Squashing maximum packings of 6-cycles into maximum packings of triples

Curt C. Lindner

*Department of Mathematics and Statistics, Auburn University,
Auburn, AL 36849-5307, U.S.A.*

Giovanni Lo Faro *

*Dipartimento di Matematica e Informatica, Università di Messina,
98166 Messina, Italia*

Antoinette Tripodi †

*Dipartimento di Matematica e Informatica, Università di Messina,
98166 Messina, Italia*

Received 21 July 2014, accepted 28 September 2014, published online 24 January 2015

Abstract

A 6-cycle is said to be squashed if we identify a pair of opposite vertices and name one of them with the other (and thereby turning the 6-cycle into a pair of triples with a common vertex). The squashing problem for 6-cycle systems was introduced by C. C. Lindner, M. Meszka and A. Rosa and completely solved by determining the spectrum. In this paper, by employing PBD and GDD-constructions and filling techniques, we extend this result by squashing maximum packings of K_n with 6-cycles into maximum packings of K_n with triples. More specifically, we establish that for each $n \geq 6$, there is a max packing of K_n with 6-cycles that can be squashed into a maximum packing of K_n with triples.

Keywords: Maximum packing with triples, maximum packing with 6-cycles.

Math. Subj. Class.: 05B07

*Supported by P.R.I.N., P.R.A. and I.N.D.A.M.(G.N.S.A.G.A.)

†Supported by P.R.I.N., P.R.A. and I.N.D.A.M.(G.N.S.A.G.A.)

E-mail addresses: lindncc@auburn.edu (Curt C. Lindner), lofaro@unime.it (Giovanni Lo Faro), atripodi@unime.it (Antoinette Tripodi)

1 Introduction

Let G be a graph. A G -design of order n is a pair (S, B) where B is a collection of subgraphs (*blocks*), each isomorphic to G , which partitions the edge set of the complete undirected graph K_n with vertex set S . After determining the spectrum for G -designs for different graphs G , many problems have been studied also recently (for example, see [1]–[7]).

A Steiner triple system (more simply, triple system) of order n is a G -design of order n where G is the graph K_3 . It is well known that the spectrum for triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod{6}$ [9], and that if (S, T) is a triple system of order n , then $|T| = n(n-1)/6$. Similarly, a 6-cycle system of order n is a G -design of order n where G is 6-cycle. The spectrum for 6-cycle systems is precisely the set of all $n \equiv 1$ or $9 \pmod{12}$ [15], and if (X, C) is a 6-cycle system of order n , then $|C| = n(n-1)/12$. It is worth noting that if (S, T) and (X, C) have order n , then $|T| = 2|C|$.

Given the fact that triple systems and 6-cycle systems coexist for all $n \equiv 1$ or $9 \pmod{12}$, an obvious question to ask is: are there any connections between the two when $n \equiv 1$ or $9 \pmod{12}$? The answer, of course, is yes. One much studied connection is that of 2-perfect 6-cycle systems. A 6-cycle system is 2-perfect provided the collection of triples obtained by replacing each 6-cycle (a, b, c, d, e, f) with the two triples (a, c, e) and (b, d, f) is a Steiner triple system. Such systems exist for all $n \equiv 1$ or $9 \pmod{12} \geq 13$ [15].

Quite recently a new connection between triple systems and 6-cycle systems has been introduced: the *squashing* of a 6-cycle system into a Steiner triple system. A definition is in order. Let (a, b, c, d, e, f) be a 6-cycle and form the following six bowties (a pair of triples with a common vertex).

If B is any one of the six bowties in Figure 1, we say that we have *squashed* (a, b, c, d, e, f) into B . So there are six different ways to squash a 6-cycle into a bowtie. If (X, C) is a 6-cycle system, $2|C| = 2n(n-1)/12 = n(n-1)/6$ is the number of triples in a Steiner triple system. Therefore it makes sense to ask the following question: what is the spectrum for 6-cycle systems that can be squashed into Steiner triple systems? In [11], a complete solution is given to this problem by constructing for every $n \equiv 1$ or $9 \pmod{12}$ a 6-cycle system that can be squashed into a Steiner triple system.

