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Distinguishing graphs by total colourings*

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Abstract

We introduce the *total distinguishing number* $D''(G)$ of a graph G as the least number d such that G has a total colouring (not necessarily proper) with d colours that is only preserved by the trivial automorphism. This is an analog to the notion of the distinguishing number $D(G)$, and the distinguishing index $D'(G)$, which are defined for colourings of vertices and edges, respectively. We obtain a general sharp upper bound: $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$ for every connected graph G.

We also introduce the *total distinguishing chromatic number* $\chi''_D(G)$ similarly defined for proper total colourings of a graph G. We prove that $\chi''_D(G) \leq \chi''(G) + 1$ for every connected graph G with the total chromatic number $\chi''(G)$. Moreover, if $\chi''(G) \geq \Delta(G) + 2$, then $\chi''_D(G) = \chi''(G)$. We prove these results by setting sharp upper bounds for the minimal number of colours in a proper total colouring such that each vertex has a distinct set of colour walks emanating from it.

Keywords: Total colourings of graphs, symmetry breaking in graphs, total distinguishing number, total distinguishing chromatic number.

Math. Subj. Class.: 05C15, 05E18

1 Introduction and definitions

In 1996, Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least number d such that G admits a vertex colouring with d colours that is only preserved by the trivial automorphism of G. Ten years later Collins and Trenk [3] defined the *distinguishing chromatic number* $\chi_D(G)$ of a graph G for proper vertex colourings, so $\chi_D(G)$ is the least number d such that G has a proper colouring with d colours that is

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only preserved by the trivial automorphism. These concepts have already spawned tens of papers. For endomorphisms instead of automorphisms this approach was investigated in [4].

Obviously, $D(G) = 1$ for all asymmetric graphs. On the other hand, for a complete graph K_n and a complete bipartite graph $K_{p,p}$ we have $D(K_n) = n$, and $D(K_{p,p}) = p+1$. The distinguishing number of cycles C_3, C_4, C_5 equals three, while cycles C_n of length $n \geq 6$ have distinguishing number two.

This compares with a more general result of Collins and Trenk [3], and of Klavžar, Wong and Zhu [7].

Theorem 1.1. *[3],[7] If G is a connected graph with maximum degree Δ, then $D(G) \leq \Delta + 1$. Furthermore, equality holds if and only if G is a K_n, $K_{p,p}$ or C_5.*

In the same paper [3], Collins and Trenk obtained a general bound for the distinguishing chromatic number.

Theorem 1.2. *[3] If G is a connected graph with maximum degree Δ, then $\chi_D(G) \leq 2\Delta$. Furthermore, equality is achieved if and only if G is a $K_{p,p}$ or C_6.*

Edge colourings breaking automorphisms were investigated by the first two authors in [5]. If a graph G does not contain K_2 as a connected component, then the *distinguishing index* $D'(G)$ of a graph G as the least number d such that G admits an edge colouring with d colours that is only preserved by the trivial automorphism. And the *distinguishing chromatic index* $\chi'_D(G)$ of a graph G is the least number d such that G has a proper edge colouring with d colours that is not preserved by any nontrivial automorphism of G. A general upper bound for the distinguishing index was proved therin.

Theorem 1.3. *[5] If G is a connected graph of order $n \geq 3$ with maximum degree Δ, then $D'(G) \leq \Delta$ unless G is C_3, C_4 or C_5.*

It was also proved in [5] that $D'(G) \leq D(G) + 1$ for any connected graph of order $n \geq 3$, and this bound is sharp for each n. Actually, quite frequently $D'(G) < D(G)$. For a complete graph $D'(K_n) = 2$ for any $n \geq 6$, and also for a complete bipartite graph $D'(K_{p,p}) = 2$ for $p \geq 4$, whereas $D(K_n)$ and $D(K_{p,p})$ are equal to $\Delta + 1$.

The following theorem gives a sharp upper bound for the distinguishing chromatic index of connected graphs.

Theorem 1.4. *[5] If G is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

except for four graphs of small order C_4, K_4, C_6, $K_{3,3}$.

