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On the split structure of lifted groups

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Abstract

Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs with the group of covering transformations $\mathrm{CT}_\wp$ being abelian. Assuming that a group of automorphisms $G \le \mathrm{Aut}\, X$ lifts along $\wp$ to a group $\tilde{G} \le \mathrm{Aut}\, \tilde{X}$, the problem whether the corresponding exact sequence $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ splits is analyzed in detail in terms of a Cayley voltage assignment that reconstructs the projection up to equivalence.

In the above combinatorial setting the extension is given only implicitly: neither $\tilde{G}$ nor the action $G \to \mathrm{Aut}\, \mathrm{CT}_\wp$ nor a 2-cocycle $G \times G \to \mathrm{CT}_\wp$, are given. Explicitly constructing the cover $\tilde{X}$ together with $\mathrm{CT}_\wp$ and $\tilde{G}$ as permutation groups on $\tilde{X}$ is time and space consuming whenever $\mathrm{CT}_\wp$ is large; thus, using the implemented algorithms (for instance, `HasComplement` in MAGMA) is far from optimal. Instead, we show that the minimal required information about the action and the 2-cocycle can be effectively decoded directly from voltages (without explicitly constructing the cover and the lifted group); one could then use the standard method by reducing the problem to solving a linear system of equations over the integers. However, along these lines we here take a slightly different approach which even does not require any knowledge of cohomology. Time and space complexity are formally analyzed whenever $\mathrm{CT}_\wp$ is elementary abelian.

Keywords: Algorithm, abelian cover, Cayley voltages, covering projection, graph, group extension, group presentation, lifting automorphisms, linear systems over the integers, semidirect product.

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1 Introduction

A large part of algebraic graph theory is devoted to analyzing structural properties of graphs with prescribed degree of symmetry in order to classify, enumerate, construct infinite families, and to produce catalogs of particular classes of interesting graphs up to a certain reasonable size. References are too numerous to be listed here, but see for instance [2, 6, 8, 9, 10, 11, 12, 14, 16, 24, 25, 26, 29, 31, 36, 39, 40, 42, 46, 47, 52, 53, 55], and the references therein.

It is not surprising, then, that the techniques employed in these studies are fairly rich and diverse, ranging from pure combinatorial and computational methods to methods from abstract group theory, permutation groups, combinatorial group theory, linear algebra, representation theory, and algebraic topology.

Covering space techniques, and lifting groups of automorphisms along regular covering projections in particular, play a prominent role in this context. (See Section 2 for exact definitions of all notions used in this Introduction.) The idea goes back to Djoković [9] (and to an unpublished work of Conway, see [3, Corollary 19.6]), who constructed first examples of infinite families of graphs of small valency and maximal degree of transitivity, and to Biggs [3, Proposition 19.3], who gave a sufficient condition for a group of automorphisms to lift as a semidirect product. While Djoković's approach is classical in terms of fundamental groups, Biggs expressed his particular lifting condition combinatorially. A combinatorial approach to covering projections of graphs in terms of voltages was systematically developed in the early 70' by Gross and Tucker, see [20], after having been introduced by Alpert and Gross [18, 19] in the context of maps on surfaces.

A systematic combinatorial treatment of lifting automorphisms along covering projections (either in the context of graphs, maps on surfaces, or cell complexes) has been considered by several authors, see [1, 21, 32, 33, 48, 50] and the references therein. More specific types of covers, say, with cyclic or elementary abelian groups of covering transformations, have been extensively studied in [22, 35, 37, 49]; for the applications we refer the reader to [8, 11, 13, 14, 26, 27, 28, 30, 36, 40, 41, 52, 55]. For some recent results on arc transitive cubic graphs arising as regular covers with an abelian group of covering transformations we refer the reader to [7].

Basic lifting techniques in terms of voltages are now well understood, yet several important issues still remain to be considered. In view of the fact that structural properties of graphs often rely on the structure and the action of their automorphism groups, one such topic is investigating the structure of lifted groups – although certain particular questions along these lines have been addressed, see [3, 15, 33, 51]. Other points of interest are algorithmic and complexity aspects of lifting automorphisms, which have so far received even less attention. Certain aspects, but not those considered here, were touched upon in [34, 48].

Specifically, let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs given in terms of voltages. Assuming that a group $G \leq \operatorname{Aut} X$ lifts along $\wp$, it is of particular interest to study the corresponding exact sequence $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$. A natural question in this context is to ask whether the extension is split: on one hand, split extensions are the most easy ones to analyze, while on the other hand, a restrictive situation stemming from the fact that the group G, acting on X, acts also on $\tilde{X}$ via its isomorphic complement

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$\overline{G}$ to $\mathrm{CT}_\wp$ within $\tilde{G}$, implies that a lot more information about symmetry properties of $\tilde{X}$ can be derived; moreover, split extensions are frequently encountered in many concrete examples of graph covers. Describing efficient methods for testing whether a given group lifts as a split extension of $\mathrm{CT}_\wp$ is the main objective of this paper.

Methods for testing whether a given extension $1 \to K \to E \to Q \to 1$ of finite groups is split, are known, see [5] and [23, Chapters 7 and 8]. In some way or another, all these methods use the fact that a set of coset representatives of K in E is a complement to K if and only if these representatives satisfy the defining relations of Q.

The essential case to be resolved in the first place is that of K being (elementary) abelian. The idea is to modify an arbitrarily chosen set of coset representatives of K so that the defining relations of Q are satisfied, if possible. Since K is normal and abelian, this modification can be traced in the frame of a certain group algebra, which finally leads to a system of linear equations over the integers (or rather, over prime fields); the complement exists if and only if such a system has a solution.

In practice, the extension can be given in several different ways: either (i) in terms of an epimorphism $E \to Q$, or (ii) in terms of E and the generators of a normal subgroup K, or (iii) via an action $\theta\colon Q \to \operatorname{Aut} K$ together with a 2-cocycle $\tau\colon Q \times Q \to K$. In cases (i) and (ii), an essential requirement is that one must have enough information about the extended group E; at least one must know its generators and must be able to perform basic computations in E. In contrast with (i) and (ii), explicit knowledge about E is not needed in case (iii) since the extension can be, up to equivalence of extensions, reconstructed as $K \times Q$ with multiplication rule $(a, x)(b, y) = (a + \theta_x(b) + \tau(x, y), xy)$.

In our setting of graph covers, however, the situation is different and does not fall in any of the above three cases. Namely, the extension $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is given only implicitly: all the information is encoded in the base graph in terms of voltages that allow G to lift; in particular, neither $\tilde{G}$ nor the action of G on $\mathrm{CT}_\wp$ nor a 2-cocycle are given. Naively translating our setting into the frame of (i) or (ii) and then applying the algorithm already implemented in MAGMA [4] in terms of permutation groups would mean to first compute the covering graph $\tilde{X}$ together with $\mathrm{CT}_\wp$ and $\tilde{G}$ acting on $\tilde{X}$, which, unfortunately, is time and space consuming whenever $\mathrm{CT}_\wp$ is large.

Our situation best fits into the frame of (iii). But in order to follow the approach described in [5] and [23, Chapters 7 and 8] we first need to compute the action $G \to \operatorname{Aut} \mathrm{CT}_\wp$ and the 2-cocycle $G \times G \to \mathrm{CT}_\wp$. As we here show, the minimal required information about these data can indeed be effectively decoded directly from voltages (without explicitly constructing the cover and the lifted group). In the actual algorithm, however, we take an approach which is slightly different and even does not require any knowledge of cohomology. Namely, instead of modifying an initial transversal and working within an appropriate group algebra, a potential complement is constructed directly in terms of certain parameters – in view of the fact that a lift of an automorphism is uniquely determined by the mapping of a single vertex – from which the required system of equations is obtained. Although the method works whenever $\mathrm{CT}_\wp$ is abelian, it can be adapted – similarly as in the general context – to treat the case when $\mathrm{CT}_\wp$ is solvable as well.

The paper is organized as follows. In Section 2 we review some basic facts about regular covering projections and lifting automorphisms. In Section 3 we show how to recapture the lifted group $\tilde{G}$ as a crossed product of $\mathrm{CT}_\wp$ by G via reconstructing the coupling $G \to \operatorname{Out} \mathrm{CT}_\wp$ and the factor set $G \times G \to \mathrm{CT}_\wp$ in terms of voltages, see Theorem 3.1. In Section 4 we give the necessary and sufficient conditions for G to lift as a split extension of $\mathrm{CT}_\wp$, see Theorem 4.1, and a similar result regarding direct product extensions, see Theorem 4.3. In Section 5 we provide an algorithm for testing whether the extension is split whenever $\mathrm{CT}_\wp$ is abelian, see Subsection 5.1 and Theorem 5.2 in particular. The special case when $\mathrm{CT}_\wp$ is elementary abelian together with formal analysis of time and space complexity is treated in Subsection 5.2, see Theorem 5.6. In Subsection 5.3 we briefly mention the case when $\mathrm{CT}_\wp$ is solvable. Concluding remarks are given in Section 6. For a more substantial account on complexity issues and further applications we refer the reader to [44, 45].

2 Preliminaries

Graphs. Formally, a graph is an ordered 4-tuple $X = (D, V; \mathrm{beg}, ^{-1})$, where $D(X) = D$ and $V(X) = V$ are disjoint sets of *darts* and *vertices*, respectively, beg is the function assigning to each dart its *initial vertex*, and $^{-1}$ is an arbitrary involution on D that creates *edges* arising as orbits of $^{-1}$. For a dart x, its *terminal vertex* is the vertex $\mathrm{end}(x) = \mathrm{beg}(x^{-1})$. An edge $e = \{x, x^{-1}\}$ is called a *link* whenever $\mathrm{beg}(x) \neq \mathrm{end}(x)$. If $\mathrm{beg}(x) = \mathrm{end}(x)$, then the respective edge is either a *loop* or a *semi-edge*, depending on whether $x \neq x^{-1}$ or $x = x^{-1}$, respectively.

There are several reasons for treating graphs formally in a manner just described. For one thing, it is quite versatile for writing down formal proofs; moreover it is indeed natural, even necessary, to consider graphs with semi-edges in different contexts, for instance when dealing with graph covers or when studying graphs that are embedded into surfaces. For a nice use of semi-edges in the context of Cayley graphs we refer the reader to [17].