Example 1.1. (A 6-cycle system of order 9 squashed into a triple system [11].)

$$\begin{array}{ll}
 (0,1,2,3,4,5) & (0,1,2)(0,4,5) \\
 (3,6,0,2,4,1) & (3,6,0)(3,4,1) \\
 (2,8,4,0,3,7) & \text{SQUASH } (2,8,4)(2,3,7) \\
 (7,0,8,6,5,1) & \longrightarrow (7,0,8)(7,5,1) \\
 (6,1,8,5,7,4) & (6,1,8)(6,7,4) \\
 (5,2,6,7,8,3) & (5,2,6)(5,8,3)
 \end{array}$$

Now if $n \equiv 3$ or $7 \pmod{12}$ there does not exist a 6-cycle system of order n . However, there does exist a maximum packing (max packing) of K_n with 6-cycles with leave a triple (i.e., a pair (X, C) and a set L , the *leave*, where C is a collection of edge disjoint 6-cycles with vertices in X , L is the set of the edges of K_n not belonging to any 6-cycle of C and $|L|$ is as small as possible) and so the following question makes sense. Does there exist for each $n \equiv 3$ or $7 \pmod{12}$ a max packing of K_n with 6-cycles which can be squashed into bowties so that the bowties plus the leave (a triple) form a Steiner triple system?

Figure 1

Example 1.2. (A max packing of K_7 squashed into a triple system [11].)

$$\begin{array}{lll}
 (2,3,4,5,0,1) & \text{SQUASH} & (2,1,0)(2,3,4) \\
 (4,6,0,2,5,1) & \longrightarrow & (4,1,5)(4,0,6) \\
 (5,6,2,4,0,3) & & (5,3,0)(5,6,2) \\
 (1,3,6) \text{ leave} & \longrightarrow & (1,3,6)
 \end{array}$$

The following theorem is proved in [11].

Theorem 1.3. [11] *There exists a 6-cycle system of every order $n \equiv 1$ or $9 \pmod{12}$ that can be squashed into a triple system and a 6-cycle maximum packing that can be squashed into a triple system for every $n \equiv 3$ or $7 \pmod{12}$, $n \geq 7$.* $\square$

The object of this paper is to finish off the problem of squashing maximum packings of K_n with 6-cycles into maximum packings of K_n with triples. We need to be a bit more precise.

Let (X, C) be a maximum packing of K_n with 6-cycles with leave L . In what follows, to keep the vernacular from getting out of hand, to say that C has been squashed means that the resulting collection $S(C)$ of bowties is a partial triple system.

Further, if t is a triple belonging to L and $S(C) \cup \{t\}$ is a maximum packing of K_n with triples (or a triple system), we will say that we have squashed (X, C) into a maximum

packing of K_n with triples. So, for example, Example 1.2 is the squashing of a maximum packing of K_7 with 6-cycles into a triple system of order 7.

The following easy to read table gives the leaves for max packings for both 6-cycles and triples not covered by Theorem 1.3. (See [8] and [13].)

K_n	6-cycles leave	triples leave
$n \equiv 0, 2, 6, 8 \pmod{12}$	1-factor	1-factor
$n \equiv 5 \pmod{12}$	4-cycle	4-cycle
$n \equiv 11 \pmod{12}$	<p>4 leaves are possible</p>
$n \equiv 4 \text{ or } 10 \pmod{12}$	<p>22 leaves are possible for $n \geq 16$</p>	<p>tripole [13]</p>

We remark that if $n \equiv 0, 2, 6, 8$ or $5 \pmod{12}$ and a 6-cycle maximum packing can be squashed, there are no triples to be added; i.e., the resulting collection of bowties is a maximum packing of K_n with triples. If $n \equiv 4, 10$ or $11 \pmod{12}$ and a 6-cycle maximum packing can be squashed, then a triple is taken from the 6-cycle leave in order to obtain a maximum packing of K_n with triples.

2 Preliminaries

From now on to say that the 6-cycle (a, b, c, d, e, f) is squashed we will always mean that it has been squashed into the bowtie $(a, b, c)(a, e, f)$; see Figure 2.

Figure 2

So, for example, in Example 1.1 we can simply list the 6-cycles (without listing the bowties they have been squashed into) and say they can be squashed into a triple system.

The following three examples are used repeatedly in what follows.

Example 2.1. (A max packing of K_6 with 6-cycles squashed into a max packing of K_6 with triples.)

