Z_3 -connectivity of $K_{1,3}$ -free graphs without induced cycle of length at least 5

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Abstract

Jaeger *et al.* conjectured that every 5-edge-connected graph is Z_3 -connected. In this paper, we prove that every 4-edge-connected $K_{1,3}$ -free graph without any induced cycle of length at least 5 is Z_3 -connected, which partially generalizes the earlier results of Lai [Graphs and Combin. 16 (2000) 165–176] and Fukunaga [Graphs and Combin. 27 (2011) 647–659].

Keywords: Z_3 -connectivity, $K_{1,3}$ -free, nowhere-zero 3-flow.

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1 Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. Terminology and notations not defined here are from [1].

For a graph G and $v \in V(G)$, denote by $N_G(v)$ (or shortly $N(v)$) the set of neighbors of v in G . Let $d_G(v) = |N_G(v)|$ and $N[v] = N(v) \cup \{v\}$. For $A \subset V(G)$, let $N(A) = \bigcup_{v \in A} N(v) \setminus A$. A graph G is *trivial* if $|V(G)| = 1$, and *non-trivial* otherwise. An n -*cycle* is a cycle of length n . A path P_n is a path on n vertices. The complete graph on n vertices is denoted by K_n , and K_n^- is obtained from K_n by deleting an edge. For two vertex-disjoint subgraphs H_1 and H_2 of G , denote by $e_G(H_1, H_2)$ (or simply $e(H_1, H_2)$) the number of edges with one end vertex in H_1 and the other one in H_2 . If $V(H_1) = \{a\}$, we use $e_G(a, H_2)$ (or simply $e(a, H_2)$) instead of $e_G(H_1, H_2)$. For simplicity, if V_1, V_2 are two disjoint subsets of $V(G)$, we use $e_G(V_1, V_2)$ for $e_G(G[V_1], G[V_2])$. Similarly, we define $e(V_1, V_2)$ and $e(a, V_2)$. For graphs $H_1, \dots, H_s$, a graph G is $\{H_1, \dots, H_s\}$ -free if for each $i \in \{1, 2, \dots, s\}$, G has no induced subgraph H_i .

Let G be a graph and let D be an orientation of G . If an edge $e = uv \in E(G)$ is directed from a vertex u to a vertex v , then u is a tail of e , v is a head of e . For a vertex $v \in V(G)$, let $E^+(v) = \{e \in E(D) : v \text{ is a tail of } e\}$, and $E^-(v) = \{e \in E(D) : v \text{ is a head of } e\}$. Let A be an abelian group with identity 0 and $A^* = A - \{0\}$. Define $F(G, A) = \{f : E(G) \rightarrow A\}$ and $F^*(G, A) = \{f : E(G) \rightarrow A^*\}$. For each $f \in F(G, A)$, the boundary of f is a function $\partial f : V(G) \rightarrow A$ given by,

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where “ $\sum$ ” refers to the addition in A .

A function $b : V(G) \rightarrow A$ is called an A -valued zero-sum function on G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A -valued zero-sum functions on G is denoted by $Z(G, A)$. A graph G is A -connected if G has an orientation D such that for any $b \in Z(G, A)$, there is a function $f \in F(G, A^*)$ such that $\partial f(v) = b$. In particular, if $\partial f(v) = 0$ for each vertex $v \in V(G)$, then f is called a nowhere-zero A -flow of G . More specifically, a nowhere-zero k -flow is a nowhere-zero Z_k -flow, where Z_k is the cyclic group of order k . Tutte [16] proved that G admits a nowhere-zero A -flow with $|A| = k$ if and only if G admits a nowhere-zero k -flow.

Integer flow problems were introduced by Tutte in [16]. Group connectivity was introduced by Jaeger *et al.* in [7] as a generalization of nowhere-zero flows. The following longstanding conjecture is due to Jaeger *et al.* and is still open.

Conjecture 1.1. (Jaeger *et al.* [7]) *Every 5-edge-connected graph is Z_3 -connected.*

Conjecture 1.1 was extensively studied over thirty years. For the literature, some results can be seen in [3, 4, 10, 13, 17, 18] and so on. Recently, Thomassen [15] proved that every 8-edge-connected graph is Z_3 -connected, which improved by Lovász, Thomassen, Wu and Zhang [12] as follows.

Theorem 1.2. *Every 6-edge-connected graph is Z_3 -connected.*

However, Conjectures 1.1 is still open. A graph is *chordal* if every cycle of length at least 4 has a chord. A graph G is *bridged* if every cycle C of length at least 4 has two vertices x, y such that $d_G(x, y) < d_C(x, y)$. A graph is *HHD-free* if any k -cycle for $k \geq 5$ in the graph has at least two chords. Lai [9] characterized Z_3 -connectivity of 3-edge-connected chordal graphs. Li *et al.* [11] and Fukunaga [6] generalized this result to bridged graphs and 4-edge-connected HHD-free graphs.

Theorem 1.3. (Fukunaga[6]) *Every 4-edge-connected HHD-free graph is Z_3 -connected.*

Figure 1: 2 forbidden graphs

On the other hand, it is easy to see that a graph G is HHD -free if and only if G contains no induced subgraph isomorphic to house, domino and k -cycle where $k \geq 5$. Note that a domino contains a $K_{1,3}$ as a subgraph. One naturally ask whether both house and domino may be replaced by a $K_{1,3}$. On the other hand, Xu [14] proved that Conjecture 1.1 is true if and only if every 5-edge-connected $K_{1,3}$ -free graph is Z_3 -connected. Thus, we consider Z_3 -connectivity of $K_{1,3}$ -free graphs without induced cycle of length at least 5 and prove the following theorem in this paper.

Theorem 1.4. *Let G be a 4-edge-connected, $K_{1,3}$ -free simple graph. If G does not contain any induced cycle of length at least 5, then G is Z_3 -connected.*

Theorem 1.4 cannot be implied by Theorem 1.2 in the sense that there are infinite graphs which is Z_3 -connected by Theorem 1.4 but not by Theorem 1.2 as follows. Let H_1 be a copy of K_5 and H_2 be a copy of K_m where $m \geq 5$. Pick a vertex u of H_1 and a vertex v of H_2 . Define G_m to be the graph obtained from H_1 and H_2 by identifying u and v . It is easy to see that for each $m \geq 5$, G_m is a 4-edge-connected $K_{1,3}$ -free graph without any induced cycle of length at least 5. Thus, G_m is Z_3 -connected by Theorem 1.4. Clearly, G_m has an edge cut of size 4 which implies Theorem 1.2 does not show that G_m is Z_3 -connected.

