, G for quadrangle \(\Box(i, j, k, m)\)](image)

The analog of Theorem 1.6 for general 1-point extensions is the following.

Theorem 2.4. Let \((\Sigma', \Pi')\) be an \((n + 1)_3\)-configuration containing an ordered cycle \(C = (a_0, \ell_2, a_1, \ell_3, a_2, \ell_4, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0), \) where \(m \geq 4\); \(a_0, a_1, \ldots, a_{m-1}\) are distinct points; and \(\ell_0, \ell_2, \ell_3, \ldots, \ell_{m-1}\) are distinct lines. Let \(\ell_1\) denote the third line containing \(a_0\) and let \(a_m\) denote the third point on \(\ell_0\). Suppose that \(\ell_1\) is distinct from \(\ell_0, \ell_2, \ell_3, \ldots, \ell_{m-1}\) and that \(a_2 \not\in \ell_1\). Let \(\ell_i'\) denote the third line containing \(a_i\), for \(i = 1, 2, \ldots, m - 1\). Suppose that \(\ell_i'\) does not contain the third point of \(\ell_i\), for \(i = 2, \ldots, m - 1\); that \(\ell_i' \cap \ell_i = \emptyset\); and that \(a_m\) is not collinear with the third point of \(\ell_m\). Then \((\Sigma', \Pi')\) can be derived from an \(n_3\)-configuration by a 1-point extension.

Proof. The incidences of the ordered cycle can be represented by the following table.
We can then delete $a_0$ and $\ell_0$, and change the incidences to the following.

$$
\begin{bmatrix}
\ell_1 & \ell_2 & \ell_3 & \cdots & \ell_{m-1} & \ell_m \\
a_1 & a_2 & a_3 & \cdots & a_{m-2} & a_{m-1} \\
. & . & . & \cdots & . & . \\
\end{bmatrix}
$$

Call the result $(\Sigma, \Pi)$. It is clear that each point of $(\Sigma, \Pi)$ is contained in exactly three lines. We have to show that any two lines intersect in at most one point in $(\Sigma, \Pi)$, and that $\ell_1, \ell_2, \ell_3, \ldots, \ell_m$ are distinct lines in $(\Sigma, \Pi)$. Any two of $\ell_1, \ell_2, \ldots, \ell_m$ intersect in at most one point because we began with an ordered cycle of distinct points, and because $a_2 \not\in \ell_1$. Let $\ell$ be any line not in this set. Suppose that $\ell$ intersects $\ell_i$ in two points, for some $i = 2, \ldots, m-1$. Now $\ell_i$ contains $a_{i-1}, a_i$ and a third point $z$. If $\ell$ contained $a_i$, then $\ell = \ell_i$, which does not intersect $\ell_i$ in $(\Sigma', \Pi')$, by assumption. Therefore $a_i \not\in \ell$. Otherwise $\ell$ must contain $a_{i-1}$ and $z$. But these points are in $\ell_i$ in $(\Sigma', \Pi')$, and $\ell$ is unchanged. It follows that $\ell$ intersects $\ell_2, \ldots, \ell_{m-1}$ in at most one point each.

Suppose that $\ell$ intersects $\ell_1$ in two points in $(\Sigma, \Pi)$. Now $\ell_1$ contains $a_1$ and two other points $u, v$. As $u$ and $v$ are both on $\ell_1$ in $(\Sigma', \Pi')$, it follows that $\ell$ does not contain both $u$ and $v$. Therefore $\ell = \ell'_1$. But by assumption, $\ell'_1 \cap \ell_1 = \emptyset$ in $(\Sigma', \Pi')$.

Suppose that $\ell$ intersects $\ell_m$ in two points in $(\Sigma, \Pi)$. The two points cannot be $a_{m-1}, a_m$, because these points occur on $\ell_0$ in $(\Sigma', \Pi')$. They cannot be $a_{m-1}$ and a third point $w$, because these points occur on $\ell_m$ in $(\Sigma', \Pi')$. And they cannot be $a_m$ and the third point $w$, because by assumption, $a_m$ is not collinear with the third point of $\ell_m$ in $(\Sigma', \Pi')$. We conclude that $(\Sigma, \Pi)$ is an $n_3$-configuration to which the conditions of Theorem 2.1 apply.

**Corollary 2.5.** Let $(\Sigma', \Pi')$ be an $(n+1)_3$-configuration containing a quadrangle $\square(i, j, k, m)$. If conditions $D, E, F$ and $G$ do not apply to some ordering of the quadrangle, and if the third line through $i$ does not contain $k$, then $(\Sigma', \Pi')$ can be derived from an $n_3$-configuration by a 1-point extension.

**Proof.** The conditions $D, E, F, G$, and $a_2 = k \not\in \ell_1$ are the conditions of Theorem 2.4 applied to an ordered quadrangle.

**Theorem 2.6.** Let $(\Sigma', \Pi')$ be an $(n + 1)_3$-configuration. If $(\Sigma', \Pi')$ does not contain a triangle, then it can be derived by a 1-point extension from an $n_3$-configuration.

**Proof.** Choose a cycle of smallest possible length in $(\Sigma', \Pi')$. Denote the cycle by

$$
(a_0, \ell_2, a_1, \ell_3, a_2, \ell_4, \cdots, a_{m-2}, \ell_m, a_{m-1}, \ell_0),
$$

where $m \geq 4$. Let $\ell_1$ be the third line containing $a_0$, and let $a_m$ be the third point on $\ell_0$. This can be denoted in tabular from by
\begin{align*}
(\Sigma, \Pi) & \quad \ell_0 \quad \ell_1 \quad \ell_2 \quad \ell_3 \quad \ldots \quad \ell_{m-1} \quad \ell_m \\
& \quad a_{m-1} \quad a_0 \quad a_0 \quad a_1 \quad \ldots \quad a_{m-3} \quad a_{m-2} \\
& \quad a_m \quad \cdot \quad a_1 \quad a_2 \quad \ldots \quad a_{m-2} \quad a_{m-1} \\
& \quad a_0 \quad \cdot \quad \cdot \quad \cdot \quad \ldots \quad \cdot \quad .
\end{align*}

Let \( \ell'_i \) denote the third line containing \( a_i \), where \( i = 1, 2, \ldots, m - 1 \). If \( \ell'_i \) were to intersect \( \ell_i \) in a point \( z \), where \( i = 2, \ldots, m - 1 \), this would create a triangle \( \Delta(a_{i-1}, a_i, z) \). If \( \ell'_1 \) were to intersect \( \ell_1 \) in a point \( u \), this would create a triangle \( \Delta(a_0, a_1, u) \). If \( a_m \) were collinear with the third point \( w \) of \( \ell_m \), this would create a triangle \( \Delta(a_{m-1}, a_m, w) \). If \( \ell_1 \) contained \( a_2 \), this would create a triangle \( \Delta(a_0, a_1, a_2) \). It follows that the conditions of Theorem 2.4 apply, so that \( (\Sigma', \Pi') \) can be derived by a 1-point extension from an \( n_3 \)-configuration.