A *walk* $W\colon u \to v$ of *length* $n \geq 0$ from a vertex $u_0 = u$ to a vertex $u_n = v$ is a sequence of vertices and darts $W = u_0\, x_1\, u_1\, x_2\, u_2\, \ldots\, u_{n-1}\, x_n\, u_n$ where $\mathrm{beg}(x_j) = u_{j-1}$ and $\mathrm{end}(x_j) = u_j$ for all indices $j = 1, \ldots, n$. Its *inverse walk* $W^{-1}\colon v \to u$ is the walk obtained by listing the vertices and darts appearing in W in reverse order. The walk u is the *trivial walk* at the vertex u. Walks of length 1 are sometimes referred to as *arcs*. A graph is *connected* if any two vertices are connected by a walk. A walk is *reduced* if no two consecutive darts in the walk are inverse to each other. Clearly, each walk W has an associated reduced walk $\underline{W}$ obtained by recursively deleting all appearances $uxvx^{-1}$ of consecutive pairs of inverse darts (together with the respective vertices). Two walks $W, W'\colon u \to v$ with the same reduction are called *homotopic*. *Homotopy* is an equivalence relation on the set of all walks, with homotopy classes denoted by $[W]$. Observe that the naturally defined product of walks W_1W_2 by 'concatenation', when defined, carries over to homotopy classes, $[W_1][W_2] = [W_1W_2]$. Assuming the graph X to be connected, the set of homotopy classes of *closed walks* $u \to u$, equipped with the above product, defines the *first homotopy group* $\pi(X, u)$. The trivial class $1_u = [u]$ consists of all walks *contractible* to u. Note that the isomorphism class of $\pi(X, u)$ does not depend on u. More precisely, $\pi(X, u)$ is isomorphic to the free product of cyclic groups $\mathbb{Z}$ or $\mathbb{Z}_2$ (where the $\mathbb{Z}_2$ factors correspond bijectively to the set of all semi-edges in X). A generating set for $\pi(X, u)$ is provided by *fundamental closed walks at* u relative to an arbitrarily chosen spanning tree.

Let X and X' be graphs. A *graph homomorphism* $f\colon X \to X'$ is an adjacency preserving mapping taking darts to darts and vertices to vertices, or more precisely, $f(\mathrm{beg}(x)) = \mathrm{beg}(f(x))$ and $f(x^{-1}) = f(x)^{-1}$. Homomorphisms are composed as functions, $(fg)(x) = f(g(x))$. Given a graph X we frequently need to consider the restricted (and the induced) left action of its group of automorphisms $\mathrm{Aut}\, X$ on certain subsets of X, for instance the sets of vertices, darts, edges etc. As $\mathrm{Aut}\, X$ is by definition a permutation group on the union $V(X) \cup D(X)$ of disjoint sets of vertices and darts, it acts faithfully on $V(X) \cup D(X)$. However, its action on $V(X)$ need not be faithful unless X is *simple*, that is, if it has no parallel links, loops, or semi-edges. We say that a group $G \leq \mathrm{Aut}\, X$ acts *semiregularly on* X whenever it acts freely on $V(X)$ (meaning that if $g \in G$ fixes a vertex it must be the identity on vertices and darts).

Covers. To fix the notation and terminology, and for easier reading, we quickly review some essential facts about covers. The interested reader is referred to [20, 33, 54] for more information.

A *covering projection* of graphs is a surjective homomorphism $\wp\colon \tilde{X} \to X$ mapping the set of darts with a common initial vertex in the *covering graph* $\tilde{X}$ bijectively to the set of darts at the image of that vertex in the *base graph* X. The preimages $\mathrm{fib}_u = \wp^{-1}(u)$, $u \in V(X)$, and $\mathrm{fib}_x = \wp^{-1}(x)$, $x \in D(X)$, are the *vertex-* and *dart-fibres*, respectively. From the definition of a covering projection it immediately follows that for any walk $W\colon u \to v$ in X and an arbitrary vertex $\tilde{u} \in \mathrm{fib}(u)$ there is a unique *lifted walk* $\tilde{W}^{\tilde{u}}$ with $\mathrm{beg}(\tilde{W}^{\tilde{u}}) = \tilde{u}$ that projects to W. This is known as the *unique-path lifting property*. Consequently, if X is connected (which will be our standard assumption without loss of generality) then all fibres have equal cardinality, usually referred to as the *number of folds*. It is also immediate that homotopic walks lift to homotopic walks, and that $\tilde{u} \cdot [W] = \mathrm{end}(\tilde{W}^{\tilde{u}})$ defines a 'right action' of homotopy classes on the vertex set of $\tilde{X}$. In particular, the fundamental group $\pi(X, u)$ acts on the right on fib_u, with the stabilizer of $\tilde{u} \in \mathrm{fib}_u$ being isomorphic to $\pi(\tilde{X}, \tilde{u})$. It is precisely this action that is responsible for the structural properties of the covering.

Covering projections that are particularly important when studying symmetry properties of covers are the *regular covering projections*. By definition, a covering projection $\wp\colon \tilde{X} \to X$ is *regular* if there exists a semiregular group $C \leq \mathrm{Aut}\, \tilde{X}$ such that its orbits on vertices and on darts coincide with vertex- and dart-fibres, respectively. In other words, C acts *regularly* on each fibre (hence the name), and so the covering projection is $|C|$-fold.

Regular covering projections can be grasped combinatorially as follows. First of all, given a graph X and an (abstract) group Γ, let $\zeta\colon D(X) \to \Gamma$ be a function such that $\zeta(x^{-1}) = (\zeta(x))^{-1}$. (For convenience we shall write $\zeta_x = \zeta(x)$ and $\zeta_x^{-1} = (\zeta_x)^{-1}$.) In this context, Γ is called a *voltage group*, ζ is a *Cayley voltage assignment* on X, and ζ_x is the *voltage* of the dart x. We remark that a voltage assignment as above is known as an *ordinary voltage assignment* in the literature [20]. With these data we may define the *derived graph* $\mathrm{Cov}(\zeta)$ with vertex set $V(X) \times \Gamma$ and dart set $D(X) \times \Gamma$, where $\mathrm{beg}(x, c) = (\mathrm{beg}(x), c)$ and $(x, c)^{-1} = (x^{-1}, c\,\zeta_x)$. The projection onto the first coordinate defines a regular covering projection $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$. The required semiregular group C is obtained by viewing $C = \Gamma$ as a group of automorphisms of $\mathrm{Cov}(\zeta)$ via its left action on the second coordinates by left multiplication on itself: an element $a \in \Gamma$ maps the vertex (u, c) to (u, ac) and the dart (x, c) to (x, ac). In addition, call the right action of Γ on itself by right multiplication a *voltage-action*. This action determines how walks of length 1 lift: a walk uxv lifts to walks $(u, c)(x, c)(v, c\,\zeta_x)$, for $c \in \Gamma$.

Conversely, with any regular covering projection $\wp\colon \tilde{X} \to X$ we can associate a Cayley voltage assignment ζ on X such that $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ 'essentially reconstructs' the projection $\wp$ in a sense to be described below. Indeed, let $\Gamma = C$ be the semiregular group from the definition of a regular covering. As Γ acts regularly on each fibre we may label the vertices and the darts of $\tilde{X}$ by elements of $V(X) \times \Gamma$ and $D(X) \times \Gamma$, respectively, as follows. Choosing arbitrarily a vertex $\tilde{u} \in \mathrm{fib}_u$ we label the vertex $c(\tilde{u}) \in \mathrm{fib}_u$, $c \in \Gamma$, by (u, c). This way we obtain a bijective labeling of fib_u by $\{u\} \times \Gamma$. Similarly, the darts in fib_x are labeled by $\{x\} \times \Gamma$, where (x, c) is the label of the dart in fib_x having its initial vertex labeled by (u, c). For $x \in D(X)$, let $\zeta_x \in \Gamma$ be such an element of the voltage group that $(\mathrm{end}(x), \zeta_x)$ is the label of the terminal vertex of the dart in fib_x labeled by $(x, 1)$. Then the terminal vertex of any dart in fib_x, say, labeled by (x, c), is labeled by $(\mathrm{end}(x), c\zeta_x)$. Clearly, $\zeta_{x^{-1}} = \zeta_x^{-1}$. The respective regular covering projection $\wp_\zeta \colon \mathrm{Cov}(\zeta) \to X$ is equivalent to $\wp$, a concept that we are now going to define.

Two covering projections $\wp\colon \tilde{X} \to X$ and $\wp'\colon \tilde{X}' \to X$ are *isomorphic* if there exists an automorphism $g \in \mathrm{Aut}\, X$ and an isomorphism $\tilde{g}\colon \tilde{X} \to \tilde{X}'$ such that the following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X}' \\ \wp \Big\downarrow & & \Big\downarrow \wp' \\ X & \xrightarrow{g} & X \end{array}$$

is commutative. If in the above diagram one can choose $g = \mathrm{id}$, then the projections are *equivalent*. Covering projections are usually studied up to equivalence, or possibly up to isomorphism (which is considerably more difficult).

A voltage assignment $\zeta\colon D(X) \to \Gamma$ can be naturally extended to walks as follows: if $W = u_0\, x_1\, u_1\, x_2\, u_2\, \ldots\, u_{n-1}\, x_n\, u_n$, then $\zeta_W = \zeta_{x_1}\zeta_{x_2}\ldots\zeta_{x_n}$. Clearly, homotopic walks carry the same voltage, and so voltages can be assigned to homotopy classes. Moreover, the 'right action' of homotopy classes via unique path-lifting along $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ is essentially the voltage-action: if $W\colon u \to v$ is a walk in X and $\tilde{u} \in \mathrm{fib}_u$ is labeled by (u, c), then $\tilde{u} \cdot [W] \in \mathrm{fib}_v$ is labeled by $(v, c\,\zeta_W)$. We may therefore say that the voltage-action faithfully represents the 'action' of homotopy classes, and in particular, the action of $\pi(X, u)$. It immediately follows that ζ defines a group homomorphism $\zeta\colon \pi(X, u) \to \Gamma$ (denoted by the same symbol for convenience).