Theorem 1.3 cannot imply Theorem 1.4 in the sense that there are infinite graphs which is Z_3 -connected by Theorem 1.4 but not by Theorem 1.3 as follows. Let H_i be a copy of K_{n_i} where $1 \leq i \leq 4$ and $n_i \geq 5$ for $i \in \{1, 2, 3, 4\}$. Pick two distinct vertices u_i and v_i of H_i . Denote by Γ_n the graph obtained from H_1, H_2, H_3, H_4 by identifying v_i with u_{i+1} for $i = 1, 2, 3$, and v_4 with u_1 . It is easy to verify that Γ_n contains a house and so Theorem 1.3 cannot guarantee that Γ_n is Z_3 -connected but Theorem 1.4 does.

The paper is organized as follows: In Section 2, the former related results are presented, and some lemmas are established. In Section 3, the main theorem is proved.

2 Lemmas

For a subset $X \subseteq E(G)$, the contraction G/X denotes the graph obtained from G by identifying the two ends of each edge in X and then deleting all the resulting loops. Note that even if G is simple, G/X may have multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G , then we write G/H for $G/E(H)$.

For $k \geq 2$, a *wheel* W_k is the graph obtained from a k -cycle by adding a new vertex, called the *center* of the wheel, which is adjacent to every vertex of the k -cycle. A wheel W_k is *odd* (*even*) if k is odd (or even). For technical reasons, we refer the wheel W_1 to a 3-cycle.

In order to prove Theorem 1.4, we need some lemmas. Some results [2, 5, 7, 8, 9, 10] on group connectivity are summarized as follows.

Lemma 2.1. Let A be an abelian group and G a simple graph. Then each of the following holds:

- (1) K_1 is Z_3 -connected.
- (2) If $e \in E(G)$ and if G is A -connected, then G/e is A -connected.
- (3) If H is a subgraph of G and if both H and G/H are A -connected, then G is A -connected.
- (4) For $n \geq 5$, K_n^- and K_n are Z_3 -connected;
- (5) An n -cycle is A -connected if and only if $|A| \geq n + 1$;

- (6) For every positive integer k , W_{2k} is Z_3 -connected and W_{2k+1} is not Z_3 -connected.
 (7) Let H be a Z_3 -connected subgraph of G . If $e(v, V(H)) \geq 2$ for $v \in V(G - H)$, then the subgraph induced by $V(H) \cup \{v\}$ is Z_3 -connected.
 (8) Let H_1, H_2 be subgraphs of G such that H_1 and H_2 are A -connected, If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is A -connected.

For a graph G with $u, v, w \in V(G)$ such that $uv, uw \in E(G)$, let $G_{[uv, uw]}$ denote the graph obtained from G by deleting two edges uv and uw , and then adding a new edge vw , that is, $G_{[uv, uw]} = G \cup \{vw\} - \{uv, uw\}$.

Lemma 2.2. (Chen *et al.* and Lai, [2, 9]) Let A be an abelian group, let G be a graph and u, v, w be three vertices of G such that $d(u) \geq 4$ and $v, w \in N(u)$. If $G_{[uv, uw]}$ is A -connected, then so is G .

A graph G satisfies the *Ore-condition* if $d_G(u) + d_G(v) \geq n$ for every pair of nonadjacent vertices u and v of G .

Theorem 2.3. (Luo *et al.*[13]) Let G be a simple graph on n vertices, where $n \geq 3$. If G satisfies the Ore-condition, then G is not Z_3 -connected if and only if G is one of $\{G_1, G_2, \dots, G_{12}\}$ shown in Figure 2.

Figure 2: 14 specified graphs

Lemma 2.4. Suppose that H is one graph of $\{G_7, G_{13}, G_{14}\}$. Denote by G the graph obtained from H by adding an edge $e = xy$ which is neither of H nor parallel to any existing edge of H . Then G is Z_3 -connected.

Proof. We use the same notation of G_{13}, G_{14} shown in Figure 2. Let $H = G_7$, then G satisfies the Ore-condition. By Theorem 2.3, G is Z_3 -connected.

Let $H = G_{13}$. If $x_2 \in \{x, y\}$, then G satisfies the Ore-condition. By Theorem 2.3, G is Z_3 -connected. Thus, assume that $x_2 \notin \{x, y\}$. By symmetry, let $e = x_1x_5$. Contracting 2-cycle in $G_{[x_1x_2, x_1x_3]}$ and contracting all 2-cycles generated in the process, we get an even wheel W_4 with the center at x_5 , which is Z_3 -connected by Lemma 2.1 (6) and so G is Z_3 -connected by Lemma 2.2.

Let $H = G_{14}$. If $e = x_2x_8$, then G satisfies the Ore-condition. Since $|V(H)| = 8$, by Lemma 2.3, G is Z_3 -connected. Thus, assume that $e \neq x_2x_8$. By symmetry, assume that $e = x_1x_5$ or $e = x_2x_6$. In the former case, contracting 2-cycle in $G_{[x_1x_2, x_1x_3]}$ and contracting all 2-cycles generated in the process, we obtain an even wheel W_4 induced by $\{x_1, x_4, x_5, x_6, x_7\}$ with the center at x_5 . Contracting this W_4 into one vertex and contracting 2-cycle generated in the process, finally we get a K_1 which is Z_3 -connected. By Lemmas 2.1 (7) and 2.2, G is Z_3 -connected. In the latter case, contracting 2-cycle in $G_{[x_1x_2, x_1x_3]}$ and contracting all 2-cycles generated in the process, we obtain an even wheel W_4 induced by $\{x_4, x_5, x_6, x_7, x_8\}$ with the center at x_5 , which is Z_3 -connected by Lemma 2.1. Note that x_1 has two neighbors in this even wheel. By Lemma 2.1(7), $G_{[x_1x_2, x_1x_3]}$ is Z_3 -connected. By Lemma 2.2, G is Z_3 -connected. $\square$

3 Proof of Theorem 1.4

Throughout this section, we assume that $\kappa'(G) \geq 4$, $K_{1,3}$ -free simple graph and G does not contain any induced cycle of length at least 5. We argue our proof by contradiction, assume that G is a counterexample to Theorem 1.4 with $|V(G)|$ minimized.