This theorem immediately implies the following interesting fact.

Corollary 1.5. *[5] Every connected Class 2 graph G admits an edge colouring with $\chi'(G)$ colours that is not preserved by any nontrivial automorphism of G.*

It has to be noted that Theorem 1.4 was a consequence of Theorem 1.6, the main result of [6]. To formulate it we need some definitions.

Let $f : E \to K$ be a proper edge colouring of a graph $G = (V, E)$. For a given vertex $x \in V$, each walk emanating from x defines a sequence of colors (α_i). We then say that the sequence (α_i) is *realizable* at a vertex x. The set of all sequences realizable at x is denoted by $W(x)$. We say that two vertices x and y of a graph G are *similar* if $W(x) = W(y)$, and the coloring f *personalizes the vertices* of G if no two vertices are similar. The minimum number of colours we need to obtain this property is denoted by $\mu(G)$ and called the *vertex distinguishing index by colour walks* of a graph G.

Theorem 1.6. *Let G be a connected graph of order $n \geq 3$. Then*

$$\mu(G) \leq \Delta(G) + 1$$

except for four graphs of small orders: C_4, K_4, C_6 and $K_{3,3}$.

The aim of this paper is to present analogous results for total colourings. We give general bounds, and an interesting relationship between the total distinguishing chromatic number and the total chromatic number.

Definition 1.7. The total distinguishing number $D''(G)$ of a graph G is the least number d such that G has a total colouring with d colours that is preserved only by the identity automorphism of G.

Observe that $D''(G) \leq \min\{D(G), D'(G)\}$. Clearly the equality holds for asymmetric graphs. And also for graphs with $\min\{D(G), D'(G)\} = 2$. The following observation can easily be verified.

Proposition 1.8. $D''(P_n) = D''(C_n) = D''(K_n) = 2$ *for* $n \geq 3$. $D''(K_{p,p}) = 2$ *for* $p \geq 1$.

However, quite frequently $D''(G) < \min\{D(G), D'(G)\}$. For instance, for a star $K_{1,n}$ of size $n \geq 3$, we shall show in the next section that $D''(K_{1,n}) = \lceil \sqrt{n} \rceil$, while $D(K_{1,n}) = D'(K_{1,n}) = n$.

We shall also investigate this concept for proper total colourings. A *proper total colouring* f of a graph G is an assignment of colours to the vertices and edges of G such that no two adjacent edges, no two adjacent vertices and no incident edges and vertices are assigned the same colour. The least number of colours among all such colourings is called the *total chromatic number* denoted by $\chi''(G)$.

Definition 1.9. The total distinguishing chromatic number $\chi''_D(G)$ of a graph G is the least number d such that G has a proper total colouring with d colours that is preserved only by the identity automorphism of G.

The total chromatic number of some simple classes of graphs was investigated first by Rosenfeld in [11]. He showed that $\Delta(G) + 2$ colours are enough for cliques, for complete bipartite and tripartite graphs, for balanced complete k-partite graphs and for graphs with maximum degree at most three. Next Kostochka proved the same bound for graphs with maximum degree at most four and five (see [8] and [9]). In the general case the following famous Behzad-Vizing conjecture is still open.

Conjecture 1.10. *[2] For every graph G, the total chromatic number satisfies the inequality*

$$\chi''(G) \leq \Delta(G) + 2.$$

So far, the best result in this direction was proved by Molloy and Reed in 1998.

Theorem 1.11. *[10] For every graph $G = (V, E)$, the total chromatic number satisfies the inequality*

$$\chi''(G) \leq \Delta(G) + 10^{26}.$$

In the next section we investigate total colourings, not necessarily proper. We prove a sharp upper bound $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$ for all connected graphs.

In Section 3 we investigate total proper colourings. We show how one can personalize vertices of a graph by colour walks in total colourings. This approach is analogous to that of [6] for edge colourings.