$C = \{(5, 0, 1, 2, 4, 3), (2, 3, 1, 5, 4, 0)\}$, leave $L = \{(0, 3), (1, 4), (2, 5)\}$. (There are no triples in the leave.)

Example 2.2. (A max packing of K_8 with 6-cycles squashed into a max packing of K_8 with triples.)

$X = Z_4 \times Z_2$; $C = \{(0_0, 3_0, 1_1, 2_0, 3_1, 0_1), (1_0, 0_0, 2_1, 3_0, 0_1, 1_1), (2_0, 1_0, 3_1, 0_0, 1_1, 2_1), (3_0, 2_0, 0_1, 1_0, 2_1, 3_1)\}$, leave $L = \{(0_0, 2_0), (1_0, 3_0), (0_1, 2_1), (1_1, 3_1)\}$. (There are no triples in the leave.)

Example 2.3. (Decomposition of $K_{4,4,4}$ into 6-cycles squashed into triples.) (An obvious definition.)

$X = Z_4 \times \{1, 2, 3\}$; $C = \{(1_2, 1_3, 0_1, 0_2, 0_3, 1_1), (0_2, 2_3, 0_1, 1_2, 0_3, 2_1), (1_1, 0_2, 3_3, 0_1, 2_2, 1_3), (0_1, 3_2, 3_3, 1_1, 2_2, 0_3), (3_2, 1_1, 2_3, 1_2, 3_1, 0_3), (1_2, 2_1, 2_3, 3_2, 3_1, 3_3), (3_1, 0_2, 1_3, 2_1, 2_2, 2_3), (2_1, 3_2, 1_3, 3_1, 2_2, 3_3)\}$. (There is no leave.)

3 Basic Lemmas

With the examples of Section 2 in hand we can go to the general constructions, where we shall make use of GDDs. Let H be a set of integers and X be a set of size n ; a $\text{GDD}(n, H, k)$ is a triple (X, G, B) where G is a partition of X into subsets called *groups* of size in H , B is a set of subsets of X (called *blocks*) of size k , such that a group and a block contain at most one common point and every pair of points from distinct groups occurs in exactly one block. A PBD is a $\text{GDD}(n, \{1\}, k)$.

We break the constructions into the eight cases: 2, 6, 8; 0; 11; 4, 10; 5 (mod 12).

3.1 $n \equiv 2, 6$ and 8 (mod 12)

These are the easiest cases, so a good place to start.

$n \equiv 2 \pmod{12}$ Write $12k + 2 = 2(6k + 1)$ and let (X, T) be a Steiner triple system of order $6k + 1$. Let $S = X \times \{1, 2\}$ and define a collection C of 6-cycles as follows: For each triple $t = \{a, b, c\} \in T$ define a copy of Example 2.1 on $\{a, b, c\} \times \{1, 2\}$ with leave $L_t = \{(a_1, a_2), (b_1, b_2), (c_1, c_2)\}$ and put these 6-cycles in C . Then C is a max packing of K_{12k+2} with 6-cycles with leave $L = \{L_t \mid t \in T\}$. Trivially, C can be squashed into a max packing of K_{12k+2} with triples with leave L .

$n \equiv 6 \pmod{12}$ The case for $n = 6$ is handled with Example 2.1. So now write $12k + 6 = 2(6k + 3)$ and proceed exactly as in the case $n \equiv 2 \pmod{12}$.

$n \equiv 8 \pmod{12}$ Write $12k + 8 = 2(6k + 4)$. The case $n = 8$ is handled by Example 2.2. So let $12k + 8 \geq 20$. It is well known that there is a PBD with block sizes 3 and 4 for every $n \equiv 4 \pmod{6}$ [13]. Let (X, B) be such a PBD, $|X| \equiv 4 \pmod{6}$, and proceed exactly as in the cases for $n \equiv 2$ or 6 (mod 12), using Example 2.2 as well as Example 2.1.

Lemma 3.1. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 2, 6, 8 \pmod{12} \geq 6$.* $\square$

3.2 $n \equiv 0 \pmod{12}$

We begin with an example.