Lifts of automorphisms. An automorphism $g \in \mathrm{Aut}\, X$ *lifts along* a covering projection $\wp\colon \tilde{X} \to X$ if there exists an automorphism $\tilde{g} \in \mathrm{Aut}\, \tilde{X}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \\ \wp \Big\downarrow & & \Big\downarrow \wp \\ X & \xrightarrow{g} & X \end{array}$$

is commutative. The automorphism $\tilde{g}$ then *projects* to g. A group $G \leq \mathrm{Aut}\, X$ lifts if all $g \in G$ lift. We call such a covering projection *G-admissible*. The collection of all lifts of all elements in G form a subgroup $\tilde{G} \leq \mathrm{Aut}\, \tilde{X}$, the *lift* of G. In particular, the lift of the trivial group is known as the *group of covering transformations* (or self-equivalences of $\wp$) and denoted by $\mathrm{CT}_\wp$. Moreover, the sequence

$$\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$$

is short exact. In other words, $\tilde{G}$ is an extension of $\mathrm{CT}_\wp$ by G, and hence $g \in G$ has exactly $|\mathrm{CT}_\wp|$ distinct lifts, a coset of $\mathrm{CT}_\wp$ within $\tilde{G}$. Furthermore, if G lifts along a given projection $\wp$, then it lifts along any covering projection equivalent to $\wp$. This allows us to study lifts of automorphisms combinatorially in terms of voltages, see for instance [1, 21, 32, 33, 48, 50]. Also note that if $\tilde{G}$ and $\tilde{G}'$ are the lifts of G along equivalent projections $\wp$ and $\wp'$, respectively, then the short exact sequences $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ and $\mathrm{id} \to \mathrm{CT}_{\wp'} \to \tilde{G}' \to G \to \mathrm{id}$ are isomorphic. Thus, structural properties of lifted groups can be studied combinatorially in terms of voltages as well. In this paper we focus on G-admissible covering projections such that the extension $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is split. We call such a covering projection *G-split-admissible*. Note, however, that the lifted group $\tilde{G}$ might contain a subgroup H isomorphic to $\mathrm{CT}_\wp$ such that the extension $\mathrm{id} \to H \to \tilde{G} \to G \to \mathrm{id}$ is split even if $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is not.

From now on we shall be assuming that covering projections are regular and moreover, that covering graphs are connected as well. By assuming connectedness we essentially do not loose on generality, however, we technically gain a lot. Let us remark that if X is connected, then the covering graph is connected if and only if the fundamental group $\pi(X, u)$ acts transitively on fib_u. An equivalent requirement in terms of a voltage assignment $\zeta\colon D(X) \to \Gamma$ is that the homomorphic image $\zeta(\pi(X, u)) \leq \Gamma$ acts transitively relative to its voltage-action on Γ, that is, $\zeta(\pi(X, u)) = \Gamma$, which in turn amounts to saying that the voltages assigned to fundamental closed walks at u generate the voltage group Γ. With the assumption on connectedness, the group $\mathrm{CT}_\wp$ acts regularly on each fibre, and hence any lift $\tilde{g}$ of $g \in \mathrm{Aut}\, X$, if it exists, is uniquely determined by the mapping of just one vertex (or dart). Also, the semiregular group C from the definition of a regular covering is now $C = \mathrm{CT}_\wp$, and the voltage assignment $\zeta\colon D(X) \to \Gamma$ that reconstructs the projection takes values in an abstract group $\Gamma \cong \mathrm{CT}_\wp$.

Consider now a regular covering projection $\wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ of connected graphs, where X is assumed to be finite, and let $u_0 \in V(X)$ be an arbitrarily chosen base vertex. By the *basic lifting lemma*, see [32, Theorem 4.2] and [33, Theorem 7.1], a group $G \leq \mathrm{Aut}\, X$ lifts along $\wp_\zeta$ if and only if any closed walk W at u_0 with $\zeta_W = 1$ is mapped to a walk with $\zeta_{gW} = 1$, for all $g \in G$. This is equivalent to requiring that for each $g \in G$ there exists an *induced automorphism* $g^{\#_{u_0}} \in \mathrm{Aut}\, \Gamma$ of the voltage group defined locally at u_0 by

$$g^{\#_{u_0}}(\zeta_W) = \zeta_{gW}, \qquad W \in \pi(X, u_0).$$

Note that if the condition is satisfied at u_0, it holds locally at any vertex. In general, for $g, h \in G$ the automorphisms $g^{\#_u}$ and $g^{\#_v}$ at distinct vertices, as well as the automorphisms $(gh)^{\#_u}$ and $g^{\#_u} h^{\#_u}$, differ by an inner automorphism of Γ. More precisely, the following holds. (Throughout the paper, Ψ_t denotes the inner automorphism $\Psi_t(a) = tat^{-1}$, whatever the group. Note further that all automorphisms are composed as functions.)

Proposition 2.1. *Let $G \leq \mathrm{Aut}\, X$ be a group of automorphisms that lifts along a regular covering projection of connected graphs $\wp\colon \tilde{X} \to X$ given in terms of a voltage assignment $\zeta\colon D(X) \to \Gamma$. Then for any $g, h \in G$ we have*

$$\begin{aligned} \Psi_{g^{\#_u}(\zeta_Q)\zeta_{gQ}^{-1}}\, g^{\#_v} &= g^{\#_u}, \quad Q\colon v \to u, \\ \Psi_{g^{\#_u}(\zeta_Q)\zeta_{gQ}^{-1}}\, (gh)^{\#_u} &= g^{\#_u} h^{\#_u}, \quad Q\colon hu \to u. \end{aligned}$$

Proof. Let W be a closed walk at v and $Q\colon v \to u$ an arbitrary walk. Then $Q^{-1}WQ$ is a closed walk at u, and by the definition of induced automorphisms of Γ at v and u we have $g^{\#_v}(\zeta_W) = \zeta_{gW}$ and $g^{\#_u}(\zeta_{Q^{-1}WQ}) = \zeta_{g(Q^{-1}WQ)}$. Clearly, $\zeta_{Q^{-1}WQ} = \zeta_Q^{-1}\zeta_W\zeta_Q$ and $\zeta_{g(Q^{-1}WQ)} = \zeta_{gQ}^{-1}\zeta_{gW}\zeta_{gQ}$. Since $g^{\#_u}$ is an automorphism we have

$$g^{\#_u}(\zeta_Q)^{-1}g^{\#_u}(\zeta_W)g^{\#_u}(\zeta_Q) = \zeta_{gQ}^{-1}\zeta_{gW}\zeta_{gQ}.$$

Hence $\Psi_{g^{\#_u}(\zeta_Q)\zeta_{gQ}^{-1}}\, g^{\#_v} = g^{\#_u}$, and the first part is proved. For the second part, let W be a closed walk at u and $Q\colon hu \to u$ an arbitrary walk. Then

$$(gh)^{\#_u}(\zeta_W) = \zeta_{ghW} = g^{\#_{hu}}(\zeta_{hW}) = g^{\#_{hu}}(h^{\#_u}(\zeta_W)).$$

Hence $(gh)^{\#_u} = g^{\#_{hu}}h^{\#_u}$. By the first part we have $\Psi_{g^{\#_u}(\zeta_Q)\zeta_{gQ}^{-1}}\, g^{\#_{hu}} = g^{\#_u}$, and consequently, $\Psi_{g^{\#_u}(\zeta_Q)\zeta_{gQ}^{-1}}\,(gh)^{\#_u} = g^{\#_u}h^{\#_u}$, as required. □

Clearly, the function

$$\#_{u_0}\colon G \to \operatorname{Aut}\Gamma, \quad g \mapsto g^{\#_{u_0}},$$

is not a group homomorphism in general. But if we define $g^{\#} = g^{\#_{u_0}} \bmod \operatorname{Inn}\Gamma$, then, by Proposition 2.1, $g^{\#}$ does not depend on u_0, and $\#\colon G \to \operatorname{Out}\Gamma$, $g \mapsto g^{\#}$, is a homomorphism. In particular, if the covering projection is *abelian*, meaning that $\Gamma \cong \mathrm{CT}_\wp$ is abelian, then $\# = \#_{u_0}\colon G \to \operatorname{Aut}\Gamma$ is a homomorphism, which turns Γ into a $\mathbb{Z}[G]$-module. We shall make substantial use of this fact later on.

If g lifts, denote by $\Phi_{v,\tilde{g}}$ the permutation on the voltage group Γ corresponding to the restriction $\tilde{g}\colon \mathrm{fib}_v \to \mathrm{fib}_{gv}$. In other words,

$$\tilde{g}(v,c) = (gv, \Phi_{v,\tilde{g}}(c)). \tag{2.1}$$

As it was shown in [32, 33], the mappings of labels at different fibres relate to each other as follows:

$$\Phi_{u,\tilde{g}}(c) = \Phi_{u,\tilde{g}}(1)\, g^{\#_u}(c) \tag{2.2}$$

$$\Phi_{v,\tilde{g}}(c) = \Phi_{u,\tilde{g}}(c)\, g^{\#_u}(\zeta_Q)\zeta_{gQ}^{-1}, \tag{2.3}$$

where $Q\colon v \to u$ is an arbitrary walk. Finally, for $t \in \Gamma$ we denote by $\tilde{g}_t$ the uniquely defined lift of g mapping the vertex in fib_{u_0} labeled by $1 \in \Gamma$ to the vertex in fib_{gu_0} labeled by $t \in \Gamma$, that is,

$$\tilde{g}_t(u_0, 1) = (gu_0, t).$$

In particular, $\tilde{\mathrm{id}}_t$ is the covering transformation acting on the second coordinates in $\mathrm{Cov}(\zeta)$ by left multiplication by t on Γ. Indeed, since $\mathrm{id}^{\#_u} = \mathrm{id}$ for all $u \in V(X)$, it follows from (2.2) and (2.3) that

$$\tilde{\mathrm{id}}_t(u,c) = (u, tc).$$

3 Extensions in terms of voltages

The method how to recapture a given group extension $1 \to K \to E \to Q \to 1$ in the form of a crossed product is known and goes back to Schreier (cf. [38]). First choose a *system of coset representatives* of K within E (also called an *algebraic transversal*) $T = \{t_x \mid x \in Q\}$ (and usually *normalized* in the sense that $t_1 = 1$). Then compute the *factor set* $\mathcal{F}\colon Q \times Q \to K$ defined by

$$\mathcal{F}\colon (x, y) \mapsto t_x t_y t_{xy}^{-1},$$

and the function $\Psi\colon Q \to \operatorname{Aut} K$, $x \mapsto \Psi_{t_x}$ (recall that $\Psi_{t_x}(a) = t_x a t_x^{-1}$); in general, Ψ is not a group homomorphism, and is often referred to as the *weak action* of Q on K (which, when reduced modulo inner automorphisms of K, gives rise to a homomorphism $Q \to \operatorname{Out} K$ known as the *coupling* or the *twisting map*). These data determine a group operation on $K \times Q$ defined by

$$(a, x)(b, y) = (a\Psi_{t_x}(b)\mathcal{F}(x, y), xy).$$

The resulting group is called the *crossed product* of K by Q and denoted $K \times_{\Psi,\mathcal{F}} Q$. The mapping $K \times_{\Psi,\mathcal{F}} Q \to E$ defined by $(a, x) \mapsto at_x$ is an isomorphism taking $K \times 1$ onto K and $1 \times Q$ onto the algebraic transversal T, and establishes an *equivalence* of short exact sequences

$$\begin{tikzcd}
 & & K \times_{\Psi,\mathcal{F}} Q \arrow[dd] \arrow[dr] & & \\
1 \arrow[r] & K \arrow[ur] \arrow[dr] & & Q \arrow[r] & 1. \\
 & & E \arrow[ur] & &
\end{tikzcd}$$