In the last section we show that $\chi''(G)+1$ colours suffice to find a total proper colouring preserved only by the trivial automorphism. We shall infer this from the results of Section 3. However, it can also be easily shown using another argument. Namely, given a proper edge colouring of a graph G, the subgroup of $\mathrm{Aut}(G)$ preserving the colouring acts freely on vertices, i.e., the only element fixing a vertex is the identity. This follows since all paths beginning at a given vertex v are uniquely determined by the sequence of edge colours (which in effect give directions of where to go next at each vertex in the path). Thus any color preserving automorphism fixing v must fix all vertices. This immediately implies that $\chi''_D(G) \leq \chi''(G) + 1$ (just colour one vertex by an additional extra colour in a total colouring of G).

A much more intricate result of Section 4 states that $\chi''_D(G) = \chi''(G)$ whenever $\chi''(G) \geq \Delta(G) + 2$ (recall that if the Behzad-Vizing conjecture is true, then every graph has a total colouring with $\Delta(G) + 1$ or $\Delta(G) + 2$ colours). This will be proved using the main result of Section 3 concerning personalizing vertices by colour walks in proper total colorings.

2 Total distinguishing number

Every finite tree T has either a central vertex or a central edge which is fixed by every automorphism of T. For $k \geq 0$, let $S_k(x)$ denote a sphere of radius k with a center x, i.e., the set of all vertices at distance k from x.

Theorem 2.1. *If T is a tree of order $n \geq 3$, then $D''(T) \leq \lceil \sqrt{\Delta(T)} \rceil$.*

Proof. If T has a central vertex v_0, then the colour of v_0 can be arbitrary. Having $\lceil \sqrt{\Delta(T)} \rceil$ colours, we have at least $\Delta(T)$ different pairs (c_1, c_2) of colours, as the colouring need not be proper. Every edge incident to v_0 and its end vertex in the first sphere $S_1(v_0)$ obtain a distinct pair of colours (c_1, c_2). Hence, all vertices adjacent to v_0 are fixed by every automorphism of T preserving this colouring. Next, we colour edges going to subsequent spheres of T by pairs of colours in the same way as for the first sphere. By induction on the distance from v_0, all vertices of T are fixed.

If T has a central edge e_0, let T_1, T_2 be subtrees obtained by deleting the edge e_0. If we put distinct colours on the end vertices of e_0, then these verices are fixed by every automorphism. Next, for $i = 1, 2$, we colour the tree T_i using the same method as in the previous case. □

To see that the bound in Theorem 2.1 is sharp, observe that for any star $K_{1,n}$ we have $D''(K_{1,n}) = \lceil \sqrt{\Delta(K_{1,n})} \rceil = \lceil \sqrt{n} \rceil$. Indeed, if we used less than $\lceil \sqrt{n} \rceil$ colours then we

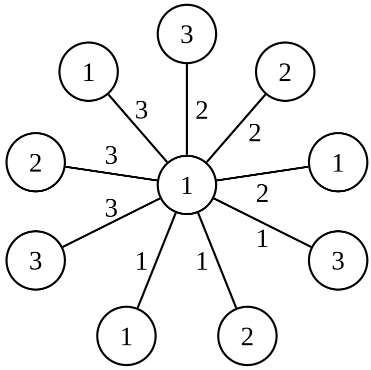

Figure 1: A total colouring of the star $K_{1,9}$ with three colours preserved only by the identity.

have less than n pairs of colours, so there would exist at least two edges coloured identically (together with their end vertices), thus a transposition of them would be a nontrivial automorphism preserving such a colouring.

Theorem 2.2. *If G is a connected graph of order $n \geq 3$, then $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.*

Proof. Denote $\Delta = \Delta(G)$. Clearly, $\Delta \geq 2$ and we have at least two colours. If G is a tree then the claim is true by Theorem 2.1. Suppose that G contains a cycle. If G is just a cycle or a complete graph, then the claim follows from Proposition 1.8.