Example 3.2. ($n = 12$)

Let $X = \{\infty_1, \infty_2\} \cup Z_{10}$ and define a collection of 6-cycles C as follows:

$$\begin{array}{llll}
(0, \infty_1, 2, 1, 3, 6), & (2, \infty_2, 4, 3, 5, 8), & (4, \infty_1, 6, 5, 7, 0), & (6, \infty_2, 8, 7, 9, 2) \\
(8, \infty_1, 9, 0, 1, 4), & (1, \infty_1, 3, 0, 2, 7), & (3, \infty_2, 5, 2, 4, 9), & (5, \infty_1, 7, 4, 6, 1), \\
(7, \infty_2, 9, 6, 8, 3), & (0, \infty_2, 1, 8, 9, 5), & &
\end{array}$$

with leave $L = \{(0, 8), (1, 9), (2, 3), (4, 5), (6, 7), (\infty_1, \infty_2)\}$. Then (X, C) is a max packing of K_{12} with 6-cycles and can be squashed into a max packing of K_{12} with triples with leave L .

We will need two constructions for $12k \geq 24$: one when k is even and one when k is odd.

$12k, k$ even Write $12k = 4(3k)$ and let (P, G, B) be a GDD $(3k, \{2\}, 3)$, set $X = P \times \{1, 2, 3, 4\}$ and define a collection of 6-cycles C as follows:

- (i) For each group $g \in G$ place Example 2.2 on $g \times \{1, 2, 3, 4\}$ with leave $L_g = \{g \times \{1\}, g \times \{2\}, g \times \{3\}, g \times \{4\}\}$ and place these 6-cycles in C .
- (ii) For each triple $t = \{a, b, c\} \in B$ place a copy of Example 2.3 on $K_{4,4,4}$ with parts $\{a\} \times \{1, 2, 3, 4\}$, $\{b\} \times \{1, 2, 3, 4\}$, and $\{c\} \times \{1, 2, 3, 4\}$ and place these 6-cycles in C .

Then (X, C) is a max packing of K_{12k} with 6-cycles with leave $L = \{g \times \{1\}, g \times \{2\}, g \times \{3\}, g \times \{4\} \mid g \in G\}$. It is straightforward to see that the 6-cycles in (i) and (ii) can be squashed into a max packing of K_{12k} with triples with leave L .

$12k, k$ odd Write $12k = 4(3k)$. Since k is odd, $3k$ is the order of a Kirkman triple system (P, T) . Let $X = P \times \{1, 2, 3, 4\}$, π a parallel class in T , and define a collection of 6-cycles C as follows:

- (i) For each triple $t = \{a, b, c\} \in \pi$, place a copy of Example 3.2 on $\{a, b, c\} \times \{1, 2, 3, 4\}$ with leave L_t and place these 6-cycles in C .
- (ii) For each triple $\{a, b, c\} \in T \setminus \pi$, place a copy of Example 2.3 on $K_{4,4,4}$ with parts $\{a\} \times \{1, 2, 3, 4\}$, $\{b\} \times \{1, 2, 3, 4\}$, and $\{c\} \times \{1, 2, 3, 4\}$, and place these 6-cycles in C .

Then (X, C) is a max packing of K_{12k} with 6-cycles with leave $L = \{L_t \mid t \in \pi\}$. Squashing these 6-cycles produces a max packing of K_{12k} with triples with leave L .

Lemma 3.3. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 0 \pmod{12}$.* $\square$

3.3 $n \equiv 11 \pmod{12}$

We begin with an example.

Example 3.4. ($n = 11$)

Let $X = Z_9 \cup \{\infty_1, \infty_2\}$ and define a collection of 6-cycles C as follows:

$$\begin{array}{llll}
(4, 8, \infty_2, 7, \infty_1, 0), & (5, 0, \infty_2, 4, \infty_1, 1), & (6, 1, \infty_2, 5, \infty_1, 2), & (7, 2, \infty_2, 6, \infty_1, 3) \\
(3, 2, 5, 7, 0, 1), & (7, 4, 5, 3, 0, 6), & (4, 3, 6, 8, 1, 2), & (8, 5, 6, 4, 1, 7)
\end{array}$$

with leave $L = \{(\infty_1, \infty_2, 3, 8), (0, 2, 8)\}$. Then (X, C) is a max packing of K_{11} with 6-cycles with leave L . Squashing these 6-cycles and adding $(0, 2, 8)$ from the leave L gives a max packing of K_{11} with triples with leave the 4-cycle $(\infty_1, \infty_2, 3, 8)$.

