On Skew Heyting Algebras

Karin Cvetko-Vah

January 15, 2015

Abstract

In the present paper we generalize the notion of a Heyting algebra to the non-commutative setting and hence introduce what we believe to be the proper notion of the implication in skew lattices. We give several examples of skew Heyting algebras, including Heyting algebras, dual skew Boolean algebras, conormal skew chains and algebras of partial maps with poset domains.

Keywords: skew lattice, Heyting algebra, non-commutative logic

Mathematical Subject Classification 2010: Primary: 06F35; Secondary: 03G27.

1 Introduction

Just like Boolean algebras represent algebraic models for classical logic, Heyting algebras represent algebraic models for intuitionistic logic. Skew Boolean algebras are non-commutative generalizations of (possibly generalized) Boolean algebras. Already in 1936 Stone proved that each Boolean algebra can be embedded into a field of sets [17]. Likewise, by Leech's result each right-handed skew Boolean algebra can be embedded into a generic example of skew Boolean algebras, the algebra of partial functions over a given set and codomain $\{0, 1\}$, see [11, 12] for details.

Though not using the categorical language, in [17] Stone essentially proved that the category of Boolean algebras with homomorphisms is dual to the category of Boolean topological spaces (i.e. compact, Hausdorff, zero-dimensional spaces) with continuous maps. Generalizations of this result within the commutative setting yield the Priestley duality [13, 14] between bounded distributive lattices and Priestley spaces (i.e. totally order disconnected Boolean spaces), and the Esakia duality [7] between Heyting algebras and Esakia spaces (i.e. Priestley spaces in which the downset of a clopen set is again clopen), see also [4] for details. In a recent paper [8] on Esakia's work, Gehrke showed that Heyting algebras may be understood as those distributive lattices for which the embedding into their Booleanisation has a right adjoint.

A recent line of research generalized Stone's and Priestley's results to the non-commutative setting. By results proved in [1] and [9] any skew Boolean algebra is dual to a sheaf of rectangular bands over a locally-compact Boolean space, while any right- [left-]handed skew Boolean algebra is dual to a sheaf (of sets) over a locally-compact Boolean space. A further generalization was given in [2] showing that any strongly distributive skew lattice (as defined below) is dual to a sheaf of rectangular bands over a locally compact Priestley space, and any right- [left-]handed strongly distributive skew lattice is dual to a sheaf (of sets) over a locally compact Priestley space.

In the present paper we introduce the notion of a skew Heyting algebra. In passing to the non-commutative setting one needs to sacrifice either the top or the bottom of the algebra (in order not to end up in the commutative setting). In the previous papers [1], [9] and [2] algebras with bottoms were considered, and hence the notion of distributivity was generalized to the notion of the so called strong distributivity . If one tried to define an implication in the setting of strongly distributive skew lattices with bottom as a right adjoint of the conjunction, that would force the skew lattice also to poses the top and hence to be commutative which would result in a usual Heyting algebra. On the other hand, one could define an additional operation in strongly distributive skew lattices with bottom as a right adjoint of the disjunction. The obtained algebras would be precisely the co-skew Heyting algebras as described in Section 4 below, a categoty isomorphic to the category of the skew Heyting algebras which is the main topic of this paper. In order to define the implication in the skew lattice setting one needs to consider the category of the upside-down duals of strongly distributive skew lattices with bottom, namely the category of strongly codistributive skew lattices with top. That is not surprising as the top plays a crucial role in logic. Choosing this setting we in fact follow the path paved by Spinks and Veroff in [16]. The category of strongly codistributive skew lattices with top is isomorphic to the category of strongly distributive skew lattices with bottom.

After stating the basic preliminary definitions and concepts in Section 2, in Section 3 we introduce the notion of a skew Heyting algebra, prove that such algebras form a variety and show that the maximal lattice image of a skew Heyting algebra is a generalized Heyting algebra (i.e. a Heyting algebra possibly without bottom). In Section 5 we explain the consequences of our results to the theory of duality, i.e. we describe how skew Heyting algebras correspond to sheaves over so called local Esakia spaces. We finish with Section 4 where we list several examples of skew Heyting algebras.

2 Preliminaries

A **skew lattice** is an algebra $(S; \wedge, \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are both idempotent and associative and they satisfy the following absorption laws:

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \text{ and } (x \wedge y) \vee y = y = (x \vee y) \wedge y.$$

On a skew lattice S one can define the **natural partial order** by stating that $x \leq y$ if and only if $x \vee y = y = y \vee x$, and the **natural preorder** by $x \preceq y$ if and only if $y \vee x \vee y = y$. **Green's equivalence relation** $\mathcal{D}$ is then defined by

$$x\mathcal{D}y \text{ if and only if } x \preceq y \text{ and } y \preceq x. \tag{1}$$

It turns out that $x \leq y$ is equivalent to $x \wedge y = x = y \wedge x$, while $x \preceq y$ is equivalent to $x \wedge y \wedge x = x$. If S is commutative then $\leq$ and $\preceq$ coincide.

Lemma 2.1 *([6]). For x and y elements of a skew lattice S the following are equivalent:*

(i) $x \leq y$,

(ii) $x \vee y \vee x = y$,

(iii) $y \wedge x \wedge y = x$.

Leech's First Decomposition Theorem for skew lattices states that the relation $\mathcal{D}$ is a congruence on a skew lattice S, $S/\mathcal{D}$ is the maximal lattice image of S, and each congruence class is a rectangular band [10]. We shall denote the $\mathcal{D}$-class containing an element x by $\mathcal{D}_x$.