Lemma 3.1. Suppose that H is a maximal Z_3 -connected subgraph of G and H_i is a component of $G - V(H)$. Let $x_1 \in V(H)$ such that $x_1y_1, \dots, x_1y_k$, where $y_1, \dots, y_k \in V(H_i)$ and $2 \leq k \leq 3$. Then each of $y_1, \dots, y_k$ is not a cut vertex of H_i .

Proof. We only prove the case that $k = 3$. The proof for that $k = 2$ is similar. Without loss of generality, we will prove that y_3 is a cut vertex of H_i . Suppose otherwise that y_3 is not a cut vertex of H_i . Since the maximality of H , $e(y_i, H) = 1$ by Lemma 2.1 (7). Since G is $K_{1,3}$ -free, $y_1y_2, y_1y_3, y_2y_3 \in E(G)$. Since $\kappa'(G) \geq 4$, let $x_4 \in V(H)$ and $y_4 \in V(H_i)$ such that $x_4y_4 \in E(G)$, and y_4 is not in the component of $H_i - y_3$ containing y_1 and y_2 .

Consider the neighbors of y_1 and y_2 . Let $N(y_1) \setminus \{x_1, y_2, y_3\} = \{u_1, u_2, \dots, u_a\}$ and $N(y_2) \setminus \{x_1, y_1, y_3\} = \{v_1, v_2, \dots, v_b\}$. Since G is $K_{1,3}$ -free, both subgraphs induced by $\{u_1, \dots, u_a\}$ and by $\{v_1, \dots, v_b\}$ are complete graphs. We assume, without loss of generality, that $a \geq b$. Since G is 4-edge-connected, $a \geq 1$ and $b \geq 1$. Note that y_3 is a cut vertex of H_i and G is $K_{1,3}$ -free. The following claim is straightforward.

Claim. All neighbors of y_3 are y_1, y_2 in the component of $H_i - y_3$ containing $\{y_1, y_2\}$.

Case 1. $\{u_1, \dots, u_a\} \cap \{v_1, v_2, \dots, v_b\} \neq \emptyset$.

If $a \geq 4$, then the subgraph induced by $\{y_1, u_1, u_2, \dots, u_a\}$ is a complete graph K_{a+1} , which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \dots, u_a\}$, contrary to the maximality of H . Thus, $a \leq 3$.

Assume that $a = 3$. If $|\{u_1, u_2, \dots, u_a\} \cap \{v_1, v_2, \dots, v_b\}| \geq 2$, then the subgraph induced by $\{y_1, y_2, u_1, \dots, u_a\}$ is K_5 or K_5^- , which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \dots, u_a\}$ which is larger than H , contrary to the choice of H . Thus, $|\{u_1, u_2, \dots, u_a\} \cap \{v_1, v_2, \dots, v_b\}| = 1$ and let $u_1 = v_1$. Assume that $3 \geq b \geq 2$. Since $\kappa'(G) \geq 4$, there is a path from $\{u_2, u_3\}$ to v_2 avoiding each vertex of $\{y_1, y_2, u_1\}$. Since G contains no induced cycle of length at least 5, $u_i v_2 \in E(G)$ where $i \in \{2, 3\}$. In this case, G contains an even wheel W_4 induced by $\{y_1, y_2, u_1, u_i, v_2\}$ with the center at u_1 , which is Z_3 -connected by Lemma 2.1 (6). By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, \dots, u_a\}$, contrary to the maximality of H . Thus, $b = 1$. In this case, since $\kappa'(G) \geq 4$, let $u_2 p_1, u_3 q_1 \in E(G)$ where $p_1 \notin \{u_1, u_3, y_1\}$ and $q_1 \notin \{u_1, u_2, y_1\}$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, $p_1 q_1, p_1 u_3, q_1 u_2 \in E(G)$. We replace p_1 with u_2 and replace q_1 with u_3 . By argument above, we obtain p_2, q_2 such that $p_2 q_2, p_2 p_1, q_2 q_1, p_2 q_1, q_2 p_1 \in E(G)$. Repeating such a way, we can obtain two infinite sequences of $p_1, p_2, \dots$ and $q_1, q_2, \dots$ such that $p_i p_{i+1}, q_i q_{i+1}, p_i q_i, p_i q_{i+1}, q_i, q_{i+1} \in E(G)$ for $i = 1, 2, \dots$. This contradicts that G is finite.

We are left to consider that $a \leq 2$. In this case, since G is 4-edge-connected, $a = b = 2$ and $\{u_1, u_2\} = \{v_1, v_2\}$. As the proof above, we also obtain a contradiction.

Case 2. $\{u_1, \dots, u_a\} \cap \{v_1, v_2, \dots, v_b\} = \emptyset$.

We claim that $a + b \geq 4$. Suppose otherwise that $a + b \leq 3$. It follows that either $a = 2, b = 1$ or $a = b = 1$. We only prove the case when $a = 2$ and $b = 1$. The proof is similar for the case that $a = b = 1$. Since $a = 2$ and $b = 1$, $y_1 u_1, y_1 u_2, y_2 v_1 \in E(G)$. By the Claim, y_3 is not adjacent to one of u_1, u_2 and v_1 . Thus, $\{y_1 u_1, y_1 u_2, y_2 v_1\}$ is an edge cut of size 3, contrary to that $\kappa'(G) \geq 4$.

Assume that $a \geq 4$. If $b \geq 4$, then G contains a path from $\{u_1, \dots, u_a\}$ to $\{v_1, \dots, v_b\}$. Note that $\kappa'(G) \geq 4$ and G has no cycle of length at least 5. If $2 \leq b \leq 3$, then each vertex of $\{v_1, v_2, \dots, v_b\}$ has a neighbor in $\{u_1, u_2, \dots, u_a\}$. If $b = 1$, then v_1 has three neighbors in $\{u_1, \dots, u_a\}$. By Lemma 2.1 (4), G contains a Z_3 -connected subgraph K_{a+1} . By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, \dots, u_a, v_1, \dots, v_b\}$, contrary to the maximality of H .