We can now give a general construction for $11 \pmod{12} \geq 23$.

$\frac{12k+11 \geq 23}{4, \{4^*, 2\}, 3}$ [13], set $X = \{\infty_1, \infty_2, \infty_3\} \cup (P \times \{1, 2\})$, and define a collection of 6-cycles C as follows:

- (i) Let b^* be the unique group of size 4 and define a copy of Example 3.4 on $\{\infty_1, \infty_2, \infty_3\} \cup (b^* \times \{1, 2\})$ with leave $L = \{(\infty_1, \infty_2, \infty_3), (x, y, z, w)\}$, where $\{x, y, z, w\} \subseteq b^* \times \{1, 2\}$ and place these 6-cycles in C .
- (ii) For each group $g \in G$ of size 2, define a copy of a max packing of K_7 with 6-cycles, with vertex set $\{\infty_1, \infty_2, \infty_3\} \cup (b \times \{1, 2\})$, that can be squashed into 6-triples with leave $(\infty_1, \infty_2, \infty_3)$ [11]. Add these 6-cycles to C .
- (iii) For each triple $t = \{a, b, c\} \in B$, place a copy of Example 2.1 on $t \times \{1, 2\}$ with leave $\{a\} \times \{1, 2\}$, $\{b\} \times \{1, 2\}$, and $\{c\} \times \{1, 2\}$ and place these 6-cycles in C .

Then (X, C) is a max packing of K_{12k+11} with 6-cycles with leave L in (i). If we squash these 6-cycles and add the triple $(\infty_1, \infty_2, \infty_3)$ from the leave L in (i) we have a max packing of K_{12k+11} with triples with leave (x, y, z, w) in (i).

Lemma 3.5. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 11 \pmod{12}$.* $\square$

3.4 $n \equiv 4$ or $10 \pmod{12}$

The following three examples are necessary for the constructions in this section.

Example 3.6. ($n = 10$)

Let $X = \{\infty\} \cup (Z_3 \times Z_3)$ and define a collection of 6-cycles C as follows: $(0_1, 1_1, 0_0, 1_2, 0_2, \infty)$, $(1_2, 2_1, 0_0, 0_1, 0_2, 1_0)$, $(1_1, 2_1, 1_0, 2_2, 1_2, \infty)$, $(2_2, 0_1, 1_0, 1_1, 1_2, 2_0)$, $(2_1, 0_1, 2_0, 0_2, 2_2, \infty)$, $(0_2, 1_1, 2_0, 2_1, 2_2, 0_0)$ with leave $L = \{\{\infty, 2_0, 1_0, 0_0\}, (0_1, 1_2), (1_1, 2_2), (2_1, 0_2)\}$. (We remark that $\{\infty, 2_0, 1_0, 0_0\}$ is a copy of K_4 and not a 4-cycle.) Then (X, C) is a max packing of K_{10} with 6-cycles with leave L . If we squash these 6-cycles and remove a triple from $\{\infty, 2_0, 1_0, 0_0\}$, the result is a max packing of K_{10} with triples with leave the tripole $K_{1,3} \cup \{(0_1, 1_2), (1_1, 2_2), (2_1, 0_2)\}$.

Example 3.7. ($n = 16$)

Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{i_j \mid i \in Z_6, j \in \{0, 1\}\}$. Further, for each $i \in Z_6$, define $\alpha(i) = \infty_1$ if i is odd and ∞_2 if i is even. For each $i \in Z_6$ define a collection of 6-cycles C as follows: $(i_1, i_0, (4+i)_1, (2+i)_1, (1+i)_1, \alpha(i))$, $(i_0, (1+i)_1, (4+i)_0, (2+i)_0, (1+i)_0, \alpha(i))$, and $(i_0, (2+i)_1, \infty_3, (1+i)_0, \infty_4, (5+i)_1)$ with leave $L = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{(i_j, (3+i)_j) \mid i \in \{0, 1, 2\}, j \in \{0, 1\}\}$. (Once again we remark that $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ is a copy of K_4 .) Then (X, C) is a max packing of K_{16} with 6-cycles with leave L . If we squash these 6-cycles and remove a triple from $\{\infty_1, \infty_2, \infty_3, \infty_4\}$, the result is a max packing of K_{16} with triples, with leave the tripole $K_{1,3} \cup \{(i_j, (3+i)_j) \mid i \in \{0, 1, 2\}, j \in \{0, 1\}\}$.