It was proved in [10] that a skew lattice always forms a **regular band** for either of the operations $\wedge$, $\vee$, i.e. it satisfies the identities

$$x \wedge u \wedge x \wedge v \wedge x = x \wedge u \wedge v \wedge x \text{ and } x \vee u \vee x \vee v \vee x = x \vee u \vee v \vee x.$$

A **skew lattice with top** is an algebra $(S; \wedge, \vee, 1)$ of type $(2, 2, 0)$ such that $(S; \wedge, \vee)$ is a skew lattice and $x \vee 1 = 1 = 1 \vee x$ holds for all $x \in S$. A skew lattice with bottom is defined dually and the bottom, if it exists, is usually denoted by 0.

Furthermore, a skew lattice is called **strongly distributive** if it satisfies the following identities:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z);$$

and it is called **strongly codistributive** if it satisfies the identities:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ and } (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

If a skew lattice S is either strongly distributive or strongly codistributive then S is **distributive** in that it satisfies the identities

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x) \text{ and } x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x).$$

A skew lattice S that is jointly strongly distributive and strongly codistributive is **binormal**, i.e. S factors as a direct product of a lattice and a rectangular band, $S \cong L \times B$, cf. [12] and [15].

Applying duality to a result of Leech [12], it follows that a skew lattice S is strongly codistributive if and only if S is jointly:

- **quasi-distributive**: the maximal lattice image $S/\mathcal{D}$ is a distributive lattice,
- **symmetric**: $x \wedge y = y \wedge x$ if and only if $x \vee y = y \vee x$, and
- **conormal**: $x \vee y \vee z \vee x = x \vee z \vee y \vee x$.

If a skew lattice is conormal then given any $u \in S$ the set

$$u\uparrow = \{u \vee x \vee u \,|\, x \in S\} = \{x \in S \,|\, u \leq x\}$$

forms a (commutative) lattice for the induced operations $\wedge$ and $\vee$, cf. [12].

The following lemma is the dual of a well known result in skew lattice theory.

Lemma 2.2 *Let S be a conormal skew lattice and let B, A be $\mathcal{D}$-classes such that $A \leq B$ holds in the lattice $S/\mathcal{D}$. Given $b \in B$ there exists a unique $a \in A$ such that $b \leq a$.*

Proof. First the uniqueness. If a and a' both satisfy the desired property then by Lemma 2.1 we have $a = b \vee a \vee b$ and likewise $a' = b \vee a' \vee b$. Now, using idempotency of $\vee$, conormality and the fact that $a \,\mathcal{D}\, a'$ we obtain:

$$a = b \vee a \vee b = b \vee a \vee a' \vee a \vee b =$$
$$b \vee a \vee a' \vee b = b \vee a' \vee a \vee a' \vee b = b \vee a' \vee b = a'.$$

To prove the existence of a take any $x \in A$ and set $a = b \vee x \vee b$. Then $a \in A$ and using the idempotency of $\vee$ we get:

$$b \vee a \vee b = b \vee (b \vee x \vee b) \vee b = b \vee x \vee b = a$$

which implies $b \leq a$. □

Recall that a **Heyting algebra** is an algebra $(H; \wedge, \vee, \rightarrow, 1, 0)$ such that $(H, \wedge, \vee, 1, 0)$ is a bounded distributive lattice that satisfies the condition:

(HA) $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.

Equivalently, the axiom (HA) can be replaced by the following set of identities:

(H1) $(x \rightarrow x) = 1$,

(H2) $x \wedge (x \rightarrow y) = x \wedge y$,

(H3) $y \wedge (x \rightarrow y) = y$,

(H4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

A **generalized Heyting algebra** is an algebra $(A; \wedge, \vee, \rightarrow, 1)$ such that $(A, \wedge, \vee, 1)$ is a distributive lattice with top 1 that satisfies:

(HA) $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.

A generalized Heyting algebra with the bottom is a Heyting algebra.

3 Skew Heyting algebras

A **skew Heyting lattice** is an algebra $(S; \wedge, \vee, 1)$ of type $(2, 2, 0)$ such that:

- $(S; \wedge, \vee, 1)$ is a strongly codistributive skew lattice with top,
- for any $u \in S$ operation $\rightarrow_u$ can be defined on each $u \uparrow$ such that $(u \uparrow ; \wedge, \vee, \rightarrow_u, 1, u)$ is a Heyting algebra with top 1 and bottom u.

Given a skew Heyting lattice one can define the operation $\rightarrow$ on S by $x \rightarrow y = y \vee x \vee y \rightarrow_y y$. A **skew Heyting algebra** is an algebra $(S; \wedge, \vee, \rightarrow, 1)$ of type $(2, 2, 2, 0)$ such that $(S; \wedge, \vee, 1)$ is a skew Heyting lattice and $\rightarrow$ is the induced implication as derived above.