Otherwise, we can always choose a vertex v_0 lying on a cycle such that the sphere $S_2(v_0)$ is nonempty. We colour v_0 with 2 and consider a BFS tree T of G rooted at v_0. We will first colour the tree T. For a given vertex v, denote

$$N_t(v) = \{(vu, u) \ : \ vu \in E(G)\}.$$

Let $S_1(v_0) = \{v_1, v_2, \ldots, v_p\}$. Without loss of generality we can assume that v_1 has a neighbour in $S_2(v_0)$. We colour both pairs (v_0v_1, v_1) and (v_0v_2, v_2) with a pair $(1, 1)$. Then we colour each pair of $N_t(v_0) \setminus \{(v_0v_1, v_1), (v_0v_1, v_2)\}$ with a distinct pair of colours different from $(1, 1)$. Thus $(1, 1)$ appears twice as a pair of colours in $N_t(v_0)$. We will then colour the graph G in such a way that v_0 will be the only vertex of G coloured with 2 such that the pair $(1, 1)$ appears twice in the neighbourhood $N_t(v_0)$. Hence v_0 will be fixed by every automorphism preserving the colouring. Therefore all vertices in $S_1(v_0)$ will also be fixed, except, possibly v_1 and v_2. To distinguish v_1 and v_2, we colour the sets $\{(v_1u, u) \in N_t(v_1) : u \in S_2(v_0)\}$ and $\{(v_2u, u) \in N_t(v_2) : v_2u \in E(T), u \in S_2(v_0)\}$ with two distinct sets of pairs of colours (this is possible since each of these sets contains at most $\Delta - 1$ elements, and we have Δ distinct pairs of colours). Therefore, every vertex adjacent to v_0, v_1 or v_2 will be fixed by every automorphism preserving our colouring. For each $i = 3, \ldots, p$, we then colour all elements of $\{(v_iu, u) : v_iu \in E(T), u \in S_2(v_0)\}$ with distinct pairs of colours different from the pair $(1, 1)$. This is again possible. Thus, all other vertices in $S_2(v_0)$ will be also fixed.

Then we proceed recursively with respect to the radius k of subsequent spheres $S_k(v_0)$ according to the ordering of vertices of the BFS tree T. Suppose all vertices of $S_i(v_0) = \{u_1, \ldots, u_{l_i}\}$, $i = 0, \ldots, k$, are fixed by every automorphism preserving colours. For each subsequent vertex $u_j, j = 1, \ldots, l_k$, we colour every pair (uju, u), where u is a descendent of u_j in T, with a distinct pair of colours except for $(1, 1)$. This is again possible since the number of pairs to be coloured is not greater than the number of admissible pairs of colours. Thus all neighbours of u_j in $S_{k+1}(v_0)$ will be also fixed.

Finally, we colour all remaining edges in $E(G) \setminus E(T)$ with 2. It is easily seen that if v is a vertex coloured with 2 such that the pair of colours $(1, 1)$ appears twice in $N_t(v)$, then $v = v_0$. Hence, all vertices of G are fixed by any automorphism preserving this colouring. □

Theorem 2.2 does not hold for disconnected graphs. For instance, consider a graph G of order n being the sum of r pairwise disjoint copies of K_2, i.e., $G = rK_2$ with $n = 2r$. It is easy to see that $D''(rK_2) = \min\{k : k^2(k-1) \geq r\}$. Hence, $D''(rK_2) \geq \sqrt[3]{\frac{n}{2}}$ while $\Delta(rK_2) = 1$.

3 Personalizing vertices by total colour walks

3.1 Total colour walks

In this section, we consider only proper colourings. Let f be a proper total colouring of a graph $G = (V, E)$. The *total palette of a vertex* v is the set

$$S(v) = \{f(u)\} \cup \{(f(vu), f(u)) : uv \in E\}.$$

For a given vertex $x \in V$, each walk emanating from x, say $xe_1x_1e_2x_2 \ldots e_px_p$, where $e_i = x_{i-1}x_i$ is an edge of G, $i = 1, 2, \ldots, p$, defines a sequence of colours $(f(x), f(e_1), f(x_1), f(e_2), f(x_2), \ldots, f(e_p), f(x_p))$. We then say that this sequence of colours is *realizable* at the vertex x. The set of all sequences realizable at x is denoted by $W(x)$.