\textbf{3 Fano-Type Configurations}

Let \( F \) denote the Fano configuration, the unique \( 7_3 \) configuration. We will use three sub-configurations to build a family of \( n_3 \) configurations which cannot be obtained by 1-point extensions.

\textbf{Definition 3.1.} Denote by \( F' \) the unique configuration obtained from \( F \) by removing a single incidence. Denote by \( F_\ell \) the unique configuration obtained from \( F \) by removing a line. Denote by \( F_p \) the unique configuration obtained from \( F \) by removing a point. Note that \( F_\ell \) and \( F_p \) are dual configurations.

The configurations \( F_\ell, F_p \) and \( F' \) are not \( n_3 \)-configurations. They can be used as building blocks of \( n_3 \) configurations, which we call Fano-type configurations. \( F' \) has one point on only two lines, and one line containing only two points. \( F_p \) has three lines containing only two points. Every point is in three lines. \( F_\ell \) has three points in only two lines. Every line contains three points. These are illustrated schematically in Figure 7, where the points missing a line are indicated as black circles, and the lines missing a point are indicated as lines.

These sub-configurations can be used as modules, which can be connected together like vertices of a graph, to create graphs representing \( n_3 \) configurations. For example, two or more copies of \( F' \) can be connected into a cycle or path of arbitrary length. If only \( F_\ell \) and \( F_p \) are used, the resulting structure is a bipartite graph.
Theorem 3.2. Let \( G \) be a multigraph which is isomorphic to either a cycle of length \( \geq 2 \), or a subdivision of a 3-regular bipartite multigraph, with bipartition \((X, Y)\). Replace each vertex of \( X \) by a configuration \( F_p \), replace each vertex of \( Y \) by a configuration \( F_\ell \), and replace each vertex of degree two by a configuration \( F' \). The result is an \( n_3 \) configuration which can not be obtained by a 1-point extension.

Proof. Refer to Figure 8, showing a cycle of length four, and a configuration constructed from the unique 3-regular bipartite multigraph on four vertices.

We must show that the \( n_3 \) configurations constructed like this cannot be obtained by a 1-point extension. Observe first that the Fano configuration \( F \) is a projective plane, so that every two points are contained in a line, and every two lines intersect in a point. Consequently, every triangle contained in \( F', F_\ell \) or \( F_p \) has an ordering which satisfies one of conditions \( A, B \) or \( C \). By Corollary 1.4, a Fano-type configuration cannot be obtained by a triangular 1-point extension (Theorem 1.2). Suppose that it can be obtained by a general 1-point extension (Theorem 2.1). By Corollary 2.2, there must be an ordered cycle \( C \) of length \( \geq 4 \) satisfying certain conditions. Let \( C = (a_0, \ell_2, a_1, \ell_3, \ldots, a_{m-2}, \ell_m, a_{m-1}, \ell_0) \) be as in Corollary 2.2, and let \( \ell'_i \) denote the third line containing \( a_i \), where \( i = 1, 2, \ldots, m-1 \). Let \( \ell_1 \) denote the third line containing \( a_0 \), and let \( a_m \) denote the third point on \( \ell_0 \). If \( C \) were contained within an \( F', F_\ell \) or \( F_p \), then \( C \) would have length 4, because any 5 points of \( F \) necessarily contain three collinear points. But in \( F', F_\ell \) or \( F_p \), every ordered quadrangle satisfies at least one of conditions \( D, E, F, G \), since the Fano configuration is a projective plane.

It follows that \( C \) is not contained within an \( F', F_\ell \) or \( F_p \). Consider the portion of \( C \) contained within some \( F', F_\ell \) or \( F_p \). It is a sequence of sequentially incident points and lines. Suppose first that it is contained within an \( F' \). Referring to Figure 6 we see that the
shortest possible portion of $C$ contained within an $F'$ is $(a_i, \ell_{i+2}, a_{i+1}, \ell_{i+3}, a_{i+2}, \ell_{i+4})$, for some $i = 0, 1, \ldots, m - 1$ where subscripts are reduced modulo $m$. If $a_{i+2} \neq a_0, a_1$, then $\ell_{i+2}'$ contains the third point of $\ell_{i+2}$, which is in $F'$. If $a_{i+2} = a_0$, then $a_{i+1} = a_{m-1}$ and $\ell_{i+2} = \ell_m$, so that $a_m$ is collinear in $F'$ with the third point of $\ell_m$. If $a_{i+2} = a_1$, then $a_{i+1} = a_0$, so that $\ell_1$ and $\ell_1'$ are in $F'$ and $\ell_1' \cap \ell_1 \neq \emptyset$. Thus, the conditions of Corollary 2.2 are never satisfied if a portion of $C$ is contained within an $F'$.

Suppose next that a portion of $C$ is contained within an $F_\ell$. Referring to Figure 6 we see that the shortest possible portion of $C$ contained within an $F_\ell$ is $(a_i, \ell_{i+2}, a_{i+1}, \ell_{i+3}, a_{i+2})$, for some $i = 0, 1, \ldots, m - 1$ where subscripts are reduced modulo $m$. If $a_{i+2} \neq a_0, a_1$, then $\ell_{i+2}'$ contains the third point of $\ell_{i+2}$, which is in $F_\ell$. If $a_{i+2} = a_0$, then $a_{i+1} = a_{m-1}$ and $\ell_{i+2} = \ell_m$, so that $a_m$ is collinear in $F_\ell$ with the third point of $\ell_m$. If $a_{i+2} = a_1$, then $a_{i+1} = a_0$, so that $\ell_1$ and $\ell_1'$ are in $F_\ell$ and $\ell_1' \cap \ell_1 \neq \emptyset$. Thus, the conditions of Corollary 2.2 are never satisfied if a portion of $C$ is contained within an $F_\ell$. A similar result holds for $F_p$, which is the dual of $F_\ell$. We conclude that the Fano-type configurations can not be obtained by a 1-point extension.

\section{The Characterization Theorem}

In this section we will assume that $(\Sigma, \Pi)$ is an $n_3$-configuration which cannot be derived by a 1-point extension. It follows from Theorem 2.6 that we can assume that $(\Sigma, \Pi)$ has a triangle. Let the points of $(\Sigma, \Pi)$ be numbered $1, 2, \ldots, n$. Without loss of generality, we can assume that $\Delta(2, 3, 1)$ is a triangle in $(\Sigma, \Pi)$. This is illustrated in Figure 9. It will be convenient to omit the commas and brackets from expressions like $\Delta(2, 3, 1)$, and write simply $\Delta_{231}$.