Lemma 3.1 *Let S be a skew Heyting lattice, $\rightarrow$ as defined above and $x, y, u \in S$ such that both $x \in u \uparrow$ and $y \in u \uparrow$ hold. Then $x \rightarrow y = x \rightarrow_u y$.*

Proof. As x and y both lie in $u \uparrow$ it follows that they commute since $x \vee y = u \vee x \vee u \vee u \vee y \vee u = u \vee x \vee u \vee y \vee u = u \vee u \vee y \vee x \vee u = u \vee y \vee x \vee u$, and $y \vee x = u \vee y \vee u \vee u \vee x \vee u = u \vee y \vee u \vee x \vee u = u \vee y \vee x \vee u$. By the definition of the operation $\rightarrow$ we have $x \rightarrow y = x \vee y \rightarrow_y y \geq y$ by (H3). On the other hand, as $\rightarrow_u$ is the Heyting implication in the Heyting algebra $u \uparrow$ it follows that $x \rightarrow_u y = x \vee y \rightarrow_u y \geq y$. Computing in the Heyting algebra $y \uparrow$ we obtain:

$$x \vee y \rightarrow_u y \leq x \vee y \rightarrow_y y \text{ iff } (x \vee y \rightarrow_u y) \wedge (x \vee y) \leq y \text{ iff } (x \vee y) \wedge y \leq y,$$

which is true by absorption.

On the other hand, computing in the Heyting algebra $u \uparrow$ we obtain:

$$x \vee y \rightarrow_y y \leq x \vee y \rightarrow_u y \text{ iff } (x \vee y \rightarrow_y y) \wedge (x \vee y) \leq y \text{ iff } (x \vee y) \wedge y \leq y,$$

which is again true by absorption. $\square$

We shall use the axioms of Heyting algebras to derive the axiomatization of skew Heyting algebras. The reader shall find most of the axioms of Theorem 3.2 below as intuitively clear generalizations to the non-commutative case. However, two axioms should be given a further explanation. Firstly, the u in axiom (SH4) below appears as when passing to the non-commutative case we no longer have an element that is both below x and y with respect to the natural partial order (we have $x \wedge y \wedge x \leq x$ but in general not $x \wedge y \wedge x \leq y$). Similarly, axiom (SH0) is needed since in the non-commutative case it no longer follows from the other axioms, the reason for this being that in general $x \leq y \vee x \vee y$ need not hold.

Theorem 3.2 *Let* $(S; \wedge, \vee, \rightarrow, 1)$ *be an algebra of type* $(2, 2, 2, 0)$ *such that* $(S; \wedge, \vee, 1)$ *is a strongly codistributive skew lattice with top* 1*. Then* $(S; \wedge, \vee, \rightarrow, 1)$ *is a skew Heyting algebra if and only if it satisfies the following axioms:*

(SH0) $x \rightarrow y = y \vee x \vee y \rightarrow y.$

(SH1) $x \rightarrow x = 1,$

(SH2) $x \wedge (x \rightarrow y) \wedge x = x \wedge y \wedge x,$

(SH3) $y \wedge (x \rightarrow y) = y$ *and* $(x \rightarrow y) \wedge y = y,$

(SH4) $x \rightarrow u \vee (y \wedge z) \vee u = (x \rightarrow u \vee y \vee u) \wedge (x \rightarrow u \vee z \vee u).$

Proof. Assume that S is a skew Heyting algebra.

(SH0). By definition $x \rightarrow y$ and $y \vee x \vee y \rightarrow y$ are both defined as $y \vee x \vee y \rightarrow_y y$. Hence they are equal.

(SH1). This is true because $x \rightarrow_x x = 1$ is true in $x \uparrow$.

(SH2). In $y \uparrow$ we have $(y \vee x \vee y) \wedge (y \vee x \vee y \rightarrow_y y) = (y \vee x \vee y) \wedge y = y$. Hence

$$x \wedge (y \vee x \vee y) \wedge (x \rightarrow y) \wedge x = x \wedge y \wedge x,$$

but on the other hand also

$$x \wedge (y \vee x \vee y) \wedge (x \rightarrow y) \wedge x = x \wedge (y \vee x \vee y) \wedge x \wedge (x \rightarrow y) \wedge x = x \wedge (x \rightarrow y) \wedge x,$$

where we have used the regularity of $\wedge$ and the fact that $x \preceq y \vee x \vee y$.

(SH3). The identities hold because the corresponding identity holds in the Heyting algebra $y\uparrow$.

(SH4). First note that (SH4) is equivalent to

$$\text{(SH4')}\quad u\vee x\vee u\to u\vee(y\wedge z)\vee u=(u\vee x\vee u\to u\vee y\vee u)\wedge(u\vee x\vee u\to u\vee z\vee u)$$

as using axiom (SH0) and the conormality of $\vee$ we obtain

$$u\vee x\vee u\to u\vee y\vee u=u\vee x\vee y\vee u\to u\vee y\vee u$$

and likewise

$$x\to u\vee y\vee u=u\vee x\vee y\vee u\to u\vee y\vee u.$$

Hence it suffices to prove that (SH4') holds.

Observe that the distributivity implies

$$(u\vee y\vee u)\wedge(u\vee z\vee u)=u\vee(y\wedge z)\vee u. \tag{2}$$

As $u\vee x\vee u$, $u\vee y\vee u$, $u\vee z\vee u$ and $u\vee(y\wedge z)\vee u$ all lie in $u\uparrow$ we can simply compute in $u\uparrow$. Using (2) and axiom (H4) for Heyting algebras we obtain: $u\vee x\vee u\to u\vee(y\wedge z)\vee u=u\vee x\vee u\to(u\vee y\vee u)\wedge(u\vee z\vee u)=(u\vee x\vee u\to u\vee y\vee u)\wedge(u\vee x\vee u\to u\vee z\vee u)$.