Example 3.8. ($n = 28$)

Let $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (Z_{12} \times \{1, 2\})$, and let (P, G, B) be a GDD(12, {3}, 4) (equivalent to a pair of orthogonal quasigroups of order 3) and define a collection of 6-cycles C as follows:

- (i) For each group $g \in G$, place a copy of Example 3.6 on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2\})$ with leave $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup \{(x_1, x_2) \mid x \in g\}$ (with K_4 based on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$).
- (ii) For each block $b \in B$ place a copy of Example 2.2 on $b \times \{1, 2\}$ with leave $\{(x_1, x_2) \mid x \in b\}$.

Then (X, C) is a max packing of K_{28} with 6-cycles with the leave in (i). Now squashing the 6-cycles in (i) and (ii) and removing a triple from K_4 gives a max packing of K_{28} with triples with leave a tripole.

We can now go to the general constructions for $n \equiv 10 \pmod{12}$, $n \geq 22$ and $n \equiv 4 \pmod{12}$, $n \geq 40$.

$n \equiv 10 \pmod{12}$, $n \geq 22$ Write $12k + 10 = 2(6k + 5)$ and let (P, B) be a PBD($6k + 5$, {5*, 3}) [13]. Set $X = P \times \{1, 2\}$ and define a collection C of 6-cycles as follows:

- (i) Let b^* be the unique block of size 5 and define a copy of Example 3.6 on $b^* \times \{1, 2\}$ and place these 6-cycles in C . (The leave is $K_4 \cup \{1\text{-factor}\}$.)
- (ii) For each triple $t = \{a, b, c\} \in B$, define a copy of Example 2.1 on $t \times \{1, 2\}$ with leave $\{(a_1, a_2), (b_1, b_2), (c_1, c_2)\}$ and place these 6-cycles in C .

Then (X, C) is a max packing of K_{12k+10} with 6-cycles with leave $K_4 \cup \{1\text{-factor}\}$. Squashing the 6-cycles in C and removing a triple from the leave in (i) produces a max packing of K_{12k+10} with triples with leave a tripole.

$n \equiv 4 \pmod{12}$, $n \geq 40$ Write $12k + 4 = 4 + 2(6k)$ and let (P, G, B) be a GDD($6k$, {6}, 3) [13]. Set $X = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (P \times \{1, 2\})$ and define a collection of 6-cycles as follows:

- (i) For each group $g \in G$ define a copy of Example 3.7 on $\{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (g \times \{1, 2\})$ with leave $K_4 \cup \{(x_1, x_2) \mid x \in G\}$ (K_4 is based on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$).
- (ii) For each triple $t = \{a, b, c\} \in B$, define a copy of Example 2.1 on $t \times \{1, 2\}$ with leave $\{(a_1, a_2), (b_1, b_2), (c_1, c_2)\}$ and place these 6-cycles in C .

Then (X, C) is a max packing of K_{12k+4} with 6-cycles. Squashing the 6-cycles in C and removing a triple from the leave K_4 in (i) produces a max packing of K_{12k+4} with triples with leave a tripole.

Lemma 3.9. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 4$ or $10 \pmod{12}$, $n \geq 6$.* $\square$

3.5 $n \equiv 5 \pmod{12}$

This case requires three examples.

Example 3.10. ($n = 17$)

Let $X = \{\infty_1, \infty_2\} \cup Z_{15}$ and define a collection of 6-cycles C as follows:
 $\{(0, 9, 4, 5, 1, 3) + i \mid i \in Z_{15}\} \cup \{(7, 14, \infty_2, 13, \infty_1, 0), (8, 0, \infty_2, 7, \infty_1, 1), (9, 1, \infty_2, 8, \infty_1, 2), (10, 2, \infty_2, 9, \infty_1, 3), (11, 3, \infty_2, 10, \infty_1, 4), (12, 4, \infty_2, 11, \infty_1, 5), (13, 5, \infty_2, 12, \infty_1, 6)\}$ with leave the 4-cycle $(\infty_1, \infty_2, 6, 14)$. Squashing all of the 6-cycles in C produces a max packing of K_{17} with triples with leave the 4-cycle $(\infty_1, \infty_2, 6, 14)$.