Assume that $a = 3$. If $b = 3$, denote by F the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2, v_3, y_1, y_2\}$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to one of $\{v_1, v_2, v_3\}$ and each vertex of $\{v_1, v_2, v_3\}$ is adjacent to each vertex of $\{u_1, u_2, u_3\}$. Since $\kappa'(G) \geq 4$, $e(\{u_1, u_2, u_3\}, \{v_1, v_2, v_3\}) \geq 3$ and each vertex of F is of degree 4 and this subgraph satisfies the Ore-condition. By Theorem 2.3, F is Z_3 -connected. By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup V(F)$, contrary to the maximality of H .

Let $b = 2$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2, u_3\}$ is adjacent to one of $\{v_1, v_2\}$ and each vertex of $\{v_1, v_2\}$ is adjacent to two vertices of $\{u_1, u_2, u_3\}$. It follows that one, say u_3 , of $\{u_1, u_2, u_3\}$ has two neighbors in $\{v_1, v_2\}$. It implies that the subgraph induced by $\{u_1, u_2, u_3, v_1, v_2\}$ is an even wheel W_4 with the center at u_3 , which is Z_3 -connected by Lemma 2.1 (6). By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, u_3, v_1, v_2\}$, contrary to the maximality of H .

Let $b = 1$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, v_1 is adja-

cent to each vertex of $\{u_1, u_2, u_3\}$. The subgraph induced by $\{u_1, u_2, u_3, v_1, y_1\}$ is K_5^- , which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{y_1, y_2, y_3, u_1, u_2, u_3, v_1\}$, contrary to the maximality of H .

Next, assume that $a = 2$. Let $b = 2$. Since $\kappa'(G) \geq 4$ and G contains no cycle of length at least 5, each vertex of $\{u_1, u_2\}$ is adjacent to two of $\{v_1, v_2\}$ and each vertex of $\{v_1, v_2\}$ is adjacent to two vertices of $\{u_1, u_2\}$. Denote by F the subgraph induced by $\{y_1, y_2, u_1, u_2, v_1, v_2\}$. It follows that F satisfies the Ore-condition and each of 4 vertices of F is of degree 4. By Theorem 2.3, F is Z_3 -connected. By Lemma 2.1 (7), G contains a Z_3 -connected subgraph induced by $V(H) \cup V(F)$, contrary to the maximality of H . $\square$

Lemma 3.2. G does not contain a nontrivial Z_3 -connected subgraph H .

Proof. Suppose that our lemma fails and H is a maximal Z_3 -connected subgraph of G . Suppose that $H_1, H_2, \dots, H_k$ are components of $G - V(H)$, where $k \geq 1$. Let $G' = G/H$ and v' be the vertex into which H is contracted.

Observe H_i , where $i \in \{1, 2, \dots, k\}$. Let $E(H, H_i) = \{x_1y_1, x_2y_2, \dots, x_ty_t\}$, where $x_i \in V(H)$ and $y_j \in V(H_i)$ for $i, j \in \{1, 2, \dots, t\}$. Since G is 4-edge-connected, $t \geq 4$. By the maximality and by Lemma 2.1 (7), $y_1, \dots, y_t$ are distinct t vertices of H_i . Let $e_i = x_iy_i$ for $i \in \{1, 2, \dots, t\}$.

Claim 1. $E(H, H_i)$ does not contain 4 edges having a common end-vertex.

Proof of Claim 1. Suppose otherwise that without loss of generality, that e_1, e_2, e_3, e_4 have a common vertex x_1 , that is, $x_1 = x_2 = \dots = x_4$. Then the subgraph induced by $\{x_1, y_1, \dots, y_4\}$ is a complete graph K_5 since G is $K_{1,3}$ -free. By Lemma 2.1 (4), K_5 is Z_3 -connected. By Lemma 2.1 (8), G contains a Z_3 -connected subgraph induced by $V(H) \cup \{x_1, y_1, \dots, y_4\}$, contrary to the choice of H . Thus, $E(H, H_i)$ contains at most three edges having a common vertex. This proves Claim 1.

Claim 2. $E(H, H_i)$ does not contain 4 independent edges.

Proof of Claim 2. Suppose otherwise that $E(H, H_i)$ contains 4 independent edges. We assume, without loss of generality, that e_1, e_2, e_3, e_4 are independent edges. Since G has no induced cycle of length at least 5, as the argument above, $y_iy_j \in E(G)$ for $1 \leq i < j \leq 4$. This means that the subgraph induced by $\{y_1, y_2, y_3, y_4\}$ is a K_4 . In the graph G' , the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5 which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3 -connected, contrary the maximality of H . This proves Claim 2.

Claim 3. $E(H, H_i)$ does not contain 2 edges having a common end-vertex.

Proof of Claim 3. By Claim 2, we assume that $t = 4$ and e_1, e_2, e_3, e_4 have at least a pair of two edges sharing a vertex in H . Suppose otherwise that we assume, without loss of generality, that e_1, e_2 have a common vertex x_1 , that is, $x_1 = x_2$. Since $t = 4$, we need to consider e_3 and e_4 do not share a common end-vertex or e_3 and e_4 share a common end-vertex.

In the former case, the subgraph induced by $\{x_1, y_1, y_2\}$ is a K_3 since G is $K_{1,3}$ -free. Since G has no induced cycle of length at least 5, $y_3y_4 \in E(G)$, $y_3y_i, y_4y_j \in E(G)$ where $i, j \in \{1, 2\}$. By Lemma 3.1, the subgraph induced by $\{y_1, y_2, y_3, y_4\}$ is a K_4 since G has no induced cycle of length at least 5. In the graph G' , the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5 which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (3),

the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3 -connected, contrary the choice of H .

In the latter case, we assume, without loss of generality, that e_3 and e_4 share a common end-vertex x_3 . Since G is $K_{1,3}$ -free, the subgraph induced by $\{x_1, y_1, y_2\}$ is a complete graph and so is the subgraph induced by $\{x_3, y_3, y_4\}$. Since G has no induced cycle of length at least 5, as the argument above, $y_i y_j \in E(G)$ for some $i \in \{1, 2\}$ and some $j \in \{3, 4\}$. We assume, without loss of generality, that $i = 2, j = 3$. By Lemma 3.1, each vertex of $\{y_1, y_2, y_3, y_4\}$ is not a cut vertex. Since G has no induced cycle of length at least 5 and G is 4-edge-connected, y_2 is adjacent to y_4 , and y_3 is adjacent to y_1 . In the graph G' , the subgraph induced by $\{v', y_1, y_2, y_3, y_4\}$ is a K_5^- which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3 -connected, contrary the maximality of H . This proves Claim 3.