To prove the converse assume that (SH0)–(SH4) hold. Now, given arbitrary $u\in S$ and $x,y,z\in u\uparrow$ set $x\to_u y=x\to y$. Axiom (SH3) implies that $x\to y\in y\uparrow\subseteq u\uparrow$, so that $\to_u$ is well defined. Axiom (SH0) assures that $\to$ is indeed the skew Heyting implication satisfying $a\to b=b\vee a\vee b\to_b b$, for any $a,b\in S$. It remains to prove that $\to_u$ is the Heyting implication on $u\uparrow$. Since $u\uparrow$ is commutative with bottom u, axioms (SH1)–(SH4) for $\to$ simplify to (H1)–(H4) for $\to_u$, respectively. □

Corollary 3.3 *Skew Heyting algebras form a variety.*

In the remaining of the paper, given a skew Heyting algebra we shall simplify the notation $\to_u$ to $\to$ when referring to the Heyting implication in $u\uparrow$.

Note that if S contains a **bottom** element 0 such that $x\wedge 0=0=0\wedge x$ for all $x\in S$, then taking $u=0$ in the axioms (SH1)–(SH4) these axioms simplify to the axioms of a Heyting algebra. In fact, it is well known in the skew lattice theory that a strongly (co)distributive skew lattice with top and bottom is a bounded distributive lattice.

Proposition 3.4 *The relation $\mathcal{D}$ defined in* (1) *is a congruence on any skew Heyting algebra.*

Proof. Let $(S; \wedge, \vee, \rightarrow, 1)$ be a skew Heyting algebra. We need to prove that $(a \rightarrow b)\,\mathcal{D}\,(c \rightarrow d)$ holds for any $a, b, c, d \in S$ satisfying $a\,\mathcal{D}\,c$ and $b\,\mathcal{D}\,d$. Without loss of generality we may assume $b \leq a$ and $d \leq c$ (otherwise replace a with $b \vee a \vee b$ and c with $d \vee c \vee d$). Define a map $\varphi : b\uparrow \rightarrow d\uparrow$ by setting $\varphi(x) = d \vee x \vee d$. We divide the proof into four steps.

Step 1. We claim that $\varphi : (b\uparrow; \wedge, \vee) \rightarrow (d\uparrow; \wedge, \vee)$ is a lattice isomorphism with the inverse $\psi : d\uparrow \rightarrow b\uparrow$ defined by $\psi(y) = b \vee y \vee b$. It is easy to see that maps φ and ψ are inverse to each other. For instance, $\psi(\varphi(x)) = b \vee d \vee x \vee d \vee b$. Using regularity of $\vee$ this is further equal to $(b \vee d \vee b) \vee x \vee (b \vee d \vee b)$, which is equal to $b \vee x \vee b$ because b is $\mathcal{D}$-equivalent to d. Since x is an element of $b\uparrow$, $b \vee x \vee b$ equals x by Lemma 2.1.

φ preserves $\wedge$: $\varphi(x \wedge x') = d \vee (x \wedge x') \vee d$ which by distributivity equals $(d \vee x \vee d) \wedge (d \vee x' \vee d) = \varphi(x) \wedge \varphi(x')$.

φ preserves $\vee$: $\varphi(x \vee x') = d \vee (x \vee x') \vee d$ which by the regularity of $\vee$ equals $(d \vee x \vee d) \vee (d \vee x' \vee d) = \varphi(x) \vee \varphi(x')$.

Step 2. We claim that $\varphi(x)\,\mathcal{D}\,x$ for all $x \in b\uparrow$. Indeed, one obtains:

$$x \vee \varphi(x) \vee x = x \vee d \vee x \vee d \vee x = x \vee d \vee x = x,$$

where we used $d\,\mathcal{D}\,b$ and $b \preceq x$, and

$$\varphi(x) \vee x \vee \varphi(x) = d \vee x \vee d \vee x \vee d \vee x \vee d = d \vee x \vee d = \varphi(x).$$

Step 3. We claim that $\varphi(a) = c$ and $\varphi(b) = d$. Indeed, $\varphi(b) = d \vee b \vee d = d$ because $b\,\mathcal{D}\,d$, and $\varphi(a) = d \vee a \vee d$ which is the unique element in the class $\mathcal{D}_a$ that is above d with respect to the natural partial order. This element is exactly c.

Step 4. We claim that $\varphi(a \rightarrow b) = c \rightarrow d$. Once we finish the proof of this step, the assertion of the Lemma will follow by Step 2. Let $w \in b\uparrow$. Using the fact that φ is a lattice isomorphism, the definition of a Heyting algebra and Step 3 above we obtain the following chain of equivalences:

$$\varphi(w) \leq \varphi(a \rightarrow b) \Leftrightarrow w \leq a \rightarrow b \Leftrightarrow w \wedge a \leq b$$
$$\Leftrightarrow \varphi(w) \wedge \varphi(a) \leq \varphi(b) \Leftrightarrow \varphi(w) \wedge c \leq d \Leftrightarrow \varphi(w) \leq c \rightarrow d.$$

Hence $\varphi(a \to b) = c \to d$ follows. $\square$

Corollary 3.5 *Let $(S; \wedge, \vee, \to, 1)$ be a skew Heyting algebra. Then the maximal lattice image $(S/\mathcal{D}; \wedge, \vee, \to, \mathcal{D}_1)$ is a generalized Heyting algebra. If S also has a bottom $\mathcal{D}$-class B, then $(S/\mathcal{D}; \wedge, \vee, \to, \mathcal{D}_1, B)$ is a Heyting algebra.*

Corollary 3.5 has a converse in the following sense. Assume that $(S; \wedge, \vee, 1)$ is a strongly codistributive skew lattice with top 1 such that operation $\to$ is defined on the maximal lattice image $S/\mathcal{D}$ making $S/\mathcal{D}$ into a generalized Heyting algebra. Then operation $\to$ can be defined on S in the following way. Given any $x, y \in S$ and $w \in \mathcal{D}_x \to \mathcal{D}_y$ set:

$$x \to y = y \vee w \vee y.$$

Note that the above definition is independent of the choice of the representative w in the $\mathcal{D}$-class $\mathcal{D}_x \to \mathcal{D}_y$, and $x \to y$ is defined as the unique element in the $\mathcal{D}$-class $\mathcal{D}_x \to \mathcal{D}_y$ that is above y with the respect to the natural partial order.