We say that two vertices x and y of a graph G are *similar* with respect to f if $W(x) = W(y)$, and the colouring f *personalizes* the vertices of G if no two vertices are similar. The minimum number of colours we need to obtain this property is denoted by $\tau(G)$, and called the *vertex distinguishing index by total colour walks* of a graph G.

Denote by $W_k(x)$ all sequences of $W(x)$ of length $2k + 1$, i.e., generated by all walks of length k. We see that the total palette of a vertex v can be identified with $W_1(v)$.

For a given $(\alpha_i) \in W(x)$, denote by $m(x, (\alpha_i))$ the last vertex on a walk emanating from x and defining the sequence (α_i). The following observation will be used several times in the proof of our main result.

Proposition 3.1. *Two vertices x and y of G are similar if and only if for each $(\alpha_i) \in W(x)$, we have $(\alpha_i) \in W(y)$ and the vertices $m(x, (\alpha_i))$, $m(y, (\alpha_i))$ have the same total palettes.*

An analogous notion for edge colouring has been introduced in [6]. The corresponding parameter was denoted by $\mu(G)$. The main result of [6] was Theorem 1.6. In particular it follows that $\mu(G) = \chi'(G)$ if $\chi'(G) = \Delta(G) + 1$.

The aim of this section is to prove an analogous result for total colourings. More precisely we shall prove the following theorem.

Theorem 3.2. *Let G be a connected graph. Then*

$$\tau(G) \leq \chi''(G) + 1.$$

Moreover, if $\chi''(G) \geq \Delta(G)+2$ *then* $\tau(G) = \chi''(G)$.

The proof of this theorem is divided into two parts. First, in the subsection below, we prove that $\tau(G) \leq \chi''(G)+1$. In the next subsection, we prove the second part of the theorem for graphs with $\chi''(G) \geq \Delta(G)+2$.

The above inequalities concerning $\tau(G)$ need not be true for disconnected graphs. For instance, consider again a graph $G = rK_2$ with $n = 2k$. It is easy to see that $\tau(rK_2) = \min\{k : 3\binom{k}{3} \geq r\}$. Hence, $\tau(rK_2) \geq \sqrt[3]{n}$ while $\Delta(rK_2) = 1$ and $\chi''(rK_2) = 3$.

3.2 Graphs with $\chi''(G) = \Delta(G)+1$

In this subsection we prove Theorem 3.2 in case $\chi''(G) = \Delta(G)+1$. Let $f : V \cup E \to K$ be a colouring of G with $\chi''(G)$ colours. Let x be a vertex of G. We define a new colouring f' of G by replacing $f(x)$ with a new colour $0 \notin K$. We show that this colouring personalizes the vertices of G.

For, suppose that there are two similar vertices u and v. Denote by Q a shortest path from u to the vertex x. Consider now the walk Q' starting at v and inducing the same colour sequence as Q. Evidently, the walk Q' should also finish in x. The crucial observation is that since the last edges of Q and Q' are of the same colour, they cannot arrive to the same vertex and, since x is the only vertex of colour 0, we get a contradiction.

3.3 Graphs with $\chi''(G) \geq \Delta(G)+2$

Now, we shall prove Theorem 3.2 in case $\chi''(G) \geq \Delta(G)+2$. Let $f : V \cup E \to K$ be a proper total colouring of a graph $G = (V, E)$ with $\chi''(G)$ colours, and let $\chi''(G) \geq \Delta(G)+2$. Assume for the rest of this subsection that there is no proper total colouring of G using $\chi''(G)$ colours which personalizes the vertices of G. For convenience, we will formulate stages of the proof as observations.

Denote by $N(x)$ and $E(x)$ the set of vertices adjacent to x and the set of edges incident to x, respectively, and let $\tilde{N}(x) = f(N(x))$ and $\tilde{E}(x) = f(E(x))$.

Observation 3.3. For each vertex $x \in V$, the set $\{f(x)\} \cup \tilde{N}(x) \cup \tilde{E}(x)$ contains all colours of K.