By Claims 1, 2, and 3, we assume, without loss of generality, that e_1, e_2, e_3 have a common vertex x_1 , that is, $x_1 = x_2 = x_3$. Thus, $t = 4$ and $x_4 \neq x_1$. It follows that the subgraph induced by $\{x_1, y_1, y_2, y_3\}$ is a complete graph K_4 . Consider the cycle $x_1 P x_4 y_4 Q y_j$, where $V(P) \subset V(H)$, $V(Q) \subset V(H_i)$ and $j \in \{1, 2, 3\}$. Since G contains no any induced cycle of length at least 5, $V(P) = V(Q) = \emptyset$ and $x_1 x_4, y_4 y_j \in E(G)$. We assume, without loss of generality, that $j = 3$, that is, $y_3 y_4 \in E(G)$. By Lemma 3.1, each of $\{y_1, y_2, y_3\}$ is not cut vertex. Since G contains no any induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $y_1 y_4, y_2 y_4 \in E(G)$. This, in the graph G' , the subgraph induced by $\{v', y_1, y_2, y_3, y_5\}$ is a K_5 , which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.1 (3), the subgraph induced by $V(H) \cup \{y_1, y_2, y_3, y_4\}$ is Z_3 -connected, contrary the maximality of H . $\square$

Proof of Theorem 1.4

Since domino contains an induced $K_{1,3}$ and G contains no induced $K_{1,3}$, G contains no induced domino. By Theorem 1.3 and the choice of G , G contains an induced house. We use the same notations depicted in Figure 2. By symmetry, assume that $d(u) \leq d(v)$.

Claim 1. $|N(u) \cap N(v) \setminus \{w\}| \leq 1$.

Proof of Claim 1. Suppose otherwise that $|N(u) \cap N(v) \setminus \{w\}| \geq 2$. Let $u_1, v_1 \in N(u) \cap N(v) \setminus \{w\}$. Denote by F the subgraph induced by $\{u_1, v_1, w\}$. Since G is $K_{1,3}$ -free, F contains at least one edge. If F contains two edges, then the subgraph induced by $\{u_1, v_1, w, u, v\}$ contains an even wheel W_4 , which is Z_3 -connected by Lemma 2.1 (6), contrary to Lemma 3.2. Thus, F contains only one edge e . By symmetry, assume that $e = wu_1$ or $e = u_1v_1$. In each case, since G is $K_{1,3}$ -free, $xv_1, yv_1 \in E(G)$. This means that the subgraph induced by $\{v_1, u, v, x, y\}$ is an even wheel W_4 with the center at v_1 , which is Z_3 -connected by Lemma 2.1 (6), contrary to Lemma 3.2. This proves Claim 1.

Claim 2. $|N(u) \cap N(v) \setminus \{w\}| \neq 0$.

Proof of Claim 2. Suppose otherwise that $|N(u) \cap N(v) \setminus \{w\}| = 0$. Since $\kappa'(G) \geq 4$, $\delta(G) \geq 4$. First, we claim that $\max\{d(u), d(v)\} \leq 5$. Suppose otherwise that $d(u) \geq 6$. Let $u_1, u_2, u_3 \in N(u) \setminus \{w, v, x\}$. Since G is $K_{1,3}$ -free, either $G[\{u, x, u_1, u_2, u_3\}]$ or $G[\{u, w, u_1, u_2, u_3\}]$ is a complete subgraph K_5 which is Z_3 -connected by Lemma 2.1, contrary to Lemma 3.2. Thus, $4 \leq d(u), d(v) \leq 5$.

Assume first that $d(u) = d(v) = 4$. Let $N(u) \setminus \{w, v, x\} = \{u_1\}$ and $N(v) \setminus \{w, u, y\} = \{v_1\}$. Since G is $K_{1,3}$ -free and $u_1v, v_1u \notin E(G)$, $u_1x, v_1y \in E(G)$.

Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $xv_1 \in E(G)$, then $G[\{u, v, u_1, v_1, x, y\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, xv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wu_1 \notin E(G)$. Since $\kappa'(G) \geq 4$, there exists a shortest (u_1, w) -path P such that $N_P(u_1) \not\subseteq \{u, x, v_1\}$. Since $wu_1 \notin E(G)$, $u_2 \in V(P)$ such that $u_1u_2, u_2w \in E(G)$ since G contains no induced cycle of length at least 5. Consider the cycle wu_2u_1xyvw . Since G contains no induced cycle of length at least 5, $u_2y, u_2x \in E(G)$. Since $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_2v \notin E(G)$. This implies that G contains a $K_{1,3}$ induced by $\{u_2, u_1, w, y\}$, a contradiction.

Next, assume that $d(u) = 4$ and $d(v) = 5$. Let $N(u) \setminus \{w, v, x\} = \{u_1\}$ and $N(v) \setminus \{w, v, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free and $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_1x, v_1y, v_2y, v_1v_2 \in E(G)$. If $wv_1, wv_2 \in E(G)$, then G contains a K_5^- induced by $\{w, v, v_1, v_2, y\}$ which is Z_3 -connected by Lemma 2.1 (4), contrary to Lemma 3.2. Thus, assume that $wv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $u_1v_2 \in E(G)$, then $G[\{u, v, x, y, u_1, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{13} + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, u_1v_2 \notin E(G)$. As the proof above, there is u_2 such that $u_1u_2, u_2w \in E(G)$ and $u_2y, u_2x \in E(G)$. It follows that G contains a $K_{1,3}$ induced by $\{u_2, u_1, w, y\}$, a contradiction.

Finally, assume that $d(u) = d(v) = 5$. Let $N(u) \setminus \{w, v, x\} = \{u_1, u_2\}$ and $N(v) \setminus \{w, u, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free and $|N(u) \cap N(v) \setminus \{w\}| = 0$, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2, u_1v_1 \in E(G)$. If $\{u_2y, u_2v_2, u_2v_1\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, u_2, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{14} + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_2y, u_2v_2, u_2v_1 \notin E(G)$. Since G contains no induced cycle of length at least 5, $u_2w \notin E(G)$. Since $\kappa'(G) \geq 4$, as the proof above, there exists a vertex $u_3 \in V(P)$ such that $u_2u_3, u_3w \in E(G)$ and $u_3x, u_3y \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{u_3, u_2, y, w\}$, a contradiction. This proves Claim 2.