Theorem 3.6 *Let $(S; \wedge, \vee, 1)$ be a strongly codistributive skew lattice with top such that its maximal lattice image $S/\mathcal{D}$ is a generalized Heyting algebra, and let operation $\to$ be defined on S as above. Then $(S; \wedge, \vee, \to, 1)$ is a skew Heyting algebra.*

Proof. Let $u \in S$ be arbitrary. We need to prove that $(u\uparrow; \wedge, \vee, \to, 1)$ is a Heyting algebra. So, let x, y and z be elements of the distributive lattice $u\uparrow$.

(H1). $x \to x$ is the unique element in $\mathcal{D}_1$ that is above x. Since $\mathcal{D}_1$ is a singleton, $x \to x = 1$ follows.

(H2). The operation $\to$ respects $\mathcal{D}$-classes by Proposition 3.4. Therefore it follows that $x \wedge (x \to y)\ \mathcal{D}\ x \wedge y$. Take any $w \in \mathcal{D}_x \to \mathcal{D}_y$ and write $x \to y = y \vee w \vee y$. Note that $y \vee w \vee y \in y\uparrow \subseteq u\uparrow$ and hence $y \vee w \vee y$ commutes with all elements of $u\uparrow$. Hence:

$$x \wedge (x \to y) = x \wedge (y \vee w \vee y) \geq x \wedge y.$$

But since $x \wedge (x \to y)\ \mathcal{D}\ x \wedge y$, $x \wedge (x \to y) = x \wedge y$ follows by Lemma 2.2.

(H3). Again, Proposition 3.4 yields $y \wedge (x \to y) \mathcal{D} y$. Writing $x \to y = y \vee w \vee y$ as above yields $y \wedge (x \to y) = y$ by absorption.

(H4) By Proposition 3.4 it follows that the elements $x \to (y \wedge z)$ and $(x \to y) \wedge (x \to z)$ are $\mathcal{D}$-related. Thus, by Lemma 2.2 it suffices to show that they are both above $y \wedge z$ with respect to the natural partial order. Given $w_1 \in \mathcal{D}_x \to \mathcal{D}_{y\wedge z}$ we get

$$x \to (y \wedge z) = (y \wedge z) \vee w_1 \vee (y \wedge z) \geq y \wedge z.$$

On the other hand, given $w_2 \in \mathcal{D}_x \to \mathcal{D}_y$ and $w_3 \in \mathcal{D}_x \to \mathcal{D}_z$ we have

$$(x \to y) \wedge (x \to z) = (y \vee w_2 \vee y) \wedge (z \vee w_3 \vee z).$$

Since all of the elments y, z, $y \vee w_2 \vee y$ and $z \vee w_3 \vee z$ lie in $u\uparrow$, they all commute. Thus:

$$y \wedge z \wedge (y \vee w_2 \vee y) \wedge (z \vee w_3 \vee z) = y \wedge (y \vee w_2 \vee y) \wedge z \wedge (z \vee w_3 \vee z) = y \wedge z$$

which finishes the proof. □

Corollary 3.7 *Let $(S; \wedge, \vee, \to, 1)$ be a skew Heyting algebra. Then S satisfies the following equivalence:*

(SHA) $x \preceq y \to z$ *if and only if* $x \wedge y \preceq z$.

Conversely, let $(S; \wedge, \vee, \to, 1)$ be an algebra of type $(2, 2, 2, 0)$, such that the following hold:

1. *$(S; \wedge, \vee, 1)$ is a strongly codistributive skew lattice with top 1,*
2. *$y \leq x \to y$ for all $x, y \in S$,*
3. *S satisfies axiom (SHA) above.*

Then $(S; \wedge, \vee, \to, 1)$ is a skew Heyting algebra.

Proof. The first part of the corollary is clear as relation $\mathcal{D}$ respects all skew Heyting algebra operations and on the commutative algebra $S/\mathcal{D}$ the natural preorder coincides with the natural partial order. To prove the converse, first observe that the axiom (SHA) guarantees $S/\mathcal{D}$ to be a generalized Heyting algebra. By Theorem 3.6 it suffices to prove that $x \to y = y \vee w \vee y$, for all $x, y \in S$ and $w \in \mathcal{D}_x \to \mathcal{D}_y$. Axiom (SHA) assures that $x \to y \, \mathcal{D} \, y \vee w \vee y$. As both $y \leq x \to y$ and $y \leq y \vee w \vee y$ hold, $x \to y = y \vee w \vee y$ follows by Lemma 2.2. □

It follows from Corollary 3.7 that $x \to y = 1$ if and only if $x \preceq y$. The skew Heyting implication thus determines a reflexive and transitive relation on S that is antisymmetric exactly in the commutative case.

The following result is useful for computing in skew Heyting algebras.

