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Counting even and odd restricted permutations

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Abstract

Let p be a *permutation* of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. We introduce techniques for counting $N(n; k, r, I; \pi)$, the number of even or odd restricted permutations of $\mathbb{N}_n$ satisfying the conditions $-k \le p(i)-i \le r$ (for arbitrary natural numbers k and r) and $p(i)-i \notin I$ (for some set I) and $\pi = 0$ for even permutations and $\pi = 1$ for odd permutations.

Keywords: Even and odd restricted permutations, exact enumeration, recurrences, permanents.

Math. Subj. Class.: 05A15, 05A05, 11B37, 15A15

1 Introduction

Let p be a *permutation* of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. So, $p(i)$ refers to the value taken by the function p when evaluated at a point i. A class of permutations in which the positions of the marks after the permutation are restricted can be specified by an $n \times n$ $(0, 1)$-matrix $A = (a_{ij})$ in which:

$$a_{ij} = \begin{cases} 1, & \text{if the mark } j \text{ is permitted to occupy the } i\text{-th place;} \\ 0, & \text{otherwise.} \end{cases}$$

The following result is a well known fact on the number of restricted permutations.

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Theorem 1.1 ([1]). *The number of restricted permutations is given by the permanent of a square matrix A:*

$$\operatorname{per} A = \sum_{p \in S_n} a_{1p(1)} a_{2p(2)} \cdots a_{np(n)},$$

where p runs through the set S_n of all permutations of $\mathbb{N}_n$. □

Next, we will define strongly and weakly restricted permutations (for more informations see [13]).

Definition 1.2. In *strongly restricted permutations* of $\mathbb{N}_n$, the number $r_i = \sum_{j=1}^{n} a_{ij}$ is uniformly small, i.e., $r_i \leq K$ $(i = 1, 2, \ldots, n)$, where K is an integer independent of n. In *weakly restricted permutations*, $n - r_i$ is uniformly small.

Let us briefly overview historical development of the topic of restricted permutations. Probably the most well known example is the derangement problem or "le Problème des Rencontres" (see [1] or [6]). Most of the restricted permutations considered in current literature deal with pattern avoidance. For surveys of such studies, see [3] or [9]. For a related topic of pattern avoidance in compositions and words see [5].

Detailed introduction to weakly restricted permutations can be found in [1]. A general method of enumeration of permutations with restricted positions was developed by Kaplansky and Riordan in a series of papers (they developed the theory of rook polynomials for these purposes—see [6], [7], [8], [18]). Lagrange, Lehmer, Mendelsohn, Tomescu and Stanley ([12, 13, 15, 16, 20, 19]) studied particular types of strongly restricted permutations satisfying the condition $|p(i) - i| \leq d$, where d is 1, 2, or 3 (more information on their work can be found in [1]).

Lehmer [13] classified some sets of strongly restricted permutations. The first author showed in [1] how to handle *all* five types of Lehmer's permutations. For the number of restricted permutations in a circular case the following is known: Stanley [19, Example 4.7.7] explored type $k = 2$ with the transfer-matrix method, Baltić [2] used finite state automata for type $k = 2$, and Li et al. [14] explored the $k = 3$ by expanding permanents.

An explicit technique for creating a system of the recurrence equations was given in [1], based on a simple mapping φ from combinations of $\mathbb{N}_{k+r+1}$ and some crucial differences between the transfer-matrix Method and the newly proposed technique were given.

Krafft and Schaefer in [11] find the closed formula for the strongly restricted permutations of the set $\mathbb{N}_n$ satisfying the condition $|p(i) - i| \leq k$, where $k + 2 \leq n \leq 2k + 2$. Panholzer [17] and Kløve [10] made progress in symmetric cases (Panholzer used finite state automata, while Kløve used modified transfer-matrix method based on expanding permanent) and they found the asymptotic expansion and gave bounds for the denominator of corresponding generating functions.

Here we pursue the more general, asymmetric cases and the cases where more numbers are forbidden than in the ordinary derangements for even and odd strongly restricted permutations. Our method determines the number of restricted permutations that are even, and the number of restricted permutations that are odd.

In Section 2 we introduce a general technique for counting $N(n; k, r, I; \pi)$, the number of even or odd restricted permutations ($N(n; k, r, I; \pi)$ is defined in abstract). In Section 3 we illustrate it with several examples. Using a program that implements our technique, we have contributed about a hundred sequences to the Sloane's online encyclopedia of integer sequences [21].

2 Counting $N(n; k, r, I; \pi)$

We established the connection between the number of restricted permutations and the permanent function of a matrix A, $\operatorname{per} A$, in Theorem 1 from the introduction. The Laplace expansion of the permanent function (this is the same as for the determinant function) is computationally inefficient for high dimension because for $n \times n$ matrices, the computational effort scales with $n!$. Therefore, the Laplace expansion is not suitable for large n. However, the matrices obtained in the Laplace expansions for restricted permutations have the regular structure, so called band matrices (a band matrix is a sparse matrix whose nonzero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side), and their expansions can be reduced to a system of linear recurrence equations.

We present a general technique for counting $N(n; k, r, I; \pi)$, the number of even or odd ($\pi = 0$ for even permutations and $\pi = 1$ for odd permutations) restricted permutations satisfying the conditions $-k \leq p(i) - i \leq r$ and $p(i) - i \notin I$ for all $i \in \mathbb{N}_n$, where $k \leq r < n$, and I is a fixed subset of the set $\{-k+1, -k+2, \ldots, r-1\}$. Assume that I contains x elements, $|I| = x$. Our technique proceeds in six steps:

1. Create $\mathcal{C}$, a set of all $(k+1)$-element combinations of the set $\mathbb{N}_{k+r+1}$ containing element $k+r+1$.
2. Create $\mathcal{D}$, a set of all ordered pairs $D = (C, \pi)$, where $C \in \mathcal{C}$ and $\pi \in \{0, 1\}$.
3. Introduce an integer sequence $a_D(n)$ for each ordered pair $D \in \mathcal{D}$.
4. Apply the mapping φ (defined below) to each ordered pair.
5. Create a system of linear recurrence equations (later we will see that these equations correspond to the Laplace expansion of a permanent of the matrix A):

$$a_D(n) = \sum_{D' \in \varphi(D)} a_{D'}(n-1).$$

6. Solve the system to obtain equations $N(n; k, r, I; 0) = a_{((r+1,r+2,\ldots,r+k+1),0)}(n)$ and $N(n; k, r, I; 1) = a_{((r+1,r+2,\ldots,r+k+1),1)}(n)$.

We next describe these steps in detail and then prove that $N(n; k, r, I; \pi)$ is indeed equal to $a_{((r+1,r+2,\ldots,r+k+1),\pi)}(n)$.

Definition 2.1. Let $\mathcal{C}$ denote a set of all combinations with $k+1$ elements of the set $\mathbb{N}_{k+r+1}$, which contain $k+r+1$. We represent these combinations as strictly increasing ordered $(k+1)$-tuples.

For example, all such combinations with 3 elements of the set $\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$ are represented (in reverse lexicographic order) by:

$$(3,4,5), \quad (2,4,5), \quad (2,3,5), \quad (1,4,5), \quad (1,3,5), \quad (1,2,5).$$

In examples we will use easier notation:

$$345, \qquad 245, \qquad 235, \qquad 145, \qquad 135, \qquad 125.$$

Definition 2.2. Let $\alpha \pm I$ denote the set $\alpha \pm I = \{\alpha \pm i \mid i \in I\}$.

We split the set $\mathcal{C}$ in two disjoint sets

$$\mathcal{C}_1 = \{C \in \mathcal{C} \mid 1 \in C\} \qquad \text{and} \qquad \mathcal{C}_2 = \{C \in \mathcal{C} \mid 1 \notin C\},$$

but we will also separate the set $\mathcal{C}_2$ into $x+1$ disjoint sets

$$\mathcal{C}_2^m = \{C \in \mathcal{C}_2 \mid m \text{ elements of } C \text{ are in } r+1-I\}, \quad (m = 0, 1, \ldots, x).$$

Let $\mathcal{C}^{k+1-m}$ denote a Cartesian product $\mathcal{C}^{k+1-m} = \mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C}$, where $\mathcal{C}$ appears $(k+1-m)$ times. Let B denote the set $B = \{0, 1\}$.

We define the set of ordered pairs $\mathcal{D} = \{(C, \pi) \mid C \in \mathcal{C}, \pi \in B\}$ and same as in the case of $\mathcal{C}$ we will divide it into disjoint sets:

$$\mathcal{D}_1 = \{(C, \pi) \mid C \in \mathcal{C}_1, \pi \in B\}, \qquad \mathcal{D}_2 = \{(C, \pi) \mid C \in \mathcal{C}_2, \pi \in B\},$$

$$\mathcal{D}_2^m = \{(C, \pi) \mid C \in \mathcal{C}_2^m, \pi \in B\}, \quad (m = 0, 1, \ldots, x).$$

For each $D \in \mathcal{D}_2$ we define ordered $(k+1)$-tuple

$$SD = (D_1, D_2, \ldots, D_k, D_{k+1})$$

in the following manner. We get each of the combinations $C_i \in \mathcal{C}$ from the initial combination $C = (c_1, c_2, \ldots, c_k, c_{k+1})$ (pay attention that $D \in \mathcal{D}$ is $D = (C, \pi)$) by deleting c_i, decreasing all other coordinates by 1, shifting all coordinates with bigger index to one place left and putting $k+r+1$ at the end:

$$C_i = (c_1 - 1, \ldots, c_{i-1} - 1, c_{i+1} - 1, \ldots, c_{k+1} - 1, k+r+1).$$

For the parity coordinate, we have an easier condition:

$$\pi_i = \begin{cases} \pi, & i \text{ is odd,} \\ 1 - \pi, & i \text{ is even,} \end{cases}$$

i.e. if i is odd the parity coordinate stays the same and if i is even the parity coordinate changes.

In the same way as before, we also introduce D_1 for each $D \in \mathcal{D}_1$:

$$D_1 = \big((c_2 - 1, c_3 - 1, \ldots, c_k - 1, c_{k+1} - 1, k+r+1), \pi\big)$$

(in this case the parity coordinate π stays the same).

Now, we get ordered $(k+1-m)$-tuple $SD' = \big(D'_1, D'_2, \ldots, D'_{k+1-m}\big)$ from ordered $(k+1)$-tuple $SD = (D_1, D_2, \ldots, D_k, D_{k+1})$ when we delete all ordered pairs $D_y = (C_y, \pi_y)$ corresponding to elements c_y which satisfy the condition $c_y \in r+1-I$.

Finally, we introduce the mapping

$$\varphi(D) = \begin{cases} \varphi_1(D), & D \in \mathcal{D}_1 \\ \varphi_2^m(D), & D \in \mathcal{D}_2^m, \end{cases}$$

defined by $\varphi_1 : \mathcal{D}_1 \to \mathcal{D}$ and $\varphi_2^m : \mathcal{D}_2^m \to \mathcal{D}^{k+1-m}$, for $m = 0, 1, \ldots, x$, defined by

$$\varphi_1(D) = D_1, \qquad \varphi_2^m(D) = SD'.$$

We use these mappings to find a system of $2 \cdot \binom{k+r}{k}$ linear recurrence equations (one equation per ordered pair, i.e. two equations per combination – one corresponding to even permutations and another corresponding to odd permutations): if we have $\varphi_1(D) = D'$ then we have the linear recurrence equation:

$$a_D(n+1) = a_{D'}(n)$$

and if we have $\varphi_2^m(D) = (D'_1, D'_2, \ldots, D'_{k+1-m})$ then we have the linear recurrence equation:

$$a_D(n+1) = a_{D'_1}(n) + a_{D'_2}(n) + \cdots + a_{D'_{k+1-m}}(n).$$

The initial conditions are: $a_{((r+1,r+2,\ldots,r+k+1),0)}(0) = 1$ and $a_D(0) = 0$ for all $D \neq ((r+1, r+2, \ldots, r+k+1), 0)$.

This system can be easily solved, for example by using the standard method based on generating functions. From this system of linear recurrence equations we are able to get a linear recurrence equation and a generating function for $N(n; k, r, I; \pi)$. We will prove that $N(n; k, r, I; 0) = a_{((r+1,r+2,\ldots,r+k+1),0)}(n)$ and $N(n; k, r, I; 1) = a_{((r+1,r+2,\ldots,r+k+1),1)}(n)$. Thus, from the matrix of this system, S, we can find $N(n; k, r, I; \pi)$ as the element in the first row and the first column of the matrix S^n, i.e., the number of the closed paths in the digraph G whose adjacency matrix is S (this observation is important because we can apply the Transfer matrix method to the matrix S). We apply this observation to determine the computational complexity of our technique: S^n can be computed with repeated squaring [4] in $O(\log_2 n)$ operations. Hence, our technique evaluates the number of restricted permutations more efficiently than the straightforward techniques of filtering permutations or expanding the permanent $\operatorname{per} A$.

All the generating functions that we derive using our technique are rational. We have a system of $2 \cdot \binom{k+r}{k}$ linear recurrence equations which leads us to the upper bound for the degree d of the denominator polynomial: $d \leq 2 \cdot \binom{k+r}{k}$. It is sufficient to compute a finite number of values, in particular $2 \cdot \binom{k+r}{k}$ of them, to find the generating function.

Theorem 2.3. *For even permutations $N(n; k, r, I; 0) = a_{((r+1,r+2,\ldots,r+k+1),0)}(n)$ and for odd permutations $N(n; k, r, I; 1) = a_{((r+1,r+2,\ldots,r+k+1),1)}(n)$.*

Proof. We establish the correspondence between combination $C = (c_1, c_2, \ldots, c_k) \in \mathcal{C}$ and the specific matrix $M_C = f(C)$. We introduce a set $\mathcal{M}_t$ (for a fixed t) of matrices M_C that correspond to the sequences $a_{D_0}(n)$ and $a_{D_1}(n)$, where $D_0 = (C, 0)$ and $D_1 = (C, 1)$.

Let matrix $M_C = (m_{ij})$ satisfies the following conditions:

1) the first $k+1$ rows is definied by:

   for $i = 1, 2, \ldots, k+1$, $m_{ij} = \begin{cases} 1, & j+r-c_i \notin I \\ 0, & j+r-c_i \in I \end{cases}$ for $j = 1, 2, \ldots, c_i$ and $m_{ij} = 0$ for $j > c_i$;

2) elements in the last $t-(k+1)$ rows satisfy: $m_{ij} = 1$ for $-k \leq j-i \leq r$, $j-i \notin I$ and $m_{ij} = 0$ otherwise.

Denote by $\mathcal{M}_t$ the set of all $t \times t$ ($t > r$) matrices M_C for $C \in \mathcal{C}$.

From the matrix $M_C \in \mathcal{M}_t$, we can determine the corresponding combination $C = (c_1, c_2, \ldots, c_k) \in \mathcal{C}$: let c_i denotes the column of the last one in the i-th row of the matrix M, $i = 1, 2, \ldots, k+1$.

So, the function $f : \mathcal{C} \to \mathcal{M}_t$, defined by $f(C) = M_C$ is a bijection.

We associate an $n \times n$ matrix $A = (a_{ij})$ defined by:

$$a_{ij} = \begin{cases} 1, & \text{if} \quad -k \leq j - i \leq r,\ j - i \notin I, \\ 0, & \text{otherwise} \end{cases}$$

with the strongly restricted permutations satisfying $-k \leq p(i) - i \leq r$ and $p(i) - i \notin I$. As stated in the introduction, the number of all permutations (even and odd) satisfying $-k \leq p(i) - i \leq r$ and $p(i) - i \notin I$ is equal to $\operatorname{per} A$. Notice that $A \in \mathcal{M}_n$ with $c_i = r + i$, where $1 \leq i \leq k + 1$, and thus the combination corresponding to A is $(r + 1, r + 2, \ldots, r + k + 1)$.

We next observe that the recurrence equations from step 5. (see page 11) correspond to the expansion of the permanent of matrices from $\mathcal{M}_t$ by the first row (φ_1) or by the first column (in cases of all of φ_2^m; note that when we skip an element c_y, it corresponds to a zero element in the first column). During this expansion we need to take care about the parity of the permutation under construction.

First, note that at each step of construction determines the position of the smallest of the remaining elements of the permutation. Let q denote the number of already used elements in the construction of the restricted permutation. Define a monotonically increasing sequence w of positions in the permutation which have not yet been assigned values: $w = (w_1, w_2, \ldots, w_{n-q})$, where $w_1 < w_2 < \cdots < w_{n-q}$.

If we make an expansion by the first row (we have one in the first column), it corresponds to $p(w_1) = q + 1$ and the parity of the permutation under construction doesn't change because we haven't got any new inversions.

If we make an expansion by the first column and if we have 1 in the i-th row (i.e. at the position $(i, 1)$ in the matrix A is 1), it corresponds to $p(w_i) = q + 1$. There are $i - 1$ numbers: $p(w_1), p(w_2), \ldots, p(w_{i-1})$ which made new inversions with $p(w_i) = q + 1$, because all of them have not been assigned yet, so they are all greater than $q + 1$. So, the parity of the permutation under construction depends on the parity of i:

- if i is even then there are odd number $(i - 1)$ inversions, so we need to change the parity of the permutation under construction, $\pi' = 1 - \pi$;
- if i is odd then there are even number $(i - 1)$ inversions, so we don't need to change the parity of the permutation under construction, $\pi' = \pi$.

These observations lead to the main conclusions:

$$N(n; k, r, I; 0) = a_{((r+1,\ldots,r+k+1),0)}(n)$$
$$N(n; k, r, I; 1) = a_{((r+1,r+2,\ldots,r+k+1),1)}(n).$$

□

3 Examples

We illustrate the technique from the previous section on two examples.

Example 3.1. We find the number of even (odd) permutations of the set $\mathbb{N}_n$, satisfying the condition $-1 \leq p(i) - i \leq 1$ for all $i \in \mathbb{N}_n$. It is usually referred to as a permutation of length n within distance 1.

In this case we have $k = r = 1$, i.e. $k + r + 1 = 3$ and $\mathcal{C} = \{23, 13\}$.
$\varphi_2(23,0) = \{(23,0),(13,1)\}$, $\varphi_1(13,0) = \{(23,0)\}$, $\varphi_2(23,1) = \{(23,1),(13,0)\}$, $\varphi_1(13,1) = \{(23,1)\}$, from which we get the system of linear recurrence equations:

$$\begin{aligned} a_{(23,0)}(n+1) &= a_{(23,0)}(n) + a_{(13,1)}(n), \\ a_{(13,0)}(n+1) &= a_{(23,0)}(n), \\ a_{(23,1)}(n+1) &= a_{(23,1)}(n) + a_{(13,0)}(n), \\ a_{(13,1)}(n+1) &= a_{(23,1)}(n), \end{aligned}$$

with the initial conditions $a_{(23,0)}(0) = 1$, $a_{(13,0)}(0) = 0$, $a_{(23,1)}(0) = 0$, $a_{(13,1)}(0) = 0$. If we substitute $a_{(23,0)}(n) = a_n$, $a_{(13,0)}(n) = b_n$, $a_{(23,1)}(n) = c_n$ and $a_{(13,1)}(n) = d_n$ we have a simpler form:

$$a_{n+1} = a_n + d_n, \qquad b_{n+1} = a_n, \qquad c_{n+1} = c_n + b_n, \qquad d_{n+1} = c_n.$$

The initial conditions are $a_0 = 1$, $b_0 = c_0 = d_0 = 0$.

For a sequence which is denoted by a lower case letter we will denote the corresponding generating function by the same upper case letter ($a_n \leftrightarrow A(z)$, $b_n \leftrightarrow B(z)$, and so on). We find the following system of linear equations $\big($variables are $A(z), B(z), C(z), D(z)\big)$:

$$\frac{A(z)-1}{z} = A(z) + D(z), \quad \frac{B(z)}{z} = A(z), \quad \frac{C(z)}{z} = C(z) + B(z), \quad \frac{D(z)}{z} = C(z)$$

and part of its solution that we are interested in is:

$$A(z) = \frac{1-z}{1-2z+z^2-z^4}, \qquad C(z) = \frac{z^2}{1-2z+z^2-z^4}.$$

From the denominator of these generating functions $1-2z+z^2-z^4$, we can find the linear recurrence equations $a_n = 2a_{n-1} - a_{n-2} + a_{n-4}$ and $c_n = 2c_{n-1} - c_{n-2} + c_{n-4}$.

When we solve these equations we find the general terms of these sequences:

$$a_n = \frac{1}{2}\left(F_{n+1} + x_n\right), \qquad c_n = \frac{1}{2}\left(F_{n+1} - x_n\right),$$

where F_n denotes n-th Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$; A000045 at [21]), and $x_n = \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}$ (A010892 at [21]).

The number of even permutations, a_n, and odd permutations, c_n, both satisfying the condition $|p(i) - i| \le 1$, for all $i \in \mathbb{N}_n$ is determined by previous formulae or by their generating functions $A(z)$ and $C(z)$:

n	0	1	2	3	4	5	6	7	8	9	10	...
a_n	1	1	1	1	2	4	7	11	17	27	44	...
c_n	0	0	1	2	3	4	6	10	17	28	45	...

These sequences are A005252 and A024490 at [21]. ■

Example 3.2. We find the number of even (odd) permutations of the set $\mathbb{N}_n$, satisfying the condition $p(i) - i \in \{-2, 0, 2\}$.

In this case we have $k = r = 2$, i.e. $k + r + 1 = 5$, set $I = \{-1, 1\}$, which implies $(r + 1 - I) = \{2, 4\}$ and $\mathcal{C} = \{345, 245, 235, 145, 135, 125\}$, which is separated into sets:

$$\mathcal{C}_1 = \{145, 135, 125\}, \qquad \mathcal{C}_2^0 = \emptyset, \qquad \mathcal{C}_2^1 = \{345, 235\}, \qquad \mathcal{C}_2^2 = \{245\}.$$

$$\begin{array}{ll}
\varphi_2^1(345,0) = \{(345,0),(235,0)\}, & \varphi_2^1(345,1) = \{(345,1),(235,1)\}, \\
\varphi_2^2(245,0) = \{(135,0)\}, & \varphi_2^2(245,1) = \{(135,1)\}, \\
\varphi_2^1(235,0) = \{(145,1),(125,0)\}, & \varphi_2^1(235,1) = \{(145,0),(125,1)\} \\
\varphi_1(145,0) = \{(345,0)\}, & \varphi_1(145,1) = \{(345,1)\}, \\
\varphi_1(135,0) = \{(245,0)\}, & \varphi_1(135,1) = \{(245,1)\}, \\
\varphi_1(125,0) = \{(145,0)\}, & \varphi_1(125,1) = \{(145,1)\}.
\end{array}$$

If we substitute $a_{(345,0)}(n) = a_n$, $a_{(245,0)}(n) = b_n$, $a_{(235,0)}(n) = c_n$, $a_{(145,0)}(n) = d_n$, $a_{(135,0)}(n) = e_n$, $a_{(125,0)}(n) = f_n$, $a_{(345,1)}(n) = g_n$, $a_{(245,1)}(n) = h_n$, $a_{(235,1)}(n) = i_n$, $a_{(145,1)}(n) = j_n$, $a_{(135,1)}(n) = k_n$ and $a_{(125,1)}(n) = \ell_n$ we get the system of linear recurrence equations:

$$\begin{array}{ll}
a_{n+1} = a_n + c_n, & g_{n+1} = g_n + i_n, \\
b_{n+1} = e_n, & h_{n+1} = k_n, \\
c_{n+1} = j_n + f_n, & i_{n+1} = d_n + \ell_n \\
d_{n+1} = a_n, & j_{n+1} = g_n, \\
e_{n+1} = b_n, & k_{n+1} = h_n, \\
f_{n+1} = d_n, & \ell_{n+1} = j_n,
\end{array}$$

with the initial conditions $a_0 = 1$ and $b_0 = c_0 = \cdots = \ell_0 = 0$.

From this system we find the generating functions:

$A(z) = \frac{1-z-z^4}{1-2z+z^2-2z^4+2z^5-z^6+z^8}$ and $G(z) = \frac{z^3}{1-2z+z^2-2z^4+2z^5-z^6+z^8}$.

From the denominator of these generating functions

$$1 - 2z + z^2 - 2z^4 + 2z^5 - z^6 + z^8 = (1-z)(1+z)(1+z^2)(1-z+z^2)(1-z-z^2),$$

we find the linear recurrence equation $a_n = 2a_{n-1} - a_{n-2} + 2a_{n-4} - 2a_{n-5} + a_{n-6} - a_{n-8}$ and same for g_n.

When we solve this equation we find the general terms of these sequences:

$$a_n = \frac{1}{10}\left(L_{n+2} + y_n + z_n\right), \qquad g_n = \frac{1}{10}\left(L_{n+2} + y_n - z_n\right),$$

where L_n denotes n-th Lucas number ($L_1 = 1$, $L_2 = 3$, $L_{n+1} = L_n + L_{n-1}$; A000032 and A000204 at [21]), $y_n = 2\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}$ and $z_n = 5$, if n is congruent to 0, 1 or 2 modulo 6, and $z_n = 0$, if n is congruent to 3, 4 or 5 modulo 6.

The number of even permutations, a_n, and odd permutations, g_n, both satisfying the conditions $|p(i) - i| \leq 2$ and $p(i) - i \neq -1, 1$ is determined by previous formulae or by their generating functions $A(z)$ and $G(z)$:

n	0	1	2	3	4	5	6	7	8	9	10	...
a_n	1	1	1	1	2	3	5	8	13	20	32	...
g_n	0	0	0	1	2	3	4	7	12	20	32	...

■

4 Concluding remarks

We have developed a technique for generating a system of linear recurrence equations that enumerate the even and the odd strongly restricted permutations. In some cases, using the digraph corresponding to the matrix of the system we can establish a connection between restricted permutations and restricted compositions. Using a program that implements this technique, we have contributed 96 sequences, A241975–A242070, to the Online encyclopedia of integer sequences [21].

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