By Claims 1 and 2, assume that $N(u) \cap N(v) \setminus \{w\} = \{z\}$. If $xz, yz \in E(G)$, then $G[\{u, v, x, y, z\}]$ is a Z_3 -connected subgraph W_4 , contrary to Lemma 3.2. Thus, $xz \notin E(G)$ or $yz \notin E(G)$. Recall that $d(u) \leq d(v)$. We claim that $d(v) \leq 6$. Otherwise, since G is $K_{1,3}$ -free, $G[N[v] \setminus \{w, u, z\}]$ contains a complete subgraph K_m , where $m \geq 5$, which is Z_3 -connected by Lemma 2.1, contrary to Lemma 3.2. Thus, $4 \leq d(u), d(v) \leq 6$.

Case 1. $xz, yz \notin E(G)$.

Since $G[\{u, w, x, z\}]$ is not an induced $K_{1,3}$, $wz \in E(G)$. We first establish a claim.

Claim 3. If $d(u) = 4$, then $d(x) = 4$; if $d(v) = 4$, then $d(y) = 4$.

Proof of Claim 3. Suppose otherwise that $d(x) \geq 5$. Since $d(u) = 4$, each $s \in N(x) \setminus \{u\}$ is not adjacent to u . Thus, $G[N[x] \setminus \{u\}]$ is a Z_3 -connected K_m , where $m \geq 5$, since G is $K_{1,3}$ -free, contrary to Lemma 3.2. Since G is 4-edge-connected, $d(x) \geq 4$. Thus, $d(x) = 4$. The proof for the case that $d(y) = 4$ is similar. This proves Claim 3.

Assume that $d(u) = d(v) = 4$. By Claim 3, $d(x) = 4$. Let $N(x) \setminus \{u, y\} = \{x_1, x_2\}$. Since G is $K_{1,3}$ -free, $yx_1, yx_2, x_1x_2 \in E(G)$. Since $\kappa'(G) \geq 4$, G contains a path from x_1 to w which does not contain any vertex of $\{x_2, x, y, u, v\}$. Since G contains no induced cycle of length at least 5, this path is an edge, that is, $x_1w \in E(G)$ or $x_1z \in E(G)$. Similarly, we can prove that $x_2z \in E(G)$ or $x_2w \in E(G)$. In each case, $H =$

$G[\{u, v, x, y, x_1, x_2, w, z\}]$ satisfies the Ore-condition. By Lemma 2.3, H is Z_3 -connected, contrary to Lemma 3.2.

Assume that $d(u) = 4$ and $d(v) = 5$. Let $N(v) \setminus \{u, w, z, y\} = \{v_1\}$. Since G is $K_{1,3}$ -free, $yv_1 \in E(G)$. By the Claim, $d(x) = 4$. Assume that $xv_1 \in E(G)$. Let $xx_1 \in E(G)$. Since G is $K_{1,3}$ -free, $x_1y, x_1v_1 \in E(G)$. Let $H = G[\{u, v, x, y, x_1, v_1, w, z\}]$. If $wv_1 \in E(G)$, contract the 2-cycle (v, v_1) in $H_{[wv, wv_1]}$ and repeatedly contract the 2-cycles generated in the process, eventually, we get a K_1 which is Z_3 -connected. By Lemmas 2.1 and 2.2, H is Z_3 -connected, contrary to Lemma 3.2. Thus, $wv_1 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $x_1w \in E(G)$. As the proof above, we can get $H_{[x_1y, x_1v_1]}$ is Z_3 -connected. By Lemma 2.2, H is Z_3 -connected, contrary to Lemma 3.2.

Thus, $xv_1 \notin E(G)$. Let $xx_1, xx_2 \in E(G)$. Since G is $K_{1,3}$ -free, $yx_1, yx_2, x_1x_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $x_1v_1, x_2v_1, wv_1, zv_1 \notin E(G)$. Since G contains no induced cycle of length at least 5 and $\kappa'(G) \geq 4$, $x_1w, x_2z \in E(G)$ or $x_1z, x_2w \in E(G)$. In each case, $L = G[\{u, v, x, y, x_1, x_2, w, z\}]$ satisfies the Ore-condition. By Lemma 2.3, L is Z_3 -connected, contrary to Lemma 3.2.

If $d(u) = 4$ and $d(v) = 6$, let $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free, $v_1y, v_2y, v_1v_2 \in E(G)$. By the Claim, $d(x) = 4$. First assume that $xv_1, xv_2 \in E(G)$. In this case, G contains a Z_3 -connected subgraph K_5^- induced by $\{x, y, v, v_1, v_2\}$, contrary to Lemma 3.2. Next, assume that $xv_1 \in E(G)$ and $xv_2 \notin E(G)$. Let $xx_1 \in E(G)$. Since G is $K_{1,3}$ -free, $x_1y, x_1v_1 \in E(G)$. Let $H = G[\{u, w, v, x, y, x_1, v_1, v_2\}]$. If $wv_1 \in E(G)$ or $wv_2 \in E(G)$ or $x_1z \in E(G)$, we can prove that $H_{[wv, wv_1]}$ or $H_{[wv, wv_2]}$ or $H_{[x_1y, x_1v_1]}$ is Z_3 -connected. By Lemma 2.2, H is Z_3 -connected, contrary to Lemma 3.2. If $x_1v_2 \in E(G)$, then G contains a Z_3 -connected subgraph K_5^- induced by $\{x_1, y, v, v_1, v_2\}$, a contradiction. Thus, $wv_1, wv_2, x_1z, x_1v_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wx_1 \in E(G)$. As the argument above, $H_{[x_1y, x_1v_1]}$ is Z_3 -connected. By Lemma 2.2, H is Z_3 -connected, contrary to Lemma 3.2. Finally, assume that $xv_1, xv_2 \notin E(G)$. Let $xx_1, xx_2 \in E(G)$. Since G is $K_{1,3}$ -free, $x_1x_2, yx_1, yx_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2, zv_1, zv_2 \notin E(G)$ and $e(\{x_1, x_2\}, \{v_1, v_2\}) = 0$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wx_1, zx_2 \in E(G)$ or $wx_2, zx_1 \in E(G)$. In each case, $L = G[\{u, w, v, x, y, x_1, x_2, z\}]$ satisfies the Ore-condition, by Lemma 2.3, L is Z_3 -connected, contrary to Lemma 3.2.