Proposition 3.8 *Let* $(S; \wedge, \vee, \to, 1)$ *be a skew Heyting algebra and* $x, y, z \in S$. *Then*

$$x \vee y \vee x \to z = (x \to z) \wedge (y \to z) \wedge (x \to z).$$

Proof. As $S/\mathcal{D}$ is a generalized Heyting algebra and relation $\mathcal{D}$ respects all skew Heyting algebra operations, it follows that $x \vee y \vee x \to z \, \mathcal{D} \, (x \to z) \wedge (y \to z) \wedge (x \to z)$. However, both $x \vee y \vee x \to z$ and $(x \to z) \wedge (y \to z) \wedge (x \to z)$ are above z with respect to the natural partial order, and hence must be equal by Lemma 2.2. □

Following [5] a **commuting subset** of a skew lattice is a non-empty subset whose elements both join and meet commute. A skew lattice S is **meet [join] complete** if each commuting subset possesses an infimum [a supremum] in S. It follows by a result of [5] that if S is a meet complete strongly distributive skew lattice with 1 then S is complete. We call a skew Heyting algebra **complete** if it is complete as a skew lattice.

4 Connections to duality

A **skew Boolean algebra** is an algebra $(T; \wedge, \vee, \setminus, 0)$ such that $(T; \wedge, \vee, 0)$ is a strongly distributive skew lattice with bottom 0 and the operation $\setminus$ satisfies the properties

$$(x \setminus y) \wedge (x \wedge y \wedge x) = 0 = (x \wedge y \wedge x) \wedge (x \setminus y)$$
$$(x \setminus y) \vee (x \wedge y \wedge x) = x = (x \wedge y \wedge x) \vee (x \setminus y).$$

Given any u in a skew Boolean algebra, the set

$$u \downarrow = \{u \wedge x \wedge u \,|\, x \in T\} = \{x \in T \,|\, x \leq u\}$$

is a Boolean algebra with top u.

A **dual skew Boolean algebra** is an algebra $(S; \wedge, \vee, c, 1)$ of type (2,2,2,0) such that $(S; \wedge, \vee, 1)$ is a strongly codistributive skew lattice with top and the operation c satisfies the identities

$$\begin{aligned} c(x,y) \wedge (y \vee x \vee y) = y = (y \vee x \vee y) \wedge c(x,y), \\ c(x,y) \vee (y \vee x \vee y) = 1 = (y \vee x \vee y) \vee c(x,y). \end{aligned} \tag{3}$$

Given any u in a dual skew Boolean algebra S, $u\uparrow$ becomes a Boolean algebra with top 1 and bottom u and the complement defined as $x' = c(x,u)$, for all $x \in u\uparrow$.

Dual skew Boolean algebras are order duals (upside-downs) to usually studied skew Boolean algebras. Skew Boolean algebras and dual skew Boolean algebras are categorically isomorphic. Right-handed (dual) skew Boolean algebras are dually equivalent to sheaves over locally compact Boolean spaces by results of [1] and [9]. Here A skew lattice S is said to be **right- [left-]handed** if it satisfies $x \vee y \vee x = x \vee y$ $[x \vee y \vee x = y \vee x]$, and a *locally compact Boolean space* is a topological space whose one-point-compactification is a Boolean space. The obtained duality yields that any right- [left-]handed skew Boolean algebra is isomorphic to the skew Boolean algebra of compact open sections (i.e. sections over compact open subsets) of the étale map over some locally compact Boolean space. Let us note that the restriction to right- [left-]handed algebras is not a major restriction since Leech's Second Decomposition Theorem yields that any skew lattice is a pull back of a left-handed and a right-handed skew lattice over their common maximal lattice image [10]. However, the general two-sided case was also covered in [1].

Bounded distributive lattices are dual to Priestley space; in this duality each bounded distributive lattice is represented as the distributive lattice of all clopen (i.e. closed and open) upsets of a Priestley space. The Esakia duality established in [7] yields that Heyting algebras are dual to **Esakia spaces**, i.e. those Priestley spaces in which the downset of each clopen set is again clopen. Moreover, if $(X, \leq, \tau)$ is an Esakia space then given clopen subsets U and V in X the implication is defined by

$$U \to V = X \setminus \downarrow (U \setminus V).$$

Duality for strongly distributive skew lattices was recently established in [2]. It yields that right-handed strongly distributive skew lattices with bottom are dual to the sheaves over locally Priestley spaces, where by a **locally Priestley space** we mean an ordered topological space whose one-point-compactification is a Priestley space. Via the obtained duality each right-handed strongly distributive skew lattice with bottom is represented as a skew lattice of sections over copen (i.e. compact and open) downsets of a locally Priestley space, with the operations being defined as follows:

$$\begin{aligned} 0 &= \emptyset, \\ r \wedge s &= s|_{\mathrm{dom}r \cap \mathrm{dom}s}, \\ r \vee s &= r \cup s|_{\mathrm{dom}s - \mathrm{dom}r}, \\ r \setminus s &= r|_{\mathrm{dom}r - \mathrm{dom}s}. \end{aligned}$$

Given a distributive lattice L denote by L^c the distributive lattice that is obtained from L by reversing the order. Denote by **DL** the category of all distributive lattices, by **PS** the category of all locally Priestley spaces and consider the functors:

$$\begin{array}{cccc} ^c : & \mathbf{DL} & \to & \mathbf{DL} \\ & L & \mapsto & L^c \end{array} \quad \text{and} \quad \begin{array}{cccc} ^r : & \mathbf{PS} & \to & \mathbf{PS} \\ & (X, \leq) & \mapsto & (X, \geq). \end{array}$$

Restricting the functors ^{c} and r to the categories **HA** of all Heyting algebras and **ES** of all Esakia spaces, respectively, yields the following isomorphism of categories:

$$\begin{array}{cccc} ^c : & \mathbf{HA} & \to & \mathbf{coHA} \\ & L & \mapsto & L^c \end{array} \quad \text{and} \quad \begin{array}{cccc} ^r : & \mathbf{ES} & \to & \mathbf{coES} \\ & (X, \leq) & \mapsto & (X, \geq), \end{array}$$

where **coHA** denotes the category of all *co-Heyting algebras* (defined as order-inverses of Heyting algebras) and **coES** denotes the category of all *co-Esakia spaces* the latter being introduced in [3] as Priestley spaces in which an upset of a clopen is again clopen.