![Figure 9: Triangle $\Delta_{231}$ with condition A](image)

We divide the analysis into two cases according to whether or not $(\Sigma, \Pi)$ has a triangle satisfying condition $A$. The theorem obtained will be the following.

**Theorem 4.1.** If $(\Sigma, \Pi)$ is an $n_3$-configuration which cannot be obtained from a 1-point extension, then either:

1) $(\Sigma, \Pi)$ is one of the Fano, Pappus, or Desargues configurations; or

2) $(\Sigma, \Pi)$ is a Fano-type configuration.

**Proof.** The proof of this theorem is very long, involving an analysis of many possible cases.

**Case A.** $(\Sigma, \Pi)$ has a triangle satisfying condition $A$. 

Let the ordered triangle be $\Delta 231$, as above. Condition $A$ tells us that the third line through 1 intersects $\ell_23$. Call the point of intersection 4. This is shown in Figure 9. We will show that any $n_3$ configuration that cannot be obtained by a 1-point extension, and which satisfies Condition $A$, is either a Fano-type configuration, or the Fano configuration. Now consider $\Delta 142$. It currently does not satisfy conditions $A, B, \text{ or } C$. Since every triangle must satisfy at least one of these conditions, there are three possibilities, which we indicate by $\Delta 142A, \Delta 142B, \text{ and } \Delta 142C$. These are shown in Figure 10. In $\Delta 142A$, the third line through 4 intersects $\ell_{12}$ (in point 5). In $\Delta 142B$, the third lines through 1 and 4 intersect (in point 5). In $\Delta 142C$, the third points on $\ell_{12}$ (point 5) and $\ell_{24}$ (point 3) are collinear.

These three structures are easily seen to be isomorphic, by relabelling the points. Each structure is self-dual, having two points incident on 3 lines each, and two lines each containing 3 points. Thus, without loss of generality, we can assume that the subconfiguration $\Delta 142A$ exists in $(\Sigma, \Pi)$ in Case A. Consider triangle $\Delta 124$. It currently does not satisfy condition $A, B, \text{ or } C$. Since it must satisfy at least one of these conditions, there are three possibilities, which we indicate by $\Delta 142A\Delta 124A, \Delta 142A\Delta 124B, \text{ and } \Delta 142A\Delta 124C$. These are shown in Figure 11.

The structures $\Delta 142A\Delta 124B$ and $\Delta 142A\Delta 124C$ are duals of each other. The first has 6 points and 5 lines, while the other has 5 points and 6 lines. It can be verified by exhaustion that every ordered triangle in these structures satisfies at least one of conditions $A, B, \text{ or } C$.

**Case $\Delta 142A\Delta 124A$.**

Consider the quadrangle $\Box 6431$ in $\Delta 142A\Delta 124A$. It must satisfy at least one of
conditions $D, E, F, G$ (see Figure 5). Condition $D$ is possible only if $\ell_{25}$ intersects $\ell_{13}$. Condition $E$ is not possible. Condition $F$ is possible only if the third line through 3 intersects $\ell_{46}$. Condition $G$ is possible only if there is a line $\ell_{56}$. These cases are illustrated in Figure 12.

![Figure 12: $\triangle 142A\triangle 124A \square 6431D$, $\triangle 142A\triangle 124A \square 6431F$, $\triangle 142A\triangle 124A \square 6431G$](image)

Now the diagrams $\triangle 142A\triangle 124A \square 6431D$ and $\triangle 142A\triangle 124A \square 6431G$ are duals of each other, for the mapping which sends points 1, 2, 3, 4, 5, 6, 7 of $D$ to $\ell_{15}, \ell_{16}, \ell_{25}, \ell_{24}, \ell_{46}, \ell_{13}, \ell_{56}$ of $G$ is an isomorphism. Therefore we need only consider cases $D$ and $F$.

**Case $\triangle 142A\triangle 124A \square 6431D$.**

It can be verified that all triangles of the diagram satisfy one of conditions $A, B, C$. Consider the quadrangle $\square 3164$. Condition $D$ is only possible if point 7 lies on line $\ell_{46}$. Condition $E$ is not possible. Condition $F$ is only possible if there is a line $\ell_{67}$. Condition $G$ is only possible if there is a line $\ell_{35}$. These cases are illustrated in Figure 13.

![Figure 13: $\triangle 142A\triangle 124A \square 6431D \square 3164D$, $F$, and $G$](image)

**Case $\triangle 142A\triangle 124A \square 6431D \square 3164D$.**

It can be verified that every triangle satisfies at least one of conditions $A, B, C$, and every quadrangle satisfies at least one of conditions $D, E, F, G$. This configuration is isomorphic to the Fano configuration, with one line removed ($\ell_{356}$), which we denote as $F_{\ell}$. The dual configuration is the Fano configuration, with one point removed, which we denote as $F_{p}$.

**Case $\triangle 142A\triangle 124A \square 6431D \square 3164F$.**

Consider the quadrangle $\square 2376$. Condition $D$ requires that $\ell_{15}$ intersects $\ell_{67}$, which
is impossible. Condition $E$ requires that $\ell_{46}$ contains point 1, which is impossible. Condition $F$ requires that $\ell_{75}$ contains point 4, which is impossible. Condition $G$ requires a line $\ell_{35}$. The result is illustrated in Figure 14.

![Figure 14: Case $\Delta 142A\Delta 124A\square 6431D\square 3164F\square 2376G$](image)

We then consider quadrangle $\square 6237$. Condition $D$ requires that $\ell_{15}$ intersects $\ell_{67}$, which is impossible. Condition $E$ requires that $\ell_{75}$ contains point 4, which is impossible. Condition $F$ requires that $\ell_{35}$ contains point 1, which is impossible. Condition $G$ requires that $\ell_{46}$ and $\ell_{25}$ intersect in point 5, which is impossible. We conclude that case $\Delta 142A\Delta 124A\square 6431D\square 3164F$ is not possible.

**Case $\Delta 142A\Delta 124A\square 6431D\square 3164G$.**

Consider the quadrangle $\square 4316$. Condition $D$ requires that $\ell_{25}$ intersects $\ell_{46}$. The point of intersection can only be 7. Condition $E$ requires that $\ell_{75}$ contains point 6, which is impossible. Condition $F$ requires that $\ell_{15}$ contains point 2, which is impossible. Condition $G$ requires a line $\ell_{356}$. These cases are illustrated in Figure 15.