If $d(u) = 5$ and $d(v) = 5$, let $N(u) \setminus \{v, w, z, x\} = \{u_1\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1\}$. Since G is $K_{1,3}$ -free, $u_1x, v_1y \in E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $u_1v_1 \in E(G)$. If $u_1y \in E(G)$ or $v_1x \in E(G)$, then $G[\{u, v, x, y, u_1, v_1\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $u_1y, v_1x \notin E(G)$. Assume that $u_1z \in E(G)$. Since G is $K_{1,3}$ -free, $v_1z \in E(G)$. It follows that G contains a Z_3 -connected subgraph W_4 induced by $\{u, v, u_1, v_1, z\}$ with the center at z , contrary to Lemma 3.2. Thus, by symmetry, we assume that $u_1z, v_1z \notin E(G)$ and $wu_1, wv_1 \notin E(G)$. As $\kappa'(G) \geq 4$, there is w_1 such that $u_1w_1, w_1w \in E(G)$. Observe cycle ww_1u_1xyvw . Since G contains no induced cycle of length at least 5, $w_1y \in E(G)$. It follows that G contains a $K_{1,3}$ induced by $\{w_1, u_1, w, y\}$, a contradiction.

If $d(u) = 5$ and $d(v) = 6$, let $N(u) \setminus \{v, w, z, x\} = \{u_1\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free, $u_1x, v_1y, v_2y, v_1v_2 \in E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2, zv_1, zv_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains

no induced cycle of length at least 5, by symmetry, we assume that $u_1v_1 \in E(G)$. If $\{u_1y, u_1v_2, v_1x, v_2x\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{13} + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, assume that $u_1y, u_1v_2, v_1x, v_2x \notin E(G)$. Since G has no induced cycle of length at least 5, $u_1z, wu_1 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there is w_1 such that $u_1w_1, w_1w \in E(G)$. Since G is $K_{1,3}$ -free, $w_1y, w_1x \in E(G)$. This implies that $G[\{w_1, u_1, w, y\}]$ is an induced $K_{1,3}$, a contradiction.

If $d(u) = 6$ and $d(v) = 6$, let $N(u) \setminus \{v, w, z, x\} = \{u_1, u_2\}$ and $N(v) \setminus \{u, w, z, y\} = \{v_1, v_2\}$. since G is $K_{1,3}$ -free, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2 \in E(G)$. If either $e(\{u_1, u_2\}, \{v_1, v_2\}) \geq 2$ or $e(\{u_1, u_2\}, \{v_1, v_2\}) = 1$ and $\{u_1y, u_1v_2, u_2y, u_2v_2, u_2v_1\} \cap E(G) \neq \emptyset$, then $G[\{u, v, x, y, u_1, u_2, v_1, v_2\}]$ contains a subgraph isomorphic to $G_{14} + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $e(\{u_1, u_2\}, \{v_1, v_2\}) \leq 1$. Moreover, if $e(\{u_1, u_2\}, \{v_1, v_2\}) = 1$ and $u_1y, u_1v_2, u_2y, u_2v_2, u_2v_1 \notin E(G)$. In this case, let $u_1v_1 \in E(G)$. Since G contains no induced cycle of length at least 5, $wu_1, wu_2, wv_1, wv_2, u_2z \notin E(G)$. Consider the case that $e(\{u_1, u_2\}, \{v_1, v_2\}) = 0$. By Lemmas 2.4 and 3.2, $e(x, \{v_1, v_2\}) \leq 1$ and $e(y, \{u_1, u_2\}) \leq 1$. Since G contains no induced cycle of length at least 5, $wu_2, u_2z \notin E(G)$. In each case, since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there is w_1 such that $u_2w_1, w_1w \in E(G)$ and $w_1y, w_1x \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{w_1, u_2, w, y\}$, a contradiction.

Case 2. one edge of $\{xz, yz\}$ is not in $E(G)$.

We assume, without loss of generality, that $xz \in E(G)$ and $yz \notin E(G)$. Since G is $K_{1,3}$ -free, $wz \in E(G)$. Consider that $d(u) = d(v) = 4$. Since $\delta(G) \geq 4$ and G is $K_{1,3}$ -free, $d(y) = 4$. Let $\{y_1, y_2\} \subseteq N(y) \setminus \{x, v\}$. Assume that one edge of y_1z, y_2z is in G , without loss of generality, assume that $y_1z \in E(G)$. Since G is $K_{1,3}$ -free, $y_1x, y_2x, y_1y_2 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, y_1, y_2\}]$. Contracting the 2-cycle (y_1, y_2) in $H_{[y_1, y_2]}$ and repeatedly contracting the 2-cycles generated in the process, eventually, we get a K_1 which is Z_3 -connected. By Lemmas 2.1 and 2.2, H is Z_3 -connected, contrary to Lemma 3.2. Thus, $y_1z, y_2z \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, $wy_1 \in E(G)$ or $wy_2 \in E(G)$. In each case, Contracting 2-cycle (u, w) and contracting all 2-cycle generated in the process in $H_{[wu, wz]}$, we obtain a K_5^- which is Z_3 -connected by Lemma 2.1 (1). By Lemma 2.2, H is Z_3 -connected, contrary to Lemma 3.2.

If $d(u) = 4$ and $d(v) = 5$, let $v_1 \in N(v) \setminus \{w, u, y, z\}$. Since G is $K_{1,3}$ -free, $v_1y \in E(G)$. Since $\kappa'(G) \geq 4$, let $yy_1 \in E(G)$. Let H be the subgraph induced by $\{u, v, x, y, w, z, y_1, v_1\}$. Since G is $K_{1,3}$ -free, $xy_1 \in E(G)$. Since G contains no induced cycle of length at least 5, $v_1w \notin E(G)$. We claim that $v_1x \notin E(G)$ for otherwise, assume that $v_1x \in E(G)$. Since G is $K_{1,3}$ -free, $y_1v_1 \in E(G)$. Contracting 2-cycle (y_1, v_1) and contracting all 2-cycles generated in the process in $H_{[xy_1, xv_1]}$, we get a K_5^- which is Z_3 -connected by Lemma 2.1 (4). By Lemma 2.2, H is Z_3 -connected, contrary to Lemma 3.2. If $v_1z \in E(G)$, by Lemma 3.2, $v_1u, v_1w \notin E(G)$. In this case, the subgraph induced by $\{z, x, w, v_1\}$ is a $K_{1,3}$, a contradiction. Thus, $v_1z \notin E(G)$. If $wy_1 \in E(G)$, then $H_{[wu, wz]}$ contains a 2-cycle (u, z) . Contracting this 2-cycle and contracting all 2-cycles generated in the process, finally we obtain a K_1 . By Lemma 2.1 (1) (3) (5), and by Lemma 2.2, H is Z_3 -connected, contrary to Lemma 3.2. Thus, $wy_1 \notin E(G)$. Recall that $wx \notin E(G)$. Since $\kappa'(G) \geq 4$, there is a vertex w_1 such that $ww_1, w_1v_1 \in E(G)$. Since $d(u) = 4$