We introduce the following categories:

SDSL : strongly distributive skew lattices with 0,

SCDSL : strongly codistributive skew lattices with 1,

SHA : skew Heyting algebras,

coSHA : co-skew Heyting algebras,

with the latter being defined as the category of all algebras of the form S^c, where S is a skew Heyting algebra and

$$\begin{array}{rlcl} {}^c : & \mathbf{SDSL} & \to & \mathbf{SCDSL} \\ & S & \mapsto & S^c \end{array}$$

is the isomorphism of categories that turns a skew lattice upside-down. The restriction of the functor ^{c} to the categories **coSHA** and **SHA** yields the isomorphism:

$$\begin{array}{rlcl} {}^c : & \mathbf{coSHA} & \to & \mathbf{SHA} \\ & S & \mapsto & S^c \end{array}.$$

The isomorphism of categories induces an isomorphism of concepts:

SHA	**coSHA**
$\wedge$	$\vee$
$\vee$	$\wedge$
1	0
strong codistributivity	strong distributivity
filter	ideal
prime filter	prime ideal

It follows from Corollary 3.5 and Theorem 3.6 that the skew Heyting algebra structure can be imposed exactly on those strongly codistributive skew lattices with top whose maximal lattice image is a generalized Heyting algebra. Therefore the duality for right-handed skew Heyting algebras yields that they are dual to sheaves over **local Esakia spaces**, i.e. ordered topological spaces whose one-point-compactification is an Esakia space.

Let $(B, \leq)$ be an Esakia space, E a topological space and $p : E \to B$ a surjective étale map. Consider the set S of all sections of p over copen upsets in B, i.e. an element of S is a map $s : U \to E$, where U is a copen upset in B, that satisfies the property $p \circ s = \mathrm{id}_U$. A section $s \in S$ is considered to be below a section $r \in S$ when s extends r. The skew Heyting operations are defined on

S by:

$$r \vee s = s|_{\text{dom}r \cap \text{dom}s},$$
$$r \wedge s = r \cup s|_{\text{dom}s \setminus \text{dom}r},$$
$$r \to s = r|_{\uparrow(\text{dom}s \setminus \text{dom}r)}$$
$$1 = \emptyset.$$

Theorem 4.1 *Let $p : E \to B$ be a surjective étale map over a locally Esakia space B. Then the set S of all sections of p over copen upsets in B is a skew Heyting algebra.*

5 Examples of skew Heyting algebras

The class of skew Heyting algebras obviously contains the classes of Heyting and generalized Heyting algebras. Below we list some further examples of skew Heyting algebrs.

5.1 Dual skew Boolean algebras

Let $(S; \wedge, \vee, c, 1)$ be a dual skew Boolean algebra. Given any u in S, $u\uparrow$ becomes a Boolean algebra with top 1 and bottom u and the complement defined as $x' = c(x, u)$, for all $x \in u\uparrow$.

We will see that dual skew Boolean algebras allow a skew Heyting structure. Take $u \in S$. We want to define a Heyting implication on $u\uparrow$. So, given $x, y \in u\uparrow$ set

$$x \to y = c(x, u) \vee y.$$

We claim that $(u\uparrow; \wedge, \vee, \to, 1, u)$ is a Heyting algebra. Let $x, y, z \in u\uparrow$.

(H1). $x \to x = c(x, u) \vee x = 1$ by (3).

(H2). $x \wedge (x \to y) = x \wedge (c(x, u) \vee y)$: this can be computed in the Boolean algebra $u\uparrow$ and hence equals $(x \wedge c(x, u)) \vee (x \wedge y) = u \vee (x \wedge y) = x \wedge y$, where the latter equality holds because u is the bottom of $u\uparrow$.

(H3). $y \wedge (x \to y) = y \wedge (c(x, u) \vee y)$ which equals y be absorption.

(H4). $x \to (y \wedge z) = c(x, u) \vee (y \wedge z) = (c(x, u) \vee y) \wedge (c(x, u) \vee z)) = (x \to y) \wedge (x \to z)$.

The following Lemma shows that one can define the operation $\to$ independently from the choice of u.

Lemma 5.1 *Let S be a dual skew Boolean algebra $x, y, u \in S$, $u \leq y$ and let $\to$ be defined on $u\uparrow$ as above. Then*

$$(y \vee x \vee y) \to y = c(x, y).$$

Proof. Take any $u \leq y$. We need to prove that $(y \vee x \vee y) \to y$ is the complement of $y \vee x \vee y$ in the Boolean algebra $y\uparrow$. Indeed,

$$((y \vee x \vee y) \to y) \wedge (y \vee x \vee y) = (c(y \vee x \vee y, u) \vee y) \wedge (y \vee x \vee y) =$$
$$(c(y \vee x \vee y, u) \wedge (y \vee x \vee y)) \vee y = u \vee y = y$$

and

$$((y \vee x \vee y) \to y) \vee (y \vee x \vee y) = (c(y \vee x \vee y, u) \vee y) \vee (y \vee x \vee y) =$$
$$c(y \vee x \vee y, u) \vee (y \vee x \vee y) = 1.$$

□

Theorem 5.2 *Let $(S; \wedge, \vee, c, 1)$ be a dual skew Boolean algebra and define operation $\to$ on S by*

$$x \to y = c(x, y).$$

Then $(S; \wedge, \vee, \to, 1)$ is a skew Heyting algebra.

