ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 8 (2015) 297–318

Spherical tilings by congruent quadrangles: Forbidden cases and substructures

Yohji Akama *

Mathematical Institue, Graduate School of Science,
Tohoku University, Sendai, Miyagi 980-0845, Japan

Nico Van Cleemput †

NTIS - New Technologies for the Information Society,
Faculty of Applied Sciences, University of West Bohemia,
Univerzitní 22, 306 14 Plzeň

Received 10 December 2013, accepted 18 April 2014, published online 6 February 2015

Abstract

In this article we show the non-existence of a class of spherical tilings by congruent quadrangles. We also prove several forbidden substructures for spherical tilings by congruent quadrangles. These are results that will help to complete of the classification of spherical tilings by congruent quadrangles.

Keywords: Spherical tiling by congruent quadrangles, monohedral tiling, quadrangulation.

Math. Subj. Class.: 05B45, 05C10, 51M20, 52C20

1 Introduction

In this paper we prove the non-existence of a subclass of spherical tilings by congruent quadrangles which have three equal sides and one side different. We also list several forbidden substructures for this type of spherical tilings.

It follows from Euler's formula that spherical tilings by congruent polygons can only exist for triangles, quadrangles and pentagons.

* http://www.math.tohoku.ac.jp/akama/stcq/

† Corresponding author, This action is realized by the project NEXLIZ – CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

E-mail addresses: akama@m.tohoku.ac.jp (Yohji Akama), nico.vancleemput@gmail.com (Nico Van Cleemput)

In [5], Davies completed the classification of spherical tilings by congruent triangles. He however only gave an outline of the proof and his classification contained several duplicates. Ueno and Agaoka [9] worked out the details of