Suppose now that a regular covering projection $\wp = \wp_\zeta\colon \operatorname{Cov}(\zeta) \to X$ of connected graphs is given in terms of a Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$, and let a group $G \le \operatorname{Aut} X$ of automorphisms lift to $\tilde{G} \le \operatorname{Aut} \operatorname{Cov}(\zeta)$.

In order to recapture $\tilde{G}$ as a crossed product of $\mathrm{CT}_\wp$ by G let us choose a particular algebraic transversal by taking $t_g = 1$ for all $g \in G$, that is, $T = \{\tilde{g}_1 \mid g \in G\}$. To compute the factor set $G \times G \to \mathrm{CT}_\wp$ we need to identify $c \in \Gamma$ such that $\tilde{g}_1 \tilde{h}_1 (\widetilde{gh})_1^{-1} = \tilde{\mathrm{id}}_c$. We do that by evaluating $\tilde{g}_1 \tilde{h}_1 = \tilde{\mathrm{id}}_c (\widetilde{gh})_1$ at $(u_0, 1)$. Using (2.2) and (2.3) we get

$$c = g^{\#_{u_0}}(\zeta_Q)\zeta_{gQ}^{-1}, \quad \text{where} \quad Q\colon hu_0 \to u_0$$

is arbitrary. As for the weak action $G \to \operatorname{Aut} \mathrm{CT}_\wp$ we need to find $t \in \Gamma$ such that $\tilde{g}_1 \tilde{\mathrm{id}}_a \tilde{g}_1^{-1} = \tilde{\mathrm{id}}_t$. Evaluating $\tilde{g}_1 \tilde{\mathrm{id}}_a = \tilde{\mathrm{id}}_t \tilde{g}_1$ at $(u_0, 1)$ and using (2.2) we obtain

$$t = g^{\#_{u_0}}(a).$$

In view of the isomorphism $\mathrm{CT}_\wp \cong \Gamma$, $\tilde{\mathrm{id}}_t \mapsto t$, we have an isomorphism of short exact sequences

$$\begin{tikzcd}
\mathrm{id} \arrow[r] \arrow[d] & \mathrm{CT}_\wp \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \mathrm{id} \arrow[d] \\
1 \arrow[r] & \Gamma \arrow[r] & \tilde{G} \arrow[r] & G \arrow[r] & \mathrm{id}.
\end{tikzcd}$$

Hence the lifted group $\tilde{G}$ can be written, up to isomorphism, as a crossed product $\Gamma \times_{\Psi,\mathcal{F}} G$, where $\mathcal{F}\colon G \times G \to \Gamma$ is given by $\mathcal{F}(g,h) = g^{\#_{u_0}}(\zeta_Q)\zeta_{gQ}^{-1}$ and $\Psi\colon G \to \operatorname{Aut}\Gamma$ is defined by $\Psi_g = g^{\#_{u_0}}$. Note that the weak action Ψ is precisely $\#_{u_0}$ defined in Preliminaries. We have therefore proved the following theorem.

Theorem 3.1. *Let $\wp = \wp_\zeta\colon \operatorname{Cov}(\zeta) \to X$ be a regular covering projection of connected graphs given in terms of a Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$, and let a group $G \leq \operatorname{Aut} X$ of automorphisms lift to $\tilde{G} \leq \operatorname{Aut}\operatorname{Cov}(\zeta)$. Choosing a base vertex $u_0 \in V(X)$, let $\Psi\colon G \to \operatorname{Aut}\Gamma$ and $\mathcal{F}\colon G \times G \to \Gamma$ be functions defined by*

$$\Psi_g = g^{\#_{u_0}} \quad \textit{and} \quad \mathcal{F}(g,h) = g^{\#_{u_0}}(\zeta_Q)\zeta_{gQ}^{-1}, \;\; Q\colon hu_0 \to u_0,$$

respectively. Then there is an isomorphism

$$\Gamma \times_{\Psi,\mathcal{F}} G \to \tilde{G}, \qquad (a,g) \mapsto \tilde{g}_a$$

taking $\Gamma \times \mathrm{id}$ onto $\mathrm{CT}_\wp$ and $1 \times G$ onto the algebraic transversal $\{\tilde{g}_1 \mid g \in G\}$. □

4 Split extensions in terms of voltages

Recall that a short exact sequence $1 \to K \to E \to Q \to 1$ is *split* if there exists an algebraic transversal $T = \{t_x \,|x \in Q\}$ which is a subgroup, called a *complement* to K within E. Relative to such a complement, the respective factor set $\mathcal{F} \equiv 1$ is trivial and the weak action Ψ is in fact an action, that is, $\Psi\colon Q \to \operatorname{Aut} K$ is a homomorphism. Consequently, recapturing E as the corresponding crossed product results in a *semidirect product* $K \rtimes_\Psi Q$ with the group operation $(a,x)(b,y) = (a\Psi_{t_x}(b), xy)$.

In the next theorem, the necessary and sufficient condition for a regular covering projection $\wp$ to be G-split-admissible, together with an explicit description of the lifted group as a semidirect product of $\mathrm{CT}_\wp$ by G, are given in terms of voltages.

Theorem 4.1. *Let $\wp = \wp_\zeta\colon \operatorname{Cov}(\zeta) \to X$ be a regular covering projection of connected graphs given in terms of a Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$, and let a group $G \leq \operatorname{Aut} X$ of automorphisms lift to $\tilde{G} \leq \operatorname{Aut}\operatorname{Cov}(\zeta)$. Then $\wp$ is G-split admissible if and only if there exists a normalized function $t\colon G \to \Gamma$ (that is, $t_{\mathrm{id}} = 1$) such that*

$$t_{gh} = t_g g^{\#_{u_0}}(t_h)\, g^{\#_{u_0}}(\zeta_Q)\zeta_{gQ}^{-1} \tag{4.1}$$

where $Q\colon hu_0 \to u_0$ is an arbitrary walk. In this case there exists a homomorphism $\theta\colon G \to \operatorname{Aut}\Gamma$ given by

$$\theta_g(c) = t_g g^{\#_{u_0}}(c) t_g^{-1}, \tag{4.2}$$

and $(a,g) \mapsto \tilde{g}_{a\,t_g}$ defines an isomorphism $\Gamma \rtimes_\theta G \to \tilde{G}$ which takes $\Gamma \times \mathrm{id}$ onto $\mathrm{CT}_\wp$ and $\mathrm{id} \times G$ onto the algebraic transversal $\overline{G} = \{\tilde{g}_{t_g} \mid g \in G\}$, a complement to $\mathrm{CT}_\wp$.

Proof. Let us recover the lifted group $\tilde{G}$ as in Theorem 3.1. The extension splits if and only if there exists an algebraic transversal $\{(t_g, g),\ g \in G\}$ to $\Gamma \times \mathrm{id}$ in $\Gamma \times_{\Psi,\mathcal{F}} G$ which is a subgroup. Equivalently, we must have $(t_{gh}, gh) = (t_g, g)(t_h, h)$. By the definition of multiplication in $\Gamma \times_{\Psi,\mathcal{F}} G$ the right hand side is equal to $(t_g g^{\#_{u_0}}(t_h)\mathcal{F}(g,h), gh)$. Hence the necessary and sufficient condition (4.1) can be expressed as stated in the theorem.

That (4.2) defines a homomorphism can be shown by computation, using (4.1) and Proposition 2.1. The rest is straightforward as well. □

Remark 4.2. In the abelian case, (4.1) rewrites as $t_{gh} = t_g + g^{\#_{u_0}}(t_h) + \tau(g,h)$, where $\tau(g,h) = \mathcal{F}(g,h) = g^{\#_{u_0}}(\zeta_Q) - \zeta_{gQ}$ is the 2-cocycle. Thus, (4.1) is equivalent to the fact that $\tau(g,h) = t_{gh} - t_g - g^{\#_{u_0}}(t_h)$ must be a 2-coboundary. □

From Theorem 4.1 we readily obtain the necessary and sufficient conditions for G to lift as a *direct product extension* of $\mathrm{CT}_\wp$, that is, when $\mathrm{CT}_\wp$ has a normal complement within the lifted group $\tilde{G}$.

Theorem 4.3. *Let* $\wp = \wp_\zeta \colon \mathrm{Cov}(\zeta) \to X$ *be a regular covering projection of connected graphs given in terms of a Cayley voltage assignment* $\zeta \colon D(X) \to \Gamma$. *Then* G *lifts along* $\wp$ *as a direct product extension of* $\mathrm{CT}_\wp$ *if and only if there exists a normalized function* $t\colon G \to \Gamma$ *(that is,* $t_{\mathrm{id}} = 1$*) satisfying*

$$t_{gh} = t_h \zeta_Q t_g \zeta_{gQ}^{-1}, \tag{4.3}$$

where $Q\colon hu_0 \to u_0$ *is an arbitrary walk. In this case,* $(a,g) \mapsto \tilde{g}_{a\,t_g}$ *defines an isomorphism* $\Gamma \times G \to \tilde{G}$ *which takes* $\Gamma \times \mathrm{id}$ *onto* $\mathrm{CT}_\wp$ *and* $\mathrm{id} \times G$ *onto the algebraic transversal* $\overline{G} = \{\tilde{g}_{t_g} \mid g \in G\}$, *a normal complement to* $\mathrm{CT}_\wp$.