Example 3.11. ($n = 29$)

Let $X = \{\infty_1, \infty_2\} \cup Z_{27}$ and define a collection of 6-cycles C as follows:
 $\{(0, 3, 1, 5, 4, 9) + i, (0, 16, 10, 17, 7, 15) + i \mid i \in Z_{27}\} \cup \{(14, 0, \infty_2, 13, \infty_1, 1) + j \mid j \in \{0, 1, 2, \dots, 11\}, (13, 26, \infty_2, 25, \infty_1, 0)\}$ with leave the 4-cycle $(\infty_1, \infty_2, 12, 26)$. Squashing the 6-cycles in C gives a max packing of K_{29} with triples with leave the 4-cycle $(\infty_1, \infty_2, 12, 26)$.

Example 3.12. ($n = 53$)

Let $X = \{\infty_1, \infty_2\} \cup Z_{51}$ and define a collection of 6-cycles C as follows:
 $\{(0, 3, 1, 5, 4, 21) + i, (0, 19, 9, 20, 11, 23) + i, (0, 24, 8, 15, 7, 22) + i, (0, 20, 6, 11, 5, 18) + i \mid i \in Z_{51}\} \cup \{(26, 0, \infty_2, 25, \infty_1, 1) + j \mid j \in \{0, 1, 2, \dots, 23\}\} \cup \{(25, 50, \infty_2, 49, \infty_1, 0)\}$ with leave the 4-cycle $(\infty_1, \infty_2, 24, 50)$. Squashing these 6-cycles gives a max packing of K_{53} with triples with leave the 4-cycle $(\infty_1, \infty_2, 24, 50)$.

We can now give two general constructions to finish off the case $n \equiv 5 \pmod{12}$.
 $12k + 5, k$ odd Write $12k + 5 = 1 + 4(3k + 1)$. Since k is odd, $1 + 4(3k + 1) = 1 + 4(6t + 4)$. Let (P, G, B) be a GDD($6t, \{4^*, 2\}, 3$), set $X = \{\infty\} \cup (P \times \{1, 2, 3, 4\})$ and define a collection of 6-cycles as follows:

- (i) For the unique group b^* of size 4, define a copy of Example 3.10 on $\{\infty\} \cup (b^* \times \{1, 2, 3, 4\})$ (the leave is a 4-cycle) and place these 6-cycles in C .
- (ii) For each group g of size 2, define a copy of Example 1.1 on $\{\infty\} \cup (g \times \{1, 2\})$ and place these 6-cycles in C . (There is no leave.)
- (iii) For each triple $\{a, b, c\} \in B$, place a copy of Example 2.3 on $\{a, b, c\} \times \{1, 2, 3, 4\}$ with parts $\{a\} \times \{1, 2, 3, 4\}$, $\{b\} \times \{1, 2, 3, 4\}$, $\{c\} \times \{1, 2, 3, 4\}$, and place these 6-cycles in C . (There is no leave.)

Then (X, C) is a max packing of K_{12k+5} with 6-cycles with leave a 4-cycle. If we squash the 6-cycles in (i), (ii) and (iii), we have a max packing of K_{12k+5} with triples with leave a 4-cycle.

$12k + 5, k$ even Write $12k + 5 = 1 + 4(3k + 1)$. Since k is even, $1 + 4(3k + 1) = 1 + 4(6t + 1)$. Since $12k + 5 \geq 77$, $6t + 1 \geq 19$, and there exists a GDD($6t + 1, \{7^*, 3\}, 3$) [11] (P, G, B) . Define a collection C of 6-cycles on $X = \{\infty\} \cup (P \times \{1, 2, 3, 4\})$ as follows:

- (i) For the unique group b^* of size 7, define a copy of Example 3.11 on $\{\infty\} \cup (b \times \{1, 2, 3, 4\})$ (with leave a 4-cycle) and place these 6-cycles in C .
- (ii) For each group g of size 3, place a copy of a 6-cycle system of order 13 which can be squashed into a triple system [11] (no leave) on $\{\infty\} \cup (g \times \{1, 2, 3, 4\})$ and place these 6-cycles in C .
- (iii) For each triple $\{a, b, c\} \in B$, place a copy of Example 2.3 on $\{a, b, c\} \times \{1, 2, 3, 4\}$ with parts $\{a\} \times \{1, 2, 3, 4\}$, $\{b\} \times \{1, 2, 3, 4\}$, $\{c\} \times \{1, 2, 3, 4\}$, and place these 6-cycles in C . (There is no leave.)