Proof. Suppose that there is a vertex x and a colour α such that $\alpha \in K \setminus (\{f(x)\} \cup \tilde{N}(x) \cup \tilde{E}(x))$. We shall show that then f could be modified in such a way that the obtained colouring would personalize the vertices of G.

Denote by Y the set of all vertices y with $W_1(y) = W_1(x)$. If Y contains only the vertex x, we are done. For, we can repeat the reasoning from the previous subsection by considering the walks ending with x.

If Y contains more vertices, we replace $f(y)$ by α in each vertex $y \in Y$, $y \neq x$. In this way, x becomes the only vertex of G with the palette $W_1(x)$. Again, we can repeat the reasoning from the previous subsection by considering the walks ending with x. □

Observation 3.4. For each edge $xy \in E$ the set $\{f(x)\} \cup \{f(y)\} \cup \tilde{E}(x) \cup \tilde{E}(y)$ contains all colours of K.

Proof. Let us suppose that there is an edge xy and a colour α such that $\alpha \in K \setminus (\{f(x)\} \cup \{f(y)\} \cup \tilde{E}(x) \cup \tilde{E}(y))$.

Consider now the set F of all edges $x'y'$ such that $f(x'y') = f(xy)$ and $W_1(x) = W_1(x')$ and $W_1(y) = W_1(y')$. Assume first that there exists only one such edge, namely

xy. Then, our colouring personalizes the vertices of G. For, suppose that there are two similar vertices u and v. Denote by Q a shortest path joining u with the edge xy. Consider now the walk Q' starting at v and inducing the same colour sequence as Q. Evidently, the walk Q' should also attain the edge xy.

Since the last edges of Q and Q' are of the same colour, they cannot arrive at the same vertex. So, one of the walks Q and Q' finishes at x and the other one at y. Since the palettes at x and y are distinct, we are done by Proposition 3.1.

If F contains more edges, we replace $f(x'y')$ by α for all edges of F exept for the edge xy. In this way, xy becomes the only edge of G coloured with $f(xy)$ and having the palettes $W_1(x)$ and $W_1(y)$ on its ends. Therefore, we can repeat the reasoning from above. □

A vertex x is *α-free* if $\alpha \notin \{f(x)\} \cup \tilde{E}(x)$.

Observation 3.5. For each vertex x, there is a colour, say α, such that x is *α-free*.

Proof. It suffices to observe that the set $\{f(x)\} \cup \tilde{E}(x)$ contains exactly $d(x)+1$ elements while the number of colours is greater than $\Delta(G)+1$. □

We say that a set of edges incident to a vertex x of G forms a *cyclic structure of size* $p \geq 2$ (with respect to the colouring f) if these edges can be ordered as xy_i, $i = 1, \ldots, p$, such that the vertex y_i is $f(xy_{i+1})$-free, for $i = 1, \ldots, p$, where the indexes are taken modulo p. Then the vertex x is called *central* while the vertices y_i are *leaves* of the cyclic structure.

The significance of a cyclic structure is shown by the next two observations. The proof of the first one follows immediately from the definition of the cyclic structure.

Observation 3.6. If the edges xy_i, $i = 1, \ldots, p$, form a cyclic structure, then we can *rotate* the colours of edges, i.e., replace the colour $f(xy_i)$ on the edge xy_i by the colour $f(xy_{i+1})$, and the obtained colouring of G remains proper.

Observation 3.7. For each vertex x, the set $E(x)$ contains a cyclic structure.

Proof. Let x be a vertex of G and denote $f(x)$ by 0. Since the set $\{x\} \cup N(x)$ has at most $\Delta(G)+1 < \chi''(G)$ elements, there is a colour, say α, which does not belong to the set $\{f(x)\} \cup \tilde{N}(x)$. Then, by Observation 3.3, $\alpha \in \tilde{E}(x)$. Denote by y_0 the second end of the edge incident to x and coloured by α. By Observation 3.5, there is a colour, say γ_1, such that the vertex y_0 is γ_1-free.

If $\gamma_1 = 0$ we can put the colour 0 on the edge xy_0 and the colour α on the vertex x. In consequence, we are able to reduce the number of vertices having the same palette as x by one, and then eventually get only one such vertex. This would provide a proper total colouring personalizing the vertices of G.