Proof. Suppose that G lifts as a direct product such that $\mathrm{CT}_\wp$ has a normal complement $\overline{G} = \{\tilde{g}_{t_g} \mid g \in G\}$. By Theorem 4.1 the respective function $t\colon G \to \Gamma$ satisfies (4.1). Normality of $\overline{G}$ implies that $\theta_g(c)$ given by (4.2) must be the identity automorphism. Hence $g^{\#_{u_0}}(c) = t_g^{-1} c t_g$, and by (4.1) we have $t_{gh} = t_h \zeta_Q t_g \zeta_{gQ}^{-1}$, as required.

For the converse suppose that a function $t\colon G \to \Gamma$ satisfies $t_{gh} = t_h \zeta_Q t_g \zeta_{gQ}^{-1}$. Taking $h = \mathrm{id}$ we obtain $\zeta_{gQ} = t_g^{-1} \zeta_Q t_g$ for all closed walks $Q\colon u_0 \to u_0$. Therefore, if $\zeta_Q = 1$ then $\zeta_{gQ} = 1$ for all $g \in G$. By the basic lifting lemma G lifts, and $g^{\#_{u_0}}$ takes the form $g^{\#_{u_0}}(c) = t_g^{-1} c t_g$. It follows that $t_{gh} = t_g g^{\#_{u_0}}(t_h)\, g^{\#_{u_0}}(\zeta_Q) \zeta_{gQ}^{-1}$. By Theorem 4.1 we have $\tilde{G} \cong \Gamma \rtimes_\theta G$ where $\theta_g(c) = t_g g^{\#_{u_0}}(c) t_g^{-1} = c$. Hence $\tilde{G} \cong \Gamma \times G$, and the proof is complete. □

Remark 4.4. Notice the subtle difference in assumptions in Theorems 4.1 and 4.3. While in 4.1 we had to assume in advance that G had a lift, this assumption is not required in 4.3 as condition (4.3) does not involve $g^{\#_{u_0}}$. □

We also note the following. Suppose that G lifts as a split extension of $\mathrm{CT}_\wp$. In general, normal and non-normal complements to $\mathrm{CT}_\wp$ might exist. So a priori knowledge about a given extension being split does not make it easier to check whether the extension is actually a direct product extension. In the abelian case, however, things are different since complements are either all normal or all non-normal. This means that if $t\colon G \to \mathrm{Aut}\,\Gamma$ is just any normalized function satisfying (4.1), the extension will be a direct product extension if and only if the corresponding homomorphism θ as in (4.2) is trivial.

For later reference, see Corollary 5.5, we explicitly record the following corollary.

Corollary 4.5. *Let* $\wp = \wp_\zeta \colon \mathrm{Cov}(\zeta) \to X$ *be a regular covering projection of connected graphs given in terms of a Cayley voltage assignment* $\zeta \colon D(X) \to \Gamma$. *Suppose that* $G \le \mathrm{Aut}\,X$ *lifts along* $\wp$ *as a split extension. Then* G *lifts as a direct product extension of* $\mathrm{CT}_\wp$ *if and only if* $\zeta_{gW} = \zeta_W$ *holds for all closed walks* W *from a basis of the first homology group* $H_1(X,\mathbb{Z})$ *and all* g *from some generating set of* G.

Proof. By Theorem 4.1 there exists a normalized function $t\colon G \to \Gamma$ satisfying (4.1). Now, in the abelian case the extension is a direct product extension if and only if θ as in (4.2) is trivial. Since $\theta_g(c) = g^{\#_{u_0}}(c)$, this amounts to saying that $\zeta_{gW} = \zeta_W$ must hold for all closed walks based at u_0 and all $g \in G$. Moreover, recall that in the abelian case $g^{\#_{u_0}}$ does not depend on u_0. Hence the above necessary and sufficient condition can be replaced by only considering closed walks from a basis of the first homology group. Clearly, it is enough to consider just the automorphisms from a generating set of G. □

Remark 4.6. Note that if the covering projection is abelian and the condition $\zeta_{gW} = \zeta_W$ holds true for all closed walks and all $g \in G$, then G clearly lifts (by the basic lifting lemma). However, the extension might not be split.

As an example, let X be the 2-dipole with vertices 1 and 2 and two parallel links from 1 to 2 defined by the darts a and b. The voltage assignment $\zeta_a = \zeta_{a^{-1}} = 0$, $\zeta_b = \zeta_{b^{-1}} = 1$ in the group $\mathbb{Z}_2$ gives rise to a connected covering graph isomorphic to the 4-cycle C_4. Clearly, $\zeta_{gW} = \zeta_W$ holds for all $g \in \operatorname{Aut} X \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and all closed walks W. However, the lifted group is isomorphic to D_4, viewed as a central extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \times \mathbb{Z}_2$, and this extension is clearly not split. □

5 Algorithmic aspects

Let $\wp = \wp_\zeta\colon \operatorname{Cov}(\zeta) \to X$ be a regular covering projection of connected graphs given in terms of a Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$, where X is assumed to be finite, and let $G \leq \operatorname{Aut} X$ be a group of automorphisms. Speaking of algorithmic and complexity issues related to lifting automorphisms one would certainly first need to address the question of how difficult is to test whether G lifts at all. However, this will not be our concern here; the problem has been considered, to some extent, in [48].

Assuming that G is known to have a lift we focus on efficient algorithms (in terms of voltages) for testing if G lifts as a split extension of $\mathrm{CT}_\wp$. Testing condition (4.1) of Theorem 4.1 is hard even if Γ is abelian – as one has to take into account all group elements of G. (Indeed, Theorem 4.1 is of purely theoretical interest.) A much better alternative would be to consider just the generators, and in fact, one must then assume that G is given by a presentation, which is sensible assumption. Proposition 5.1 below is a reformulation of a standard result, c.f. [23, Lemma 2.76], tailored to our present needs. For completeness we provide the proof.

Proposition 5.1. *Let $\wp\colon \tilde{X} \to X$ be a regular covering projection of connected graphs, and let $G \leq \operatorname{Aut} X$ be a group given by the presentation $G = \langle S \mid R\rangle$, where $S = \{g_1, g_2, \ldots, g_n\}$ and the R-relations are $R_j(g_1, g_2, \ldots, g_n) = \mathrm{id}$, $j = 1, 2, \ldots, m$. Suppose that G lifts. Then the lifted group $\tilde{G}$ is a split extension of $\mathrm{CT}_\wp$ if and only if there are lifts $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ of $g_1, g_2, \ldots, g_n$, respectively, satisfying the defining relations $R_j(\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) = \mathrm{id}$, $j = 1, 2, \ldots, m$.*

Proof. Suppose first that there are lifts $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n$ of $g_1, g_2, \ldots, g_n$ satisfying the R-relations, and let $C = \langle \bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n\rangle \leq \tilde{G}$. Since the R-relations are the defining relations of G there exists an epimorphism $G \to C$, $g_i \mapsto \bar{g}_i$. On the other hand, C projects onto G, with $\bar{g}_i \mapsto g_i$. Consequently, $C \cong G$. As C isomorphically projects onto G it must intersect the kernel $\mathrm{CT}_\wp$ of the projection $\tilde{G} \to G$ trivially. Hence C is a complement to $\mathrm{CT}_\wp$ within $\tilde{G}$.

Conversely, suppose that there are lifts $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n$ of $g_1, g_2, \dots, g_n$ such that $C = \langle \bar{g}_1, \bar{g}_2, \dots, \bar{g}_n \rangle \leq \tilde{G}$ is a complement to $\mathrm{CT}_\wp$ within $\tilde{G}$. Then $C \cong G$, and since $\bar{g}_i \mapsto g_i$ we have that each automorphism $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$ projects to $R_j(g_1, g_2, \dots, g_n) = \mathrm{id}$. So $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n) \in C$ belongs to $\mathrm{CT}_\wp$. As C is the complement we have $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n) = \mathrm{id}$, and the proof is complete. $\square$

The condition $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n) = \mathrm{id}$ can be tested just by checking whether the automorphism $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$, which necessarily belongs to $\mathrm{CT}_\wp$, fixes a vertex. With our assumption that the covering graph is reconstructed as $\mathrm{Cov}(\zeta)$ we choose this vertex to be $(u_0, 1)$. Let $\bar{g}_i(u_0, 1) = (g_i u_0, t_i)$, and recall that a lift is uniquely determined by the image of a single vertex. If $t_1, t_2, \dots, t_n$ are explicitly given, then $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$ can be evaluated recursively using (2.2) and (2.3). To find whether the required lifts $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n$ exist by checking the whole set Γ^n for all possible values of $t_1, t_2, \dots, t_n$ is far from optimal. The core of the problem is therefore to evaluate $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$ efficiently when $t_1, t_2, \dots, t_n$ are seen as symbolic variables, in which case the requirements $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)(u_0, 1) = (u_0, 1)$ translate to an equivalent problem of solving a system of equations in the variables $t_1, t_2, \dots, t_n \in \Gamma$.

We are faced with two main difficulties. First, to evaluate $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$ using symbolic variables we need to express $g^{\#u_0}$ by a 'closed formula', and second, we have to solve a (possibly a non-linear) system of equations over Γ. Both are rather hopeless if Γ is nonabelian. On the other hand, if $\Gamma \cong \mathrm{CT}_\wp$ is a finitely presented abelian group, then the automorphisms of Γ can be represented by integer matrices acting on the left on integer column vectors representing group elements. (In what follows, we shall be freely using the term 'vector' for ease of expression.) Moreover, as we shall see, in the abelian case the system of equations results in a linear system over the integers.

5.1 Abelian covers

Let us therefore assume that Γ is abelian, given by a presentation $\Gamma = \langle \Delta \mid \Lambda \rangle$, where $\Delta = \{c_1, c_2, \dots, c_r\}$ is a generating set and $\Lambda_k(c_1, c_2, \dots, c_r) = 0$, where $k = 1, 2, \dots, s$, are the Λ-relations. Each element $c \in \Gamma$ can be represented by a column vector $\underline{c} \in \mathbb{Z}^{r,1}$ such that

$$\underline{c} = [\lambda_1, \lambda_2, \dots, \lambda_r]^T, \quad \text{where} \quad c = \sum_{i=1}^{r} \lambda_i c_i.$$

This representation is unique modulo the kernel (generated by the defining relations Λ_j) of the natural quotient projection $\kappa\colon \mathbb{Z}^{r,1} \to \Gamma$. Moreover, any automorphism $\phi \in \mathrm{Aut}\,\Gamma$ can be represented (again not in a unique way) as a matrix over $\mathbb{Z}$ by expressing each $\phi(c_i)$ as $\phi(c_i) = \sum_{j=1}^{r} \alpha_{ji}\, c_j$, and taking $M_\phi = [\alpha_{ij}] \in \mathbb{Z}^{r,r}$. Clearly, the following diagram

$$\begin{array}{ccc} \mathbb{Z}^{r,1} & \xrightarrow{M_\phi} & \mathbb{Z}^{r,1} \\ \kappa \big\downarrow & & \big\downarrow \kappa \\ \Gamma & \xrightarrow[\phi]{} & \Gamma \end{array}$$

is commutative, or in other words, evaluation of the automorphism ϕ is given by $\phi(c) = \kappa(M_\phi \underline{c})$.

Coming back to our original setting of evaluating the lifted automorphisms, recall from Preliminaries that in the abelian case the automorphism $g^\# = g^{\#u_0}$ does not depend on the base vertex, and that $(gh)^\# = g^\# h^\#$. Also, we shall simplify the notation for the matrix representing $g_i^\#$ by writing $M_i = M_{g_i^\#}$. In view of (2.2) and (2.3), the formula for evaluating the lifted automorphism $\bar{g}_i$ at an arbitrary vertex (v, c) is now given by

$$\Phi_{v,\bar{g}_i}(c) = t_i + g_i^\#(c) + g_i^\#(\zeta_Q) - \zeta_{g_iQ},$$

where $Q\colon v \to u_0$ is an arbitrary walk. This can be rewritten in vector form as

$$\underline{\Phi_{v,\bar{g}_i}(c)} = \underline{t_i} + M_i\underline{c} + M_i\underline{\zeta_Q} - \underline{\zeta_{g_iQ}}. \tag{5.1}$$

So far we have overcome part of the problem: representation of $g_i^\#$'s by a 'closed formula'. However, since $\underline{t_i}$'s and the vector $\underline{c}$ (which linearly depends on t_i's when the formula is applied recursively while processing R_j) are symbolic variables, the evaluation requires symbolic computation – and that is something we still want to avoid. To this end we do the following.

Let $\mathbf{t} = [\underline{t_1}^T, \underline{t_2}^T, \dots, \underline{t_n}^T]^T \in \mathbb{Z}^{rn,1}$ be the 'extended' column of all the vectors $\underline{t_1}, \underline{t_2}, \dots, \underline{t_n}$, and let $\mathbf{E}_i = [\mathbf{0}, \dots, \mathbf{0}, \mathbf{I}, \mathbf{0}, \dots, \mathbf{0}] \in \mathbb{Z}^{r,rn}$ be the matrix consisting of $n-1$ zero submatrices $\mathbf{0} \in \mathbb{Z}^{r,r}$ and one identity submatrix $\mathbf{I} \in \mathbb{Z}^{r,r}$ at 'i-th position'. Clearly, $\underline{t_i} = \mathbf{E}_i\mathbf{t}$. At each iteration step of the evaluation we can express the vector $\underline{c}$ linearly in terms of $\mathbf{t}$ as $\underline{c} = \mathbf{A}_j\mathbf{t} + \underline{b}_j$, for an appropriate matrix $\mathbf{A}_j \in \mathbb{Z}^{r,rn}$ and vector $\underline{b}_j \in \mathbb{Z}^{r,1}$, neither of which depends on $\mathbf{t}$.

Indeed. Suppose that while scanning the relator R_j from right to left we need to evaluate $\bar{g}_i$ at vertex $(v, \kappa\underline{c})$. Substituting $\underline{c}$ with $\mathbf{A}_j\mathbf{t} + \underline{b}_j$ in (5.1) we get

$$\underline{\Phi_{v,\bar{g}_i}(c)} = (\mathbf{E}_i + M_i\mathbf{A}_j)\mathbf{t} + M_i(\underline{b}_j + \underline{\zeta_Q}) - \underline{\zeta_{g_iQ}}, \quad Q\colon v \to u_0,$$

and so the label $\Phi_{v,\bar{g}_i}(\kappa\underline{c})$ (the modified $\underline{c}$ as the input at the next step) is again of the form $\mathbf{A}_j\mathbf{t} + \underline{b}_j$, with $\mathbf{A}_j$ substituted by $\mathbf{E}_i + M_i\mathbf{A}_j$ and $\underline{b}_j$ substituted by $M_i(\underline{b}_j + \underline{\zeta_Q}) - \underline{\zeta_{g_iQ}}$. Initially, $\mathbf{A}_j$ is the zero matrix and $\underline{b}_j$ the zero vector. The method for evaluating the automorphism $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)$ is formally encoded in algorithm Evaluate.

Let the evaluation $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)(u_0, 0)$ using algorithm Evaluate terminate with $\underline{\Phi_{u_0,R_j}(0)} = \mathbf{A}_j\mathbf{t} + \underline{b}_j$, for $j = 1, 2, \dots, m$. Then $R_j(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n)(u_0, 0) = (u_0, 0)$ is equivalent to $\mathbf{A}_j\mathbf{t} + \underline{b}_j \in \operatorname{Ker}\,\kappa$, for all j. Putting together we must have

$$[\mathbf{A}_1^T, \mathbf{A}_2^T, \dots, \mathbf{A}_m^T]^T\,\mathbf{t} = -[\underline{b}_1^T, \underline{b}_2^T, \dots, \underline{b}_m^T]^T \tag{5.2}$$

modulo the relations Λ_j. Introducing additional auxiliary variables $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_m \in \mathbb{Z}^{s,1}$ we finally obtain the following linear system over $\mathbb{Z}$

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{\Lambda} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}_2 & \mathbf{0} & \mathbf{\Lambda} & \dots & \mathbf{0} \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{A}_m & \mathbf{0} & \dots & & \mathbf{\Lambda} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_m \end{bmatrix} = - \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_m \end{bmatrix}, \tag{5.3}$$

where $\mathbf{\Lambda} = [\underline{\Lambda_1}, \underline{\Lambda_2}, \dots, \underline{\Lambda_s}] \in \mathbb{Z}^{r,s}$ and $\underline{\Lambda_k}$ is the vector representing the relator Λ_k.

Algorithm: Evaluate

Input: word $R_j(g_1, g_2, \ldots, g_n)$,
set T of $|V(X)|$ vectors $\underline{\zeta_Q} \in \mathbb{Z}^{r,1}$ with $Q: v \to u_0$ for all $v \in V(X)$,
list $\mathcal{Z}$ of n sets each containing $|V(X)|$ vectors $\underline{\zeta_{g_i Q}} \in \mathbb{Z}^{r,1}$,
list $\mathcal{M}$ of n matrices $M_i \in \mathbb{Z}^{r,r}$ representing $g_i^{\#}$
Output: matrix $\mathbf{A}_j \in \mathbb{Z}^{r,rn}$, vector $\underline{b}_j \in \mathbb{Z}^{r,1}$

set $\mathbf{A}_j \in \mathbb{Z}^{r,rn}$ to be the zero matrix and $\underline{b}_j \in \mathbb{Z}^{r,1}$ the zero vector; $v \leftarrow u_0$;
suppose $R_j = g_{k_l} \cdots g_{k_1}$;
for $i \leftarrow 1$ **to** l **do** (*scan word R_j from right to left*)
  $\mathbf{A}_j \leftarrow \mathcal{M}[k_i]\mathbf{A}_j$; (*multiply $\mathbf{A}_j$ on the left by M_{k_i}*)
  for $s \leftarrow 1$ **to** r **do** (*add $\mathbf{E}_{k_i}$ to $\mathbf{A}_j$*)
    $\mathbf{A}_j[s][r * k_i + s] \leftarrow \mathbf{A}_j[s][r * k_i + s] + 1$;
  let $\underline{\zeta_Q} \in T$ and $\underline{\zeta_{g_{k_i} Q}} \in \mathcal{Z}[k_i]$ with $Q: v \to u_0$;
  $\underline{b}_j \leftarrow \mathcal{M}[k_i](\underline{b}_j + \underline{\zeta_Q}) - \underline{\zeta_{g_{k_i} Q}}$;
  $v \leftarrow g_{k_i}(v)$;
return $\mathbf{A}_j$, $\underline{b}_j$

The problem of testing whether a given extension is split-admissible has now been reduced to an equivalent problem of checking whether the linear system (5.3) has a solution. Efficient algorithms for solving a system of linear equations over $\mathbb{Z}$ are long known, and are based on reducing the matrix of coefficients into Hermite or Smith normal form, see [23, Sections 9.2.3 and 9.2.4].

Theorem 5.2. *Let X be a finite connected graph and $G \le \operatorname{Aut} X$ a group of automorphisms given by a presentation $G = \langle S \mid R\rangle$, where $S = \{g_1, g_2, \ldots, g_n\}$ is a generating set and $R_j(g_1, g_2, \ldots, g_n) = \mathrm{id}$, $j = 1, 2, \ldots, m$, are the R-relations. Further, let $\wp = \wp_\zeta \colon \mathrm{Cov}(\zeta) \to X$ be an abelian G-admissible regular covering projection of connected graphs arising from a Cayley voltage assignment $\zeta\colon D(X) \to \Gamma$. Suppose that the abelian group Γ is given by a presentation $\Gamma = \langle \Delta \mid \Lambda\rangle$, where $\Delta = \{c_1, c_2, \ldots, c_r\}$ is a generating set and $\Lambda_k(c_1, c_2, \ldots, c_r) = 0$, $k = 1, 2, \ldots, s$, are the Λ-relations.*

Then the short exact sequence $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ is split if and only if the system of linear equations (5.3) has a solution in $\mathbb{Z}$. Moreover, let $\Omega \subseteq \Gamma^n$ be the set of all solutions of (5.3) reduced relative to the defining relations in Λ. Then Ω is in bijective correspondence with all complements to $\mathrm{CT}_\wp$ within $\tilde{G}$ (which correspond to all derivations $G \to \Gamma$), and two solutions in Ω correspond to conjugate complements if and only if they differ by an inner derivation.

Proof. In view of Proposition 5.1 and the above discussion it is clear that the extension splits if and only if the linear system (5.3) has an integer solution. It is also clear that each complement to $\mathrm{CT}_\wp$ in $\tilde{G}$ corresponds to some solution of (5.3), and hence to a solution in Ω. Moreover, two distinct solutions from Ω give rise to distinct complements. Indeed. Suppose that $(t_1, t_2, \ldots, t_n)$ and $(t_1', t_2', \ldots, t_n')$ are two distinct solutions from Ω giving rise to the same complement $\overline{G}$. Then there is an index i such that $t_i \ne t_i'$, that is, $\bar{g}_i \ne \bar{g}_i'$. As $\bar{g}_i$ and $\bar{g}_i'$ are two lifts of the same automorphism g_i, we must have $\bar{g}_i' = \tilde{\mathrm{id}}_c\, \bar{g}_i$, where $\tilde{\mathrm{id}}_c \in \mathrm{CT}_\wp$. But since $\overline{G}$ is a complement to $\mathrm{CT}_\wp$ we must have $\bar{g}_i' = \bar{g}_i$, and therefore

$t_i = t'_i$, contrary to the assumption. It follows that all solutions in Ω correspond bijectively to all complements.

Let $\overline{G}$ and $\overline{G}'$ be two conjugate complements. Without loss of generality we may assume they are conjugate by an element $\tilde{\mathrm{id}}_c \in \mathrm{CT}_\wp$, that is, $\overline{G}' = \tilde{\mathrm{id}}_c \, \overline{G} \, \tilde{\mathrm{id}}_c^{-1}$. Since for any $\bar{g} \in \overline{G}$ the elements $\tilde{\mathrm{id}}_c \, \bar{g} \, \tilde{\mathrm{id}}_c^{-1}$ and $\bar{g}'$ from $\overline{G}'$ both project to $g \in G$ we must have $\bar{g}' = \tilde{\mathrm{id}}_c \, \bar{g} \, \tilde{\mathrm{id}}_c^{-1}$, for all $g \in G$. Rewrite as

$$\bar{g}' \, \tilde{\mathrm{id}}_c = \tilde{\mathrm{id}}_c \, \bar{g},$$

and let $\bar{g}(u_0, 0) = (gu_0, t_g)$ and $\bar{g}'(u_0, 0) = (gu_0, t'_g)$. Then the left hand side maps the vertex $(u_0, 0)$ to $(gu_0, t'_g + g^\#(c))$, while the right hand side maps $(u_0, 0)$ to $(gu_0, t_g + c)$. Hence $t'_g + g^\#(c) = t_g + c$, and so

$$t_g - t'_g = \delta_c(g),$$

where $\delta_c \in \mathrm{Inn}(G, \Gamma)$ is an inner derivation. In particular, the above relation holds for $(t_1, t_2, \ldots, t_n)$ and $(t'_1, t'_2, \ldots, t'_n)$ from Ω giving rise to $\overline{G}$ and $\overline{G}'$.

For the converse, let $(t_1, t_2, \ldots, t_n)$ and $(t'_1, t'_2, \ldots, t'_n)$ from Ω give rise to $\overline{G}$ and $\overline{G}'$ such that $t_i - t'_i = \delta_c(g_i)$ for $i = 1, 2 \ldots, n$. Then we can work backwards to find that $\bar{g}'_i = \tilde{\mathrm{id}}_c \, \bar{g}_i \, \tilde{\mathrm{id}}_c^{-1}$ for all indices. Hence $\overline{G}$ and $\overline{G}'$ are conjugate subgroups. This completes the proof. □

Remark 5.3. Theorem 5.2 can be used to compute the first cohomology group $H^1(G, \Gamma)$, c.f. [23, Section 7.6]. Next, observe that each solution $(t_1, t_2, \ldots, t_n) \in \Omega$ extends uniquely to a function $t \colon G \to \Gamma$ satisfying condition (4.1), and that two such functions differ precisely by a derivation, $t' - t \in \mathrm{Der}(G, \Gamma)$. Thus, the set of functions $G \to \Gamma$ satisfying condition (4.1) forms a coset of $\mathrm{Der}(G, \Gamma)$ in the group of all functions $G \to \Gamma$ equipped with pointwise addition. Consequently, an alternative proof of the last statement in Theorem 5.2 can be given using the standard result which states that two derivations give rise to conjugate complements if and only if they differ by an inner derivation. □

Remark 5.4. Algorithm Evaluate requires some precomputations. First, we need to compute the vectors $\underline{\zeta_Q} \in \mathbb{Z}^{r,1}$, for some $Q : v \to u_0$ and all $v \in V(X)$, and consequently, the vectors representing voltages of fundamental walks at u_0. This can be done efficiently using breadth first search. During the search we also compute the vectors representing voltages of the mapped paths in order to obtain, upon completion of the search, the vectors $\underline{\zeta_{g_i Q}} \in \mathbb{Z}^{r,1}$ together with the vectors representing voltages of the mapped fundamental walks, for each g_i. Second, with these data in hand we then build the systems of linear equations over $\mathbb{Z}$ whose solutions give rise to the matrices $M_i \in \mathbb{Z}^{r,r}$ representing $g_i^\#$. □

Theorem 5.2 can also be used for testing whether a given group lifts as a direct product extension. In view of Corollary 4.5 we first check if the condition $\zeta_{gW} = \zeta_W$ holds for all $g \in S$ and all closed walks W from a basis of $H_1(X, \mathbb{Z})$. If true then G lifts, and we test whether the extension splits by solving the linear system (5.3) using algorithm Evaluate with all $g_i^\# = \mathrm{id}$. Algorithm Evaluate simplifies in that all matrices M_i are now equal to the identity matrix. Also, since the covering projection is abelian, recall that if some complement to $\mathrm{CT}_\wp$ is normal, then all complements are normal. We record this formally as Corollary 5.5.

Corollary 5.5. *With assumptions and notation above,* $\mathrm{id} \to \mathrm{CT}_\wp \to \tilde{G} \to G \to \mathrm{id}$ *is a direct product extension if and only if the following two conditions are satisfied:*

(i) $\zeta_{gW} = \zeta_W$ *holds for all* $g \in S$ *and all closed walks* W *from a basis of* $H_1(X, \mathbb{Z})$*;*

(ii) the (simplified) system of linear equations (5.3) has a solution in $\mathbb{Z}$*.*

Moreover, in this case the set of solutions of (5.3), reduced relative to the defining relations in Λ*, is in bijective correspondence with normal complements to* $\mathrm{CT}_\wp$ *within* $\tilde{G}$*.* □

5.2 Elementary abelian covers

One particular special case worth mentioning is that of $\mathrm{CT}_\wp$ being elementary abelian. In this case, Γ can be identified with the vector space over the corresponding prime field, and the automorphisms of Γ are then viewed as invertible linear transformations. More precisely, let $\Gamma = \mathbb{Z}_p^r$. The generating set $\{c_1, c_2, \ldots, c_r\}$ is now understood to be the standard generating set (of a vector space), and (5.1) can be viewed as a formula in this vector space. Consequently, instead of (5.3) we need to find solutions of (5.2) over $\mathbb{Z}_p$, which can be done using Gaussian elimination. This makes computation easier; in particular, we do not experience difficulties which might otherwise be present with computations over $\mathbb{Z}$ (like uncontrolled integer growth). An algorithm for testing whether the extension is split now immediately follows from the above discussion. It is formally encoded in algorithm IsSplit.

Algorithm: IsSplit

Input: Cayley voltage assignment $\zeta\colon D(X) \to \mathbb{Z}_p^d$ giving rise to connected cover, automorphism group $G = \langle g_1, g_2, \ldots, g_n \mid R_1, R_2, \ldots, R_m \rangle$ that lifts
Output: true, if the lifted group is split extension, false otherwise

set $\mathbf{A} \in \mathbb{Z}^{0,dn}$ to be the zero matrix and $\underline{b} \in \mathbb{Z}^{0,1}$ the zero vector;
take an arbitrary vertex u_0 in X;
compute set T of $|V(X)|$ vectors $\underline{\zeta_Q} \in \mathbb{Z}_p^{d,1}$ with $Q : v \to u_0$ for all $v \in V(X)$;
compute list $\mathcal{Z}$ of n sets each containing $|V(X)|$ vectors $\underline{\zeta_{g_iQ}} \in \mathbb{Z}_p^{d,1}$;
compute list $\mathcal{M}$ of n matrices $M_i \in \mathbb{Z}^{d,d}$ representing $g_i^{\#}$;
for $j \leftarrow 1$ **to** m **do**
  let $\mathbf{A}_j$ and $\underline{b}_j$ be the output of evaluating the relator R_j at $(u_0, 0)$ using the algorithm Evaluate;
  $\mathbf{A} \leftarrow \begin{bmatrix} \mathbf{A} \\ \mathbf{A}_j \end{bmatrix}$; $\underline{b} \leftarrow \begin{bmatrix} \underline{b} \\ \underline{b}_j \end{bmatrix}$;
if system $\mathbf{A}\,\mathbf{t} = -\underline{b}$ has a solution **then**
  return true
else
  return false

Theorem 5.6. *Let* $\wp = \wp_\zeta\colon \mathrm{Cov}(\zeta) \to X$ *be an elementary abelian regular covering projection of connected graphs arising from a Cayley voltage assignment* $\zeta\colon D(X) \to \mathbb{Z}_p^d$*. Further, let a given group of automorphisms* $G = \langle g_1, g_2, \ldots, g_n \mid R_1, R_2, \ldots, R_m \rangle$ *lift* *along $\wp$. Then the algorithm* IsSplit *tests whether the lifted group is a split extension of* $\mathrm{CT}_{\wp}$ *by G in*

$$\mathcal{O}(n|V(X)| + nd|D(X)| + d^3r + nd^2r + nd^3L + nd^3m^2)$$

steps using

$$\mathcal{O}(n|V(X)| + nd|D(X)| + nd^2m)$$

memory space, where r is the Betti number of X and $L = \sum_{j=1,2,\ldots,m} |R_j|$.

Proof. It remains to consider time and space complexity. The vectors $\underline{\zeta_{W_k}}$ representing the voltages of the fundamental walks W_k, $k = 1, 2, \ldots, r$, at u_0 together with the vectors $\underline{\zeta_{g_iW_k}}$ representing the voltages of the mapped fundamental walks g_iW_k, $i = 1, 2, \ldots, n$, as well as the vectors $\underline{\zeta_Q}$ and $\underline{\zeta_{g_iQ}}$ can be computed as described in Remark 5.4 using breadth first search at the cost of $\mathcal{O}(d)$ steps per edge; altogether this takes $\mathcal{O}(n|V(X)| + nd|D(X)|)$ steps. As for constructing the matrices $M_i \in \mathbb{Z}^{d,d}$ we first need to solve d systems of linear equations:

$$\begin{array}{rcl}
x_{1,1}\,\underline{\zeta_{W_1}} + x_{1,2}\,\underline{\zeta_{W_2}} + \cdots + x_{1,r}\,\underline{\zeta_{W_r}} & = & e_1 \\
x_{2,1}\,\underline{\zeta_{W_1}} + x_{2,2}\,\underline{\zeta_{W_2}} + \cdots + x_{2,r}\,\underline{\zeta_{W_r}} & = & e_2 \\
& \vdots & \\
x_{d,1}\,\underline{\zeta_{W_1}} + x_{d,2}\,\underline{\zeta_{W_2}} + \cdots + x_{d,r}\,\underline{\zeta_{W_r}} & = & e_d,
\end{array} \tag{5.4}$$

where e_i's are the standard basis vectors of $\mathbb{Z}_p^{d,1}$. Solving d systems using Gaussian elimination requires $\mathcal{O}(d^3r)$ steps. An arbitrary matrix M_i can then be computed in $\mathcal{O}(d^2r)$ steps; thus $\mathcal{O}(nd^2r)$ steps are required to compute n such matrices. Algorithm Evaluate takes $\mathcal{O}(nd^3|R_j|)$ steps for evaluating an arbitrary relator R_j. Hence all relators can be evaluated in $\mathcal{O}(nd^3L)$ steps, where $L = \sum_{j=1,2,\ldots,m} |R_j|$. It remains to solve the system $\mathbf{A}\,\underline{b} = -\mathbf{b}$ for $\mathbf{A} \in \mathbb{Z}_p^{dm,dn}$ and $\underline{b} \in \mathbb{Z}_p^{dn,1}$, which takes $\mathcal{O}(nd^3m^2)$ steps using Gaussian elimination. Hence the problem of testing whether the extension splits can be solved in $\mathcal{O}(n|V(X)| + nd|D(X)| + d^3r + nd^2r + nd^3L + nd^3m^2)$ steps.

Representing the graph X using adjacency list takes $\mathcal{O}(|V(X)|+|D(X)|)$ space, while representing a vector in $\mathbb{Z}_p^{d,1}$ takes $\mathcal{O}(d)$ space. Therefore the representation of the Cayley voltage assignment ζ takes $\mathcal{O}(|V(X)|+d|D(X)|)$ space. As for the representation of automorphisms as permutations, this takes $\mathcal{O}(n|D(X)|)$ space. During breadth first search we also need $\mathcal{O}(n|V(X)|)$ space to store the mapped vertices, and $\mathcal{O}(nd|D(X)|)$ additional space to store the voltages of the mapped walks. It takes $\mathcal{O}(nd^2)$ space to store all the matrices M_i, while storing the matrix $\mathbf{A} \in \mathbb{Z}_p^{dm,dn}$ takes $\mathcal{O}(nd^2m)$ space. Putting together, the space complexity is $\mathcal{O}(n|D(X)| + nd|D(X)| + nd^2m)$. □

Example 5.7. Let X be the 3-dipole with vertices 1 and 2 and three parallel links from 1 to 2 defined by the darts a, b and c. The voltage assignment $\zeta_a = \zeta_{a^{-1}} = (0,0)$, $\zeta_b = \zeta_{b^{-1}} = (1,0)$, $\zeta_c = \zeta_{c^{-1}} = (0,1)$ taking values in the elementary abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ gives rise to a connected covering graph $\tilde{X}$ isomorphic to the 3-cube graph. Consider the group $G = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = \tau\sigma\tau\sigma^2 = 1 \rangle$ acting as a subgroup of $\operatorname{Aut} X$, where $\sigma = (abc)(a^{-1}b^{-1}c^{-1})$ and $\tau = (aa^{-1})(bb^{-1})(cc^{-1})$. By computation, G lifts along $\wp_\zeta$. We now test whether $\wp_\zeta$ is split-admissible for the group G.

Choosing $u_0 = 1$ as the base vertex, let us work through the computation on the first relator τ^2. Observe that $M_\tau = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. Set $\mathbf{E}_\tau = \left[\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right]$. Initially we have $\mathbf{A}_1 = \left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$, $\underline{b}_1 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, and $v = 1$. Scanning the relator from right to left we start with the generator τ. We multiply $\mathbf{A}_1$ on the left by M_τ, and then add $\mathbf{E}_\tau$ to get $\mathbf{A}_1 = \left[\begin{smallmatrix}0&0&1&0\\0&0&0&1\end{smallmatrix}\right]$. For the walk $W = 1a2b^{-1}1$ we have $\underline{\mu_W} = \underline{\mu_{\tau W}} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. So we add $\underline{\mu_W}$ to $\underline{b}_1$, multiply the result on the left by M_τ, and subtract $\underline{\mu_{\tau W}}$ to obtain $\underline{b}_1 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$. Further, mapping the vertex v by τ we get $v = 2$. Moving left we scan the generator τ again. Multiplying $\mathbf{A}_1$ on the left by M_τ and adding $\mathbf{E}_\tau$ gives $\mathbf{A}_1 = \left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$. For the walk $W = 2c^{-1}1$ we have $\underline{\mu_W} = \underline{\mu_{\tau W}} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. We then add $\underline{\mu_W}$ to $\underline{b}_1$, multiply the result on the left by M_τ, and subtract $\underline{\mu_{\tau W}}$ to get $\underline{b}_1 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$.

Similarly, the computation on the second relator σ^3 gives $\mathbf{A}_2 = \left[\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right]$ and $\underline{b}_2 = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, while the computation on the third relator $\tau\sigma\tau\sigma^2$ results in $\mathbf{A}_3 = \left[\begin{smallmatrix}0&0&1&1\\0&0&1&0\end{smallmatrix}\right]$ and $\underline{b}_3 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Putting together we obtain the following system over $\mathbb{Z}_2$

$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}x\\y\\u\\v\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix},$$

which is clearly consistent. Thus the projection $\wp_\zeta$ is G-split-admissible. □

5.3 Solvable covers

The elementary abelian version of Theorem 5.2 can be used to decide whether G lifts as a split extension of $\mathrm{CT}_\wp$ whenever $\mathrm{CT}_\wp$ is solvable.

First recall, c.f. [35, 51], that if $q\colon Z \to X$ is a regular covering projection of connected graphs, and $q = r\,s$ where $r\colon Y \to X$ and $s\colon Z \to Y$ are regular covering projections with CT_s a characteristic subgroup of CT_q, then q is admissible for a group of automorphisms $G \le \operatorname{Aut} X$ if and only if G lifts along r and its lift lifts along s (in which case this lift is the lift of G along q).

The following lemma (c.f. [5, Theorem 4.2]) shows that testing whether the projection $q\colon Z \to X$ is split-admissible can be reduced to testing whether the projections $r\colon Y \to X$ and $s\colon Z \to Y$ are split-admissible. We omit the obvious proof.

Lemma 5.8. *Let $q\colon Z \to X$ be a regular covering projection of connected graphs, and let $q = r\,s$ where $r\colon Y \to X$ and $s\colon Z \to Y$ are regular covering projections with CT_s a characteristic subgroup of CT_q. Suppose that q is admissible for a group of automorphisms $G \le \operatorname{Aut} X$. Then the following statements are equivalent.*

(i) *The projection q is split-admissible for G.*

(ii) *The projection r is split-admissible for G, and s is split-admissible for some complement to CT_r within the G-lift along r.* □

Remark 5.9. Denote by $\tilde{G}$ the lift of G along $q\colon Z \to X$ and by H the lift of G along $r\colon Y \to X$. Observe that the projection $s\colon Z \to Y$ as in Lemma 5.8(ii) should be checked, at least in principle, relative to all complements of CT_r within H (in particular, this requires the construction of all such complements). However, if K is a complement, then any subgroup conjugate to K is also a complement; and if K lifts along $s\colon Z \to Y$ as a split extension, then any of its conjugate complements also lifts as a split extension. Therefore, when applying Lemma 5.8(ii) we only need to consider representatives of conjugacy classes of complements within H. A method for constructing such representatives is described in [5, 23]. □

Coming back to the case when $\mathrm{CT}_\wp$ is solvable, we first find a series of characteristic subgroups $\mathrm{CT}_\wp = K_0 > K_1 > \ldots > K_n = \mathrm{id}$ with elementary abelian factors K_{j-1}/K_j. The method is known, see [23, Chapter 8]. The covering projection $\wp$ then decomposes as $\tilde{X} = X_n \overset{\wp_n}{\to} X_{n-1} \to \ldots \to X_1 \overset{\wp_1}{\to} X_0 = X$, where $\wp_j \colon X_j \to X_{j-1}$ is a regular elementary abelian covering projection with $\mathrm{CT}_{\wp_j}$ isomorphic to K_{j-1}/K_j. At each step we may then recursively apply Lemma 5.8. To this end, one has to explicitly construct (among other things) the voltage assignments that define the intermediate projections in the above decomposition. For more details we refer the reader to [44].

6 Concluding remarks

In order to evaluate the performance of the above method for testing whether a given solvable covering projection is split-admissible the second author has implemented it in MAGMA [4], as a part of a larger package for computing with graph covers, see [43], and [44] for a more detailed account on experimental results.

We further remark that in the case of solvable covers one can take an alternative approach that even does not require explicit reconstruction of the intermediate covering projections. It is enough to first compute the automorphisms $g^{\#_{u_0}}$ and the factor set $\mathcal{F}(g,h) = g^{\#_{u_0}}(\zeta_Q)\zeta_{gQ}^{-1}$ (partially as needed) in order to reconstruct the lifted group as a crossed product, and then consider the decomposition abstractly without reference to covers. This is discussed in [45].

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