5.2 Conormal skew chains

A **skew chain** is a skew lattice whose $\mathcal{D}$-classes are totally ordered. Skew chains are trivially quasi-distributive. To see that they are symmetric, take any $y \preceq x$; then $x \vee y = y \vee x$ iff $x \vee y = x = y \vee x$. But $x \vee y = x = y \vee x$ iff (by basic duality) $x \wedge y = y = y \wedge x$. It follows that a skew chain in strongly codistributive if and only if it is conormal.

Proposition 5.3 *Let $(S; \wedge, \vee, 1)$ be a conormal skew chain with top 1. Given $x, y \in S$ set*

$$x \to y = \begin{cases} 1; & \text{if } x \preceq y; \\ y; & \text{else} \end{cases}$$

Then $(S; \wedge, \vee, \to, 1)$ is a skew Heyting algebra.

Proof. Since $S/\mathcal{D}$ is totally ordered it can be given the generalized Heyting structure by setting

$$\mathcal{D}_x \to \mathcal{D}_y = \begin{cases} \mathcal{D}_1; & \text{if } \mathcal{D}_x \leq \mathcal{D}_y; \\ \mathcal{D}_y; & else \end{cases}$$

Given any $u \in S$, $u \uparrow$ is a totally ordered lattice and $\to$ restricted to $u \uparrow$ simplifies as

$$x \to y = \begin{cases} 1; & \text{if } x \leq y; \\ y; & else \end{cases}$$

which is the Heyting implication. □

5.3 Finite strongly codistributive skew lattices

Let $(S; \wedge, \vee, 1)$ be a finite strongly codistributive skew lattice with top 1. The maximal lattice image $S/\mathcal{D}$ is a finite distributive lattice and can thus be given the Heyting structure by letting $\mathcal{D}_x \to \mathcal{D}_y$ be the join of all $\mathcal{D}$-classes Z satisfying

$$\mathcal{D}_x \wedge Z \leq \mathcal{D}_y.$$

For $x, y \in S$ let $x \to y$ be the join of all the (necessarily commuting) elements $z \in y \uparrow$ satisfying

$$(y \vee x \vee y) \wedge z \leq y. \tag{4}$$

It is elementary to show that $\to$ restricted to any $u \uparrow$ equals the join of all elements $z \in u \uparrow$ satisfying (4).

References

[1] A. Bauer, K. Cvetko-Vah, Stone duality for skew Boolean algebras with intersections, *Houston J. Math.* **39** (2013), 73–109.

[2] A. Bauer, K. Cvetko-Vah, M. Gehrke, S. van Gool and G. Kudryavtseva, A Non-Commutative Priestley Duality, *Topology and Appl.* **160** (2013), 1423–1438.

[3] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, A, Kurz, Bitopological duality for distributive lattices and Heyting algebras, preprint.

[4] N. Bezhanishvili, *Lattices of Intermediate and Cylindric Modal Logics*, PhD Thesis, ILLC, University of Amsterdam, 2006.

[5] R. J. Bignall, J. E. Leech, Skew Boolean algebras and discriminator varieties, *Algebra Universalis* **33** (1995), 387–398.

[6] J. P. Pita da Costa, On the Coset Structure of a Skew Lattice, *Demonstratio Mathematica* **XLIV** No. 4 (2011), 673–692.

[7] L. L. Esakia, Topological Kripke models, *Soviet Mathematics Doklady* **15** (1974), 147-151.

[8] M. Gehrke, Canonical extensions, Esakia spaces, and universal models, Chapter in *Leo Esakia on duality in modal and intuitionistic logics*, upcoming special volume in the series Trends in Logic (Outstanding Contributions subseries) dedicated to the achievements of Leo Esakia.

[9] G. Kudryavtseva, A refinement of Stone duality to skew Boolean algebras, *Algebra Universalis* **67** (2012), 397–416.

[10] J. Leech, Skew lattices in rings, *Alg. Universalis* **26** (1989), 48–72.

[11] J. Leech, Skew Boolean Algebras, *Alg. Universalis* **27** (1990), 497–506.

[12] J. Leech, Normal skew lattices, *Semigroup Forum 44* (1992), 1–8.

[13] H. A. Priestley, Representation of distributive lattices by means of ordered stone spaces, *Bull. Lond. Math. Soc.* **2** (1970), 186–190.

[14] H.A. Priestley, Ordered topological spaces and the representation of distributive lattices, *Proc. Lond. Math. Soc. (3)* **24** (1972), 507–530.

[15] B. M. Schein, Pseusosemilattices and Pseudolattices, *Amer. Math. Soc. Transl. (2)* **119** (1983), 1–16.

[16] M. Spinks, R. Veroff, Axiomatizing the skew Boolean propositional calculus.*J. Automat. Reason.* **37** (2006), no. 1-2, 3-20.

[17] M. H. Stone, The theory of representation for Boolean algebras, *Trans. Amer. Math. Soc.* **74**:1 (1936), 37–111.