![Figure 15: Cases $\Delta 142A\Delta 124A\square 6431D\square 3164G\square 4316D$ and $G$](image)

These two configurations are easily seen to be isomorphic, by the permutation of the points given by $(2, 3, 4)(5, 6, 7)$, mapping $D$ onto $G$. They are both isomorphic to the Fano configuration, with one incidence removed, denoted by $F'$. Every triangle satisfies at least one of conditions $A, B, C$, and every quadrangle satisfies at least one of conditions $D, E, F, G$. 
Note that we can complete $F'$ to the Fano configuration, which can not be constructed by a 1-point extension.

We summarise Case A as follows:

Consider an $n_3$ configuration $(\Sigma, \Pi)$, where $n > 7$, which cannot be constructed by a 1-point extension. Every triangle satisfying condition $A$ is contained in a unique sub-configuration isomorphic to one of $F_\ell, F_p, \text{or } F'$.

**Case B.** $(\Sigma, \Pi)$ has no triangle satisfying condition $A$.

We begin with triangle $\Delta_{231}$. It must satisfy condition $B$ or $C$. These two possibilities are shown in Figure 16.

![Figure 16: $\Delta_{231}B$ and $\Delta_{231}C$](image)

These two structures are duals of each other. Hence we can assume without loss of generality that $(\Sigma, \Pi)$ contains the structure $\Delta_{231}B$.

Consider the triangle $\Delta_{123}$. It must satisfy condition $B$ or $C$. We must take these as two separate cases, Case $B\Delta_{123}B$ and Case $B\Delta_{123}C$. They are shown in Figure 17. It will be necessary to examine a great many subcases.

![Figure 17: Cases $B\Delta_{123}B$ and $B\Delta_{123}C$](image)

**Case $B\Delta_{123}B$.**

Consider triangle $\Delta_{132}$. There are two possibilities, cases $B\Delta_{123}B\Delta_{132}B$ and $B\Delta_{123}B\Delta_{132}C$, which must both be considered. They are shown in Figure 18.

**Case $B\Delta_{123}B\Delta_{132}B$.**

Consider triangle $\Delta_{243}$. There are two choices $B\Delta_{123}B\Delta_{132}B\Delta_{243}B$ and
\[ B\Delta 123B\Delta 132B\Delta 243C \]. They are shown in Figure 19. These structures both have 7 points \( \{1, 2, \ldots, 7\} \), so that a mapping from the first to the second can be denoted by a permutation. It is easy to see that the permutation \((1, 2, 3)(4, 6, 5)(7)\) maps the first to the second. Thus, without loss of generality, we can suppose that \((\Sigma, \Pi)\) contains the structure \(B\Delta 123B\Delta 132B\Delta 243B\).

Consider triangle \(\Delta 342\). There are two possibilities, \(B\Delta 123B\Delta 132B\Delta 243B\, \Delta 342B\) and \(B\Delta 123B\Delta 132B\Delta 243B\Delta 342C\). They are shown in Figure 20. We must consider both possibilities.

This is beginning to look remarkably like the Pappus configuration.

**Case B\Delta 123B\Delta 132B\Delta 243B\Delta 342B.**

Consider the quadrangle \(\Box 1248\). At least one of conditions \(D, E, F, G\) must be satisfied. Of these, it is only possible to satisfy condition \(E\), namely the third line
through 8 must intersect \( \ell_{24} \). The point of intersection can only be 5. Therefore the left diagram of Figure 21 must exist in \((\Sigma, \Pi)\).

![Figure 21: Cases B□1248E and B□1248E□7238E](image)

Consider the quadrangle \( \Box 7238 \). At least one of conditions \( D, E, F, G \) must be satisfied. Of these, it is only possible to satisfy condition \( E \), namely the third line through 8 must intersect \( \ell_{23} \). Therefore the right diagram of Figure 21 must exist in \((\Sigma, \Pi)\).

Consider the quadrangle \( \Box 3159 \). It is only possible to satisfy condition \( E \), namely the third line through 9 must intersect \( \ell_{15} \) in point 6. Therefore the following structure (Figure 22) must exist in \((\Sigma, \Pi)\).

![Figure 22: Case B□1248E□7238E□3159E](image)

Consider the quadrangle \( \Box 1347 \). It is only possible to satisfy condition \( E \), namely the third line through 7 must intersect \( \ell_{34} \). The point of intersection must be 6, so that \( \ell_{69} \) must be extended to include point 6. Consider next quadrangle \( \Box 1783 \). It is only possible to satisfy condition \( E \), namely the third line through 3 must intersect \( \ell_{78} \). The point of intersection must be 6, so that \( \ell_{78} \) must be extended to include point 6. Consider next quadrangle \( \Box 1745 \). It is

---

Case B\( \Delta 123B\Delta 132B\Delta 243B\Delta 342C \).

This case is illustrated in Figure 20. Consider the triangle \( \Delta 274 \). There are two possibilities, \( \Delta 274B \) and \( \Delta 274C \), shown in Figure 23. These are duals of each other. The mapping which sends the points 1, 2, . . . , 8 of \( \Delta 274B \) to the lines \( \ell_{15}, \ell_{25}, \ell_{34}, \ell_{32}, \ell_{12}, \ell_{13}, \ell_{58}, \ell_{47} \) of \( \Delta 274C \) is an isomorphism. Hence we only need to consider one of them, the first, say.

Consider the quadrangle \( \Box 1783 \). It is only possible to satisfy condition \( E \), namely the third line through 3 must intersect \( \ell_{78} \). The point of intersection must be 6, so that \( \ell_{78} \) must be extended to include point 6. Consider next quadrangle \( \Box 1745 \). It is
only possible to satisfy condition \( E \), namely the third line through 5 must intersect \( \ell_{47} \). The result is illustrated in Figure 24.

Finally, consider quadrangle \( \square 7138 \). It is only possible to satisfy condition \( E \), namely the third line through 8 must intersect \( \ell_{13} \). The point of intersection must be 9, so that \( \ell_{13} \) must be extended to include point 9. Once again we have the Pappus configuration.

**Case \( B\Delta 123B\Delta 132C \).**

This case is illustrated in Figure 18. Consider the triangle \( \Delta 267 \). There are two possible ways to satisfy condition \( B \), namely the third line through 6 could contain either 4 or 5. The first of these choices is illustrated in Figure 25. The second is not allowed, as it would create a triangle \( \Delta 125 \) satisfying condition \( A \). There are two possible ways to satisfy condition \( C \), namely \( \ell_{67} \) could intersect \( \ell_{13} \) or \( \ell_{34} \). Call these two results \( C_1 \) and \( C_2 \), respectively, also shown in Figure 25.

**Case \( B\Delta 123B\Delta 132C\Delta 267B \).**

Consider the quadrangle \( \square 1673 \). It is not possible to satisfy conditions \( D \) or \( F \). Condition \( E \) can only be satisfied if \( \ell_{34} \) intersects \( \ell_{67} \). Condition \( G \) can only be satisfied if \( \ell_{15} \) intersects \( \ell_{46} \). These cases are shown in Figure 26.