and $d(v) = 5$, $w_1u, w_1v \notin E(G)$. Since G has no induced cycle of length at least 5, $w_1x \in E(G)$. In this case, the subgraph induced by $\{w, w_1, x, v_1\}$ is a $K_{1,3}$, a contradiction.

If $d(u) = 4$ and $d(v) = 6$, let $N(v) \setminus \{w, u, x, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free, $yv_1, yv_2, v_1v_2 \in E(G)$. Assume that $v_1z \in E(G)$. Observe the subgraph $G[\{z, x, w, v_1\}]$. Since G is $K_{1,3}$ -free, $xv_1 \in E(G)$ or $wv_1 \in E(G)$. In the former case, G contains a Z_3 -connected subgraph W_4 induced by $\{z, u, x, v_1, v\}$ with the center at z , contrary to Lemma 3.2. In the latter case, G contains a Z_3 -connected subgraph W_4 induced by $\{w, u, z, v_1, v\}$ with the center at v , contrary to Lemma 3.2. Thus, $v_1z \notin E(G)$. Similarly, $v_2z \notin E(G)$. If $v_1x, v_2x \in E(G)$, then G contains a Z_3 -connected subgraph K_5^- induced by $\{y, x, v_1, v, v_2\}$, contrary to Lemma 3.2. Thus, $|\{v_1x, v_2x\} \cap E(G)| \leq 1$. Assume that $v_1x \notin E(G)$. Since G contains no induced cycle of length at least 5, $wv_1, wv_2 \notin E(G)$. Since $\kappa'(G) \geq 4$ and G contains no induced cycle of length at least 5, there exists a vertex w_1 such that $ww_1, w_1v_1 \in E(G)$ and $w_1x \in E(G)$. In this case, G contains a $K_{1,3}$ induced by $\{w_1, w, x, v_1\}$, a contradiction.

If $d(u) = d(v) = 5$, let $N(u) \setminus \{w, v, x, z\} = \{u_1\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1\}$. Since G is $K_{1,3}$ -free, $u_1x, v_1y \in E(G)$. Since G is $K_{1,3}$ -free, $zu_1 \in E(G)$. Since G has no induced cycle of length at least 5, $wv_1 \notin E(G)$. We claim that $zv_1 \notin E(G)$. To the contrary, assume that $zv_1 \in E(G)$. Since G is $K_{1,3}$ -free, $u_1v_1, xv_1 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, u_1, v_1\}]$. Contracting the 2-cycle (u, x) in $H_{[u_1u, u_1x]}$ and repeatedly contracting the all 2-cycles generated in the process, eventually, we get a K_1 which is Z_3 -connected. By Lemmas 2.1 and 2.2, H is Z_3 -connected, contrary to Lemma 3.2. Thus, $v_1z \notin E(G)$. In this case, since $\kappa'(G) \geq 4$, there is a path Q from u_1 to v_1 avoiding any vertex in $\{z, w, u, v\}$. Since G has no induced cycle of length at least 5, $|E(Q)| = 1$, that is, $v_1u_1 \in E(G)$. If $u_1y \in E(G)$ or $v_1x \in E(G)$, then $G[\{u, v, x, y, u_1, v_1\}]$ contains a subgraph isomorphic to $G_7 + e$ which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $u_1y, v_1x \notin E(G)$. Since G has no induced cycle of length at least 5, $wu_1 \notin E(G)$. As $\kappa'(G) \geq 4$, there is a path P from w to v_1 . Since $wv_1 \notin E(G)$, there is $w_1 \in V(G)$ such that $w_1w, w_1v_1 \in E(G)$. Since G has no induced cycle of length at least 5, $w_1x, w_1y \in E(G)$. Since G is $K_{1,3}$ -free, $xv_1 \in E(G)$. This is a contradiction, as we have proved $xv_1 \notin E(G)$.

If $d(u) = 5$ and $d(v) = 6$, let $N(u) \setminus \{w, v, x, z\} = \{u_1\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free, $u_1x, v_1y, v_2y, v_1v_2, zu_1 \in E(G)$. Since G has no induced cycle of length at least 5, $wv_1, wv_2 \notin E(G)$. We claim that none of $\{zv_1, zv_2\}$ is in $E(G)$. Suppose otherwise that assume that $zv_1 \in E(G)$. Since G is $K_{1,3}$ -free, $u_1v_1, xv_1 \in E(G)$. Let $H = G[\{u, v, w, x, y, z, u_1, v_1, v_2\}]$. Then H is isomorphic to $G_{14} + e$, which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2. Thus, $zv_1, zv_2 \notin E(G)$. As $\kappa'(G) \geq 4$, there is a path P from u_1 to v_1 avoiding any vertex in $\{z, w, u, v, x, y\}$. Since G has no induced cycle of length at least 5, $u_1v_1 \in E(G)$. In this case, the subgraph induced by $\{u, v, x, y, z, u_1, v_1, v_2\}$ is also isomorphic to $G_{14} + e$, which is Z_3 -connected by Lemma 2.4, contrary to Lemma 3.2.

If $d(u) = d(v) = 6$, let $N(u) \setminus \{w, v, x, z\} = \{u_1, u_2\}$ and $N(v) \setminus \{w, u, y, z\} = \{v_1, v_2\}$. Since G is $K_{1,3}$ -free, $u_1x, u_2x, u_1u_2, v_1y, v_2y, v_1v_2, zu_1, zu_2 \in E(G)$. This means that the subgraph induced by $\{z, u, u_1, u_2, x\}$ is a K_5 , which is Z_3 -connected by Lemma 2.1, contrary to Lemma 3.2.

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