So, we may assume that $\gamma_1 \neq 0$. Then, by Observation 3.4, $\gamma_1 \in \tilde{E}(x)$. Let xy_1 be the edge coloured with γ_1. Again, by Observation 3.5, there is a colour, say γ_2, such that the vertex y_1 is γ_2-free.

If $\gamma_2 = 0$ we can put the colour 0 on the edge xy_1, the colour γ_1 on the edge xy_0 and the colour α on the vertex x (see Figure 2). In consequence, we are able to reduce the number of vertices having the same palette as x to obtain eventally only one such vertex. This would provide a colouring personalizing vertices of G.

If $\gamma_2 = \alpha$, the edges xy_1, xy_2 form a cyclic structure of size two.

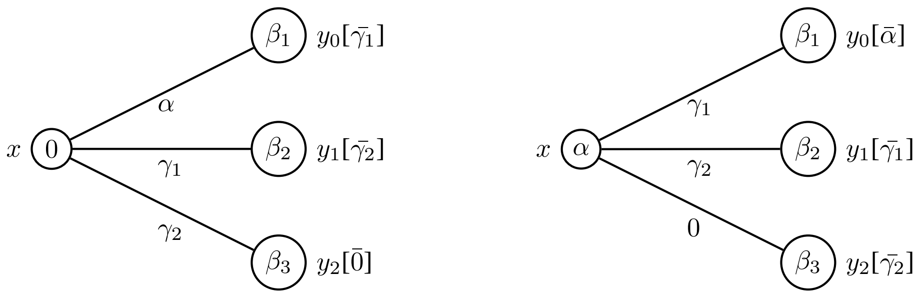

Figure 2: Before and after the change described in the proof of Observation 3.7

If $\gamma_2 \neq 0$ and $\gamma_2 \neq \alpha$, we continue the procedure of choosing at each step, as the missing colour, the first possible colour from the sequence $0, \alpha, \gamma_1, \gamma_2, \ldots$. If such a choice is possible, we can either exchange the colours and get a situation where x has a unique total palette, or we obtain a cyclic structure.

If the procedure finishes without finding 0 as a missing colour and without finding a cyclic structure, then the last vertex y_{d-1}, where $d = d(x)$, is γ_d-free for some $\gamma_d \notin \{0, \alpha, \gamma_1, \ldots, \gamma_{d-1}\}$. It means, in particular, that also the vertex x is γ_d-free, a contradiction with Observation 3.4. □

Let the set Cyc_1 of edges xy_i, $i = 1, \ldots, p$, incident to a vertex x of G, be a cyclic structure of size p (with respect to the colouring f). If all the vertices y_i, $i = 1, \ldots, p$, have the same colour, say β, then the palette at x remains unchanged after the rotation described in Observation 3.6. Therefore, we need a somewhat more complicated structure.

Suppose that a set Cyc_2 is another cyclic structure of size q with a central vertex $\hat{x}$ distinct from x. If Cyc_1 and Cyc_2 have a leave in common then we say that the sets Cyc_1 and Cyc_2 form a *double cyclic structure*.

Observation 3.8. If G has at least one double cyclic structure with respect to the colouring f then this colouring can be modified such that a new colouring personalizes the vertices of G.

Proof. Suppose that two sets of edges $\mathrm{Cyc}_1 = \{xy_i : i = 1, \ldots, p\}$ and $\mathrm{Cyc}_2 = \{\hat{x}z_j : j = 1, \ldots, q\}$ form a double cyclic structure. Without loss of generality we may assume that $y_1 = z_1$. Denote $f(y_1) = f(z_1) = \beta$ and $f(z_1\hat{x}) = \delta_1$.

Let Y be the set of all vertices y with $W_2(x) = W_2(y)$. If Y contains only the vertex x, we are done by repeating the reasoning from the previous subsection with the walks ending at x.

If Y contains more than one vertex, then each vertex y belonging to Y and different from x, is a central vertex of a cyclic structure of size p which is a part of a double cyclic structure with the second part being of size q.