Now case \( G \) (the right diagram) leads to a contradiction, for consider the quadrangle \( \square 3167 \). Conditions \( E, F, G \) are not possible. Condition \( D \) is only possible if \( 5 \in \ell_{67} \). But this creates a triangle \( \Delta 156 \) satisfying condition \( A \), a contradiction. Therefore we consider case \( E \) (the left diagram). Consider the quadrangle \( \square 3761 \). Conditions \( D, F, G \) cannot be satisfied. Condition \( E \) can only be satisfied if \( \ell_{15} \) intersects \( \ell_{67} \) in point 8, as shown in Figure 27. Consider next the quadrangle \( \square 6137 \). Conditions
Figure 25: Cases $B\Delta 123B\Delta 132C\Delta 267 B, C_1, \text{ and } C_2$

Figure 26: Cases $B\Delta 123B\Delta 132C\Delta 267 B\square 1673 E \text{ and } G$

$D, F, G$ cannot be satisfied. Condition $E$ can only be satisfied if the third line through 7 intersects $\ell_{13}$ in a point 9, also illustrated in Figure 27.

Figure 27: Cases $E\square 1673E$ and $E\square 1673E\square 6137E$

Consider now the quadrangle $\square 2685$ in the right diagram of Figure 27. Conditions $D, F, G$ cannot be satisfied. Condition $E$ can only be satisfied if the third line through 5 contains point 7, which is only possible if $5 \in \ell_{79}$. The result is isomorphic to the diagram of Figure 24. Once again, we obtain the Pappus configuration.

Case $B\Delta 123B\Delta 132C\Delta 267C_1$.

Refer to Figure 25. Consider the quadrangle $\square 2784$. Conditions $D$ and $F$ cannot be satisfied. Condition $E$ can only be satisfied if there is a line $\ell_{46}$, which gives a result identical to the left diagram of Figure 26. Condition $G$ can only be satisfied if the third line through 7 intersects $\ell_{26}$ in point 1, but this creates a triangle $\Delta 127$
satisfying condition A, which is not allowed. This completes this case.

**Case B Δ123B Δ132C Δ267C2.**

Refer to Figure 25. Consider the quadrangle □1376. Conditions D, E, F are not possible. Condition G is only possible if \( \ell_{15} \) and \( \ell_{34} \) intersect, shown in Figure 28. Consider now the quadrangle □1872. Conditions D, E, F are not possible. Condition G is possible if \( \ell_{15} \) intersects the third line through 8. The point of intersection can be either 5 or 9, resulting in \( G_1 \) and \( G_2 \), also shown in Figure 28.

![Figure 28: Cases C2 □1376G, G □1872G1 and G □1872G2](image)

Consider the quadrangle □7218 in diagram G □1872G1. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if the third line through 7 intersects \( \ell_{24} \). The point of intersection can be 4 or 5. But 4 creates a triangle \( \Delta734 \) satisfying condition A, a contradiction. Therefore the intersection must be point 5, as shown in Figure 29. Then consider quadrangle □7812. Conditions D, E, F cannot be satisfied. Condition G can only be satisfied if \( \ell_{15} \) and \( \ell_{89} \) intersect, also shown in Figure 29. Next, consider quadrangle □1572. Conditions D, E, F, G cannot be satisfied, a contradiction. This completes this case, and also case B Δ123B Δ132C Δ267C2, and case B Δ123B Δ132C and case B Δ123B.

**Case B Δ123C.**
Refer to Figure 17. Consider the triangle $\triangle 132$. Condition $B$ can be satisfied if the third line through 1 intersects $\ell_{34}$. There are two ways this can occur – the intersection can be point 4, or a new point. This gives $B_1$ and $B_2$, shown in Figure 30. Condition $C$ can be satisfied if point 6 is collinear with the third point on $\ell_{12}$. There are two ways this can occur. The line through 6 intersecting $\ell_{12}$ can be $\ell_{56}$ or a new line. This gives $C_1$ and $C_2$, shown in Figure 31.

![Figure 30: Case $B\triangle 123C\triangle 132 B_1$ and $B_2$](image)

![Figure 31: Case $B\triangle 123C\triangle 132 C_1$ and $C_2$](image)

It can be observed that $C_1$ is isomorphic to the dual of $B_1$. If we map points 1, 2, 3, 4, 5, 6, 7 of $C_1$ to lines $\ell_{12}, \ell_{23}, \ell_{13}, \ell_{56}, \ell_{14}, \ell_{34}, \ell_{24}$, respectively, of $B_1$, we have an isomorphism. Similarly, $C_2$ is isomorphic to the dual of $B_2$. An isomorphism maps points 1, 2, 3, 4, 5, 6, 7 of $C_2$ to lines $\ell_{12}, \ell_{13}, \ell_{23}, \ell_{56}, \ell_{24}, \ell_{34}, \ell_{17}$, respectively, of $B_2$. Consequently, we have only cases $B_1$ and $B_2$ to deal with.

**Case $B\triangle 123C\triangle 132 B_1$.**
Consider the quadrangle $\square 1562$. Condition $D$ can only be satisfied if the third point on $\ell_{12}$ is collinear with point 3. But then triangle $\triangle 123$ would satisfy condition $A$, which is not allowed. Condition $E$ can be satisfied if $\ell_{24}$ intersected $\ell_{56}$. This is shown in Figure 32. Condition $F$ can only be satisfied if the third line through 5 intersected $\ell_{15}$ in point 3. However, 6 and 3 are already collinear. Condition $G$ can be satisfied if the third line through 5 intersected $\ell_{14}$. The third line through 5 cannot be $\ell_{24}$, for $\triangle 124$ would then satisfy condition $A$. Thus, the third line through 5 must be a new line, as shown also in Figure 32.

**Case $B\triangle 123C\triangle 132 B_1\square 1562E$.**
Consider the triangle $\triangle 267$. Condition $B$ can be satisfied if the third line through 6 intersected $\ell_{12}$. The third line through 6 cannot be $\ell_{14}$, as the triangle $\triangle 123$ would then satisfy condition $A$. Hence, the third line through 6 must be a new line, as shown in Figure 33. Condition $C$ can only be satisfied if points 4 and 5 are collinear.
The line containing 4 and 5 cannot be $\ell_{14}$ and it cannot be $\ell_{34}$. Therefore Condition $C$ is impossible, and we must have $B\Delta 123C\Delta 132B_1\square 1562E\Delta 267B$, shown in Figure 33.