Then (X, C') is a max packing of K_{12k+5} with 6-cycles with leave the 4-cycle in (i). Squashing the 6-cycles in (i), (ii) and (iii) produces a max packing of K_{12k+5} with triples with leave the 4-cycle in (i).

Lemma 3.13. *There exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples for all $n \equiv 5 \pmod{12} \geq 17$.* $\square$

4 Main result and further developments

Putting together the results in Section 3 gives the following theorem.

Theorem 4.1. *For each $n \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$, $n \geq 6$, there exists a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples. There are no exceptions.* $\square$

Since a complete solution is also a max packing, we can combine Theorem 1.3 and Theorem 8.1 into the following corollary (giving a complete solution to the squashing of max packings of 6-cycles into max packings with triples).

Corollary 4.2. *For each $n \geq 6$, there is a max packing of K_n with 6-cycles that can be squashed into a max packing of K_n with triples.* $\square$

In this paper we give a complete solution to the problem of squashing maximum packings of K_n with 6-cycles into maximum packings of K_n with triples. An open problem is to solve the general case, i.e. squashing a maximum packing of K_n with $2m$ -cycles into a maximum packing of K_n with m -cycles; the case $m = 4$ is completely solved in [10].

References

- [1] L. Berardi, M. Gionfriddo and R. Rota, Perfect octagon quadrangle systems - II, *Discrete Math.* **312**(3) (2012), 614–620.
- [2] Y. Chang, T. Feng, G. Lo Faro and A. Tripodi, The existence spectrum of $\text{Meta}(K_4^{(3)} > K_4^{(3)} - e)$, *Science China Mathematics* **11** (53) (2010), 2865–2876.
- [3] Y. Chang, T. Feng, G. Lo Faro and A. Tripodi, The triangle intersection numbers of a pair of disjoint $S(2, 4, v)$ s, *Discrete Math.* **310**(21) (2010), 3007–3017.
- [4] Y. Chang, T. Feng, G. Lo Faro and A. Tripodi, The fine triangle intersection for $(K_4 - e)$ -designs, *Discrete Math.* **311**(21) (2011), 2442–2462.
- [5] Y. Chang, T. Feng, G. Lo Faro and A. Tripodi, The fine triangle intersection for kite systems, *Discrete Math.* **312** (3) (2012), 545–553.
- [6] Y. Chang, G. Lo Faro and A. Tripodi, Tight blocking sets in some maximum packings of λK_n , *Discrete Math.* **308** (2008), 427–438.
- [7] Y. Chang, G. Lo Faro and A. Tripodi, Determining the spectrum of $\text{Meta}(K_4 + e > K_4, \lambda)$ for any λ , *Discrete Math.* **308** (2008), 439–456.
- [8] J. A. Kennedy, Maximum packings of K_n with hexagons, *Australas. J. Combin.* **7** (1993), 101–110.
- [9] T. P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* **2** (1847), 191–204.
- [10] C. C. Lindner, G. Lo Faro, M. Meszka and A. Tripodi, Squashing maximum packings of K_n with 8-cycles into maximum packings of K_n with 4-cycles, *Filomat* **28:4** (2014), 887–896.

- [11] C. C. Lindner, M. Meszka and A. Rosa, From squashed 6-cycles to Steiner triple systems, *J. Combin. Des.* **22** (2014), 189–195.
- [12] C. C. Lindner, K. T. Phelps and C. A. Rodger, The spectrum for 2-perfect 6-cycle systems, *J. Combin. Theory Ser. A* **57** (1991), 76–85.
- [13] C. C. Lindner and C. A. Rodger, *Design Theory*, second ed., CRC Press, 2009.
- [14] A. Rosa, On cycle decompositions of the complete graph into $(4m+2)$ -gons, *Mat-fyz Časopis* **16** (1966), 349–435.
- [15] A. Rosa and C. Huang, Another class of balanced graph designs: balanced circuit designs, *Discrete Math.* **12** (1975), 269–293.