Now, for each vertex $y \in Y \setminus \{x\}$, we rotate the colours of edges of the cyclic structure of size q forming the second part of a double cyclic structure. In this new colouring f' the set $W_2'(y)$ does not contain the sequence $(f(x), \gamma_1, \beta, \delta_1, f(\hat{x}))$ which was and still is present in $W_2'(x)$. In consequence, f' is a colouring such that $W_2'(x) \neq W_2'(y)$ for every vertex y distinct from x. It follows that f' personalizes the vertices of G. □

The next observation finishes the proof of Theorem 3.2.

Observation 3.9. Each graph G has at least one double cyclic structure.

Proof. For each vertex x we choose one cyclic structure $\mathrm{Cyc}(x)$ having x as a central vertex. The existence of such a structure is assured by Observation 3.7.

Consider now an auxiliary digraph Γ defined in the following way. The vertex set $V(\Gamma)$ coincides with the vertex set $V(G)$ and the arcs of Γ are the edges of G belonging to all sets $\mathrm{Cyc}(x)$ oriented from a central vertex of a structure towards the leaves of it.

By definition of a cyclic structure we have $d^+_\Gamma(x) \geq 2$ for each x. This implies, in particular, that there exists at least one vertex, say u, with $d^-_\Gamma(u) \geq 2$. Denote by z and $\hat{z}$ two of its in-neighbours in Γ. Then, the set $\mathrm{Cyc}(z) \cup \mathrm{Cyc}(\hat{z})$ forms a double cyclic structure. □

4 Total distinguishing chromatic number

The following lemma exhibits a relationship between $\tau(G)$ and $\chi''_D(G)$.

Lemma 4.1. *Every connected graph G of order $n \geq 3$ fulfils the inequality*

$$\chi''_D(G) \leq \tau(G).$$

Proof. Let f be a proper total colouring personalizing the vertices of G by colour walks, i.e., $W(x) \neq W(y)$ if $x \neq y$. Suppose φ is a nontrivial automorphism of G preserving f. Then there exists a vertex x such that $x \neq \varphi(x)$. An automorphism φ preserves the colouring, so every sequence $(\alpha_i) \in W(x)$ belongs also to $W(\varphi(x))$. And every sequence (β_i) starting at $\varphi(x)$, starts also at $\varphi^{-1}(\varphi(x)) = x$. Hence, x and $\varphi(x)$ are not distinguished by colour walks in this colouring. □

□

As a consequence of Lemma 4.1 and Theorem 3.2 we obtain a sharp upper bound for the distinguishing chromatic number of connected graphs.

Theorem 4.2. *Every connected graph G fulfils the inequality*

$$\chi''_D(G) \leq \chi''(G) + 1.$$

Moreover, $\chi''_D(G) = \chi''(G)$ if $\chi''(G) \geq \Delta(G) + 2$.

A total proper colouring of G with $\chi''(G)$ colours is called *minimal*. This theorem immediately implies the following interesting result.

Corollary 4.3. *Every connected graph G with $\chi''(G) \geq \Delta(G) + 2$ admits a minimal total colouring that is not preserved by any nontrivial automorphism.*

For graphs with $\chi''(G) = \Delta(G) + 1$, we sometimes need one colour more for $\chi''_D(G)$ than $\chi''(G)$.

For instance, cycles of order $6k$, for all $k \geq 1$, have a unique (up to a permutation of colours) colouring with three colours and this colouring is preserved by some rotations. Thus $\chi''_D(C_{6k}) = \chi''(C_{6k}) + 1$, by Theorem 4.2.

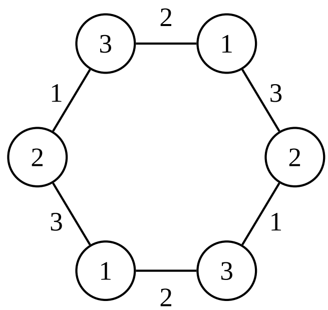

Figure 3: A minimal proper total colouring of C_6 with three colours.

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