This structure is found to be isomorphic to the dual of $B\Delta 123B\Delta 132C\Delta 267B \square 1673G$, shown in Figure 26. The isomorphism maps points 1, 2, 3, 4, 5, 6, 7, 8 to lines $\ell_{24}, \ell_{26}, \ell_{56}, \ell_{15}, \ell_{34}, \ell_{18}, \ell_{68}, \ell_{14}$. This completes case $B\Delta 123C\Delta 132B_1\square 1562E$.

**Case B\Delta 123C\Delta 132B_1\square 1562G.**

Consider the quadrangle $\square 2651$. Condition $D$ can only be satisfied if the third point on $\ell_{23}$ is collinear with point 3. However triangle $\Delta 132$ would then satisfy condition $A$. Condition $E$ can only be satisfied if $\ell_{14}$ intersected $\ell_{56}$. The point of intersection cannot be 7. If it were point 4, then $\Delta 563$ would then satisfy condition $A$. Hence condition $E$ is not possible. Condition $F$ can only be satisfied if $\ell_{57}$ intersected $\ell_{26}$ in point 3. However 5 and 3 are already collinear. Condition $G$ can be satisfied if the third line through 6 intersected $\ell_{24}$. The point of intersection cannot be 4. The only possibility is a new line through 6, as shown in Figure 34.

Consider the quadrangle $\square 4863$. Condition $D$ can only be satisfied if the third point on $\ell_{34}$ is collinear with point 2. The triangle $\Delta 342$ would then satisfy condition $A$,
which is not allowed. Condition $E$ can only be satisfied if $\ell_{13}$ intersected $\ell_{68}$ in either 1 or 5. However, 1 and 5 are already each on 3 lines. Condition $F$ can only be satisfied if $\ell_{56}$ intersected $\ell_{48}$ in 2. However 6 and 2 are already collinear. Condition $G$ can be satisfied if the third line through 8 intersected $\ell_{14}$. The point of intersection can only be 7, shown in the right diagram of Figure 34.

Consider the quadrangle $\square_{6512}$. Condition $D$ can only be satisfied if the third point on $\ell_{12}$ were collinear with point 3. But triangle $\Delta_{123}$ would then satisfy condition $A$. Condition $E$ can only be satisfied if $\ell_{24}$ intersected $\ell_{15}$ in 3. This is not possible. Condition $F$ can only be satisfied if $\ell_{24}$ intersected $\ell_{57}$ in 9. This is not possible. Condition $G$ can only be satisfied if $\ell_{57}$ intersected $\ell_{68}$. This is shown in Figure 35.

Consider the quadrangle $\square_{6512}$. Condition $D$ can only be satisfied if the third point on $\ell_{34}$ were collinear with point 1. But then triangle $\Delta_{341}$ would satisfy condition $A$. Condition $E$ can only be satisfied if $\ell_{23}$ intersected $\ell_{47}$ in point 1. This is not possible. Condition $F$ can only be satisfied if $\ell_{24}$ intersected $\ell_{57}$ in 9. This is not possible. Condition $G$ can only be satisfied if $\ell_{78}$ intersected $\ell_{56}$ in a new point, also shown in Figure 35.

Consider the triangle $\Delta_{157}$. Condition $B$ can only be satisfied if $\ell_{12}$ intersected
The point of intersection must be point 0. Condition C can only be satisfied if points 4 and 9 are collinear. The line of collinearity must be $\ell_{34}$. The resulting two structures are both isomorphic to the Desargues configuration, with one incidence missing, as can be seen from Figure 1.1. If we then consider $\Delta_{268}$, the remaining incidence is forced. This completes case $B\Delta_{123}C\Delta_{132}B_1\square_{1562}G$ and also case $B\Delta_{123}C\Delta_{132}B_1$.

Case $B\Delta_{123}C\Delta_{132}B_2$.

Refer to Figure 30. Consider the triangle $\Delta_{173}$. Condition B can be satisfied if the third line through 7 intersected $\ell_{12}$. The point of intersection cannot be point 2. Therefore it is a new point, as shown in Figure 36. Condition C can be satisfied if points 4 and 5 are collinear. The line of collinearity cannot be $\ell_{56}$, for triangle $\Delta_{453}$ would then satisfy condition A. Hence $\ell_{45}$ is a new line, also shown in Figure 36. be satisfied if $\ell_{57}$ intersected $\ell_{68}$. This is shown in Figure 35.

Now case $B\Delta_{123}C\Delta_{132}B_2\Delta_{173}C$ is isomorphic to case $B\Delta_{123}B\Delta_{132}C\Delta_{267}B$, shown in Figure 25. As both diagrams have 7 points, the isomorphism can be given by a permutation, $(1, 5, 6)(2, 3, 4)$, which maps diagram $B\Delta_{123}B\Delta_{132}C\Delta_{267}B$ to $B\Delta_{123}C\Delta_{132}B_2\Delta_{173}C$. Thus we need only consider case $B\Delta_{123}C\Delta_{132}B_2\Delta_{173}B$.

Consider the triangle $\Delta_{781}$ in the left diagram of Figure 36. Condition B can be satisfied if the third line through 8 intersected $\ell_{37}$. The point of intersection cannot be 3. Therefore there must be a line $\ell_{48}$, as shown in Figure 37. Condition C can be satisfied if the third point on $\ell_{17}$ is collinear with point 2. The line of collinearity cannot be $\ell_{26}$, for if point 6 were on $\ell_{17}$, triangle $\Delta_{136}$ would satisfy condition A. Hence $\ell_{24}$ must intersect $\ell_{17}$ in a new point. This is also shown in Figure 37.

Case $B\Delta_{123}C\Delta_{132}B_2\Delta_{173}B\Delta_{781}$. 

Consider the triangle $\Delta_{365}$. Condition B can be satisfied if the third line through 6 intersected $\ell_{37}$. The point of intersection cannot be 4, because $\ell_{48}$ would then contain 6, causing a triangle $\Delta_{682}$ satisfying condition A. Line $\ell_{17}$ cannot contain 6, for then triangle $\Delta_{136}$ would satisfy condition A. Therefore condition B requires that $\ell_{78}$ contain 6, shown in Figure 38. Condition C can be satisfied if the third point on $\ell_{56}$ were collinear with point 1. The line of collinearity must be $\ell_{17}$, also shown in Figure 38.
Refer to the left diagram of Figure 38. Consider the quadrangle $\square 2176$. Condition $D$ can only be satisfied if points 3 and 8 were collinear. This is not possible as 3 and 8 are already incident on 3 lines each. Condition $E$ can only be satisfied if $\ell_{56}$ intersected $\ell_{17}$, shown in Figure 39. Condition $F$ can only be satisfied if $\ell_{37}$ intersected $\ell_{12}$ in 8. However, 7 and 8 are already collinear. Condition $G$ can only be satisfied if $\ell_{15}$ and $\ell_{24}$ intersected. The point of intersection must be 5, making triangle $\Delta 132$ satisfy condition $A$. We conclude that only $E$ is possible.

Consider the quadrangle $\square 2156$. Condition $D$ can only be satisfied if points 3 and 9 were collinear, which is impossible. Condition $E$ can only be satisfied if $\ell_{67}$ intersected $\ell_{15}$ in point 3, which is impossible. Condition $F$ can only be satisfied if
the third line through 5 intersected \( \ell_{12} \) in point 8, which is impossible. Condition \( G \) can only be satisfied if \( \ell_{17} \) and \( \ell_{24} \) intersected. The point of intersection must be point 9, also shown in Figure 39. As can be seen from the diagram, this is the Pappus configuration with one incidence missing. We conclude that this case results in the Pappus configuration.

**Case B\(\Delta 123C\Delta 132B\Delta 173B\Delta 781B\Delta 365C\).**

Refer to the right diagram of Figure 38. Consider the quadrangle \( \square 7123 \). Condition \( D \) can only be satisfied if points 4 and 6 are collinear, which is impossible. Condition \( E \) can only be satisfied if \( \ell_{13} \) contains 8, which is impossible. Condition \( F \) can only be satisfied if \( \ell_{24} \) contains point 9. Condition \( G \) can only be satisfied if \( \ell_{78} \) intersected \( \ell_{13} \). The point of intersection must be 5, creating a triangle \( \Delta 195 \) satisfying condition \( A \), a contradiction. We conclude that only condition \( F \) is possible, shown in Figure 40.

![Figure 40: Cases B\(\Delta 123C\Delta 132B\Delta 173B\Delta 781B\Delta 365C\square 7123F \) and \( \square 2371F \)](image)

Consider the quadrangle \( \square 2371 \). Condition \( D \) can only be satisfied if points 8 and 9 are collinear, which is impossible. Condition \( E \) is only possible if \( \ell_{13} \) contains 4, which is impossible. Condition \( F \) is possible only if \( \ell_{78} \) contains 6. Condition \( G \) is only possible if \( \ell_{29} \) and \( \ell_{35} \) intersected, which is impossible. We conclude that condition \( F \) is necessary.

We next consider quadrangle \( \square 4862 \). Condition \( D \) can only be satisfied if points 9 and 3 are collinear, which is impossible. Condition \( E \) can only be satisfied if \( \ell_{21} \) contains point 7, which is impossible. Condition \( F \) is possible only if \( \ell_{69} \) and \( \ell_{48} \) intersect in point 5. Condition \( G \) is only possible if \( \ell_{81} \) and \( \ell_{47} \) intersected, which is impossible. We conclude that condition \( F \) is necessary, giving the Pappus configuration. This completes case \( B\Delta 123C\Delta 132B\Delta 173B\Delta 781B \).

**Case B\(\Delta 123C\Delta 132B\Delta 173B\Delta 781C\).**

Refer to the right diagram of Figure 37. Consider triangle \( \Delta 243 \). Condition \( B \) can only be satisfied if the third line through 4 intersected \( \ell_{28} \). The point of intersection can only be 8, as shown in Figure 41. Condition \( C \) can only be satisfied if points 6 and 7 are collinear. The line of collinearity cannot be \( \ell_{17} \), as triangle \( \Delta 231 \) would then satisfy condition \( A \). Hence, the line can only be \( \ell_{78} \), which must contain 6, as shown in Figure 41.

Case \( C \) is isomorphic to the dual of \( B\Delta 123C\Delta 132B\Delta 173B\Delta 178B\Delta 365B \), shown in Figure 38. An isomorphism maps points 1, 2, \ldots, 9 of \( C \) to lines \( \ell_{67}, \ell_{34}, \ell_{23}, \ell_{24}, \ell_{56}, \ell_{13}, \ell_{12}, \ell_{17}, \ell_{48} \), respectively, of \( B \). Thus we only need consider case \( B \).
Consider the quadrangle $\Box 8731$. Condition $D$ can only be satisfied if points 2 and 6 are collinear, which is impossible, as the line of collinearity could only be $\ell_{24}$. Condition $E$ cannot be satisfied. Condition $F$ can only be satisfied if $\ell_{36}$ intersects $\ell_{87}$. The point of intersection must be 6, as shown in Figure 42. Condition $G$ can only be satisfied if $\ell_{84}$ and $\ell_{79}$ intersect, which is impossible. Thus, only condition $F$ is possible. But this diagram is isomorphic to case $B \Delta 123C \Delta 132B \Delta 173B \Delta 178C \Delta 243B$ and $C$.

We summarise Case $B$ as follows:

An $n_3$ configuration $(\Sigma, \Pi)$, which cannot be constructed by a 1-point extension, and having no triangle satisfying condition $A$, is one of the Pappus or Desargues configurations.

We still must show that the Fano, Pappus, and Desargues configurations cannot be obtained by 1-point extensions. This is clearly so for the Fano configuration, as there are no $6_3$ configurations. Consider the Pappus configuration. One way to show that it cannot be obtained by a 1-point extension is to start with the unique $8_3$ configuration and to show that the possible 1-point extensions do not produce the Pappus configuration. Another way is to show that every ordering of every triangle and quadrilateral in the Pappus configuration satisfies one of conditions $A, B, C, D, E, F, G$, so that the Pappus configuration does not
arise by a 1-point extension. The collineation group of the Pappus configuration has order 108. It is transitive on points, lines, triangles, and quadrangles, so that only one triangle and one quadrilateral need be tested. We omit the proof.

Figure 43: The Pappus configuration

Consider next the Desargues configuration. Its collineation group has order 120. It is transitive on points, lines, triangles, quadrangles, and also on quadruples \((a_0, \ell_2, a_1, \ell_3)\), where \(a_0, a_1 \in \ell_2, a_0 \neq a_1, a_1 \in \ell_3,\) and \(\ell_2 \neq \ell_3\). It is not transitive on pentagons, hexagons, etc. Refer to Figure 44. We look for a cycle beginning \((a_0, \ell_2, a_1, \ell_3, \ldots, \ell_0) = (1, \ell_{13}, 3, \ell_{34}, \ldots)\), satisfying the conditions of Theorem 2.4. Since \(\ell'_1 \cap \ell_1 = \emptyset\), where \(\ell'_1 = \ell_{37}\), and \(\ell_1\) is the third line through \(a_0 = 1\), we must have \(\ell_1 = \ell_{15}\), so that \(\ell_0 = \ell_{17}\). Since \(a_2 \notin \ell_1\), by Theorem 2.4, we cannot have \(a_2 = 5\). Hence, \(a_2 = 4\).

Figure 44: The Desargues configuration

Then since \(\ell'_2 \cap \ell_2 = \emptyset\), we cannot have \(\ell'_2 = \ell_{42}\), as \(\ell_{42}\) intersects \(\ell_2 = \ell_{13}\) in 2. Therefore \(\ell_4 = \ell_{49}\), from which we have \(a_3 = 9\), and the cycle is \((1, \ell_{13}, 3, \ell_{34}, 4, \ell_{49}, 9, \ldots, \ell_{17})\). Since \(\ell'_3 \cap \ell_3 = \emptyset\), we cannot have \(\ell'_3 = \ell_{59}\), as \(\ell_{59}\) intersects \(\ell_3 = \ell_{34}\) in 5. It follows that \(\ell_5 = \ell_{59}\). But then \(a_4\) must be either 1 or 5, both of which are impossible. We conclude that the Desargues configuration cannot be obtained by a 1-point extension. This completes the proof of Theorem 4.1.

Observe that we have only used 1-point extensions based on triangles and quadrangles in the proof of Theorem 4.1. Hence we have proved that if an \((n+1)_3\) configuration cannot be obtained using a 1-point extensions based on triangles or quadrangles, then it is the Fano, Pappus, Desargues, or a Fano-type configuration. Therefore we have the following corollary.
Corollary 4.2. Every \((n+1)_3\) configuration that can be obtained from an \(n_3\) configuration by a 1-point extension, can be obtained using a 1-point extension based on triangles or quadrangles.

A consequence of this corollary is that the \((n+1)_3\) configurations can be constructed from the \(n_3\) configurations by constructing all sequences of sequentially incident points and lines of length at most 4, and testing whether they satisfy the conditions required for a 1-point extension. Isomorphism testing of the resulting \((n+1)_3\) configurations then gives all configurations that can be constructed by 1-point extensions. Those which cannot be constructed in this way are the Fano-type configurations, which can be constructed from cycles and subdivisions of bipartite 3-regular multigraphs, using Theorem 3.2.

One of the central problems in the theory of \(n_3\) configurations is to determine whether they are geometric, that is, whether they can be coordinatized over the reals and/or rationals. See [3, 14, 15, 16]. This means to assign homogeneous coordinates in the real and/or rational projective plane, so that the lines are straight lines, and all incidences and non-incidences are respected. The application of 1-point extensions to geometric configurations will be described in another article (in preparation).

5 The 3-Point Extension

Let \((\Sigma, \Pi)\) be an \(n_3\)-configuration. Choose a line \(\ell\), and let its points be \(a_1, a_2, a_3\). Construct a new configuration \((\Sigma', \Pi')\) as follows. \(\Sigma' = \Sigma \cup \{b_1, b_2, b_3, \ell_1, \ell_2, \ell_3\}\), where \(b_1, b_2, b_3\) are new points and \(\ell_1, \ell_2, \ell_3\) are new lines. The incidences \(\Pi'\) are constructed as follows. \(\ell_1\) contains the points \(a_1, b_2, b_3\). \(\ell_2\) contains the points \(b_1, a_2, b_3\), and \(\ell_3\) contains the points \(b_1, b_2, a_3\). Choose 3 lines \(\ell'_1, \ell'_2, \ell'_3\) \(\neq \ell\) such that \(\ell'_i\) contains \(a_i\). Remove \(a_i\) from \(\ell'_i\) and place \(b_i\) on \(\ell'_i\). This is illustrated in the following table. Then \(\Pi'\) contains all remaining incidences of \(\Pi\), except for the incidences \(a_1 \ell'_1, a_2 \ell'_2, a_3 \ell'_3\).

\[
\begin{array}{cccccccccc}
\ell & \ell_1 & \ell_2 & \ell_3 & \ell'_1 & \ell'_2 & \ell'_3 \\
a_1 & a_1 & b_1 & b_1 & b_2 & b_3 \\
a_2 & b_2 & a_2 & b_2 & \cdot & \cdot \\
a_3 & b_3 & b_3 & a_3 & \cdot & \cdot \\
\end{array}
\]

Theorem 5.1. \((\Sigma', \Pi')\) is an \((n+3)_3\)-configuration.

Proof. Note that each \(b_i\) is incident on exactly 3 lines, and that each of \(\ell'_1, \ell'_2, \ell'_3\) is incident on exactly 3 points. We must verify that any 2 lines of \((\Sigma', \Pi')\) intersect in at most one point. Clearly \(\ell, \ell_1, \ell_2, \ell_3\) intersect each other in at most one point. Similarly for \(\ell, \ell'_1, \ell'_2, \ell'_3\). The same is true for all other lines of \(\Sigma'\), because it is true for \((\Sigma, \Pi)\).

Example 5.2. The Fano configuration has 7 points and 7 lines, all of which are equivalent under automorphisms. There is one way to choose 3 points \(a_1, a_2, a_3\). The incidences of \(\ell, \ell_1, \ell_2, \ell_3\) are uniquely determined. The choice of \(\ell'_1, \ell'_2, \ell'_3\) is not unique, as each \(a_i\) is incident on two lines other than \(\ell\). There results two possible 3-point extensions of the Fano configuration. One of these is the Desargues configuration. The other is known as the “anti-Pappian” configuration [5].

A complete quadrilateral in an \(n_3\) configuration is a set of four distinct lines intersecting in six distinct points. Notice that the extended configuration \((\Sigma', \Pi')\) always contains a complete quadrilateral \(\ell, \ell_1, \ell_2, \ell_3\), intersecting in the six points \(a_1, a_2, a_3, b_1, b_2, b_3\). The
3-point extension can also be constructed from the dual point of view – rather than beginning with 3 collinear points \(a_1, a_2, a_3\), we begin with 3 concurrent lines, and so forth. This is equivalent to using the 3-point extension in the dual of \((\Sigma, \Pi)\), and then dualizing \((\Sigma', \Pi')\). In this case, the 3-point extension will always contain a complete quadrangle, that is, the dual of a complete quadrilateral.

**Theorem 5.3.** The Fano-type configurations cannot be obtained by a 3-point extension.

**Proof.** Suppose that a Fano-type configuration \((\Sigma, \Pi)\) were obtained by a 3-point extension. It would then contain a complete quadrilateral \(\ell, \ell_1, \ell_2, \ell_3\), intersecting in the six points \(a_1, a_2, a_3, b_1, b_2, b_3\). These four lines and six points must all be part of a single \(F', F_p,\) or \(F_\ell\). Refer to Figure 6. Now the points \(a_1, a_2, a_3\) must be collinear. Furthermore, there must be a line containing \(a_1, b_2, b_3\), and so forth. This determines the labelling of an \(F', F_p,\) or \(F_\ell\). But we then find there is a line containing at least one of the pairs \(a_1, b_1; a_2, b_2; a_3, b_3\), which is not possible in a 3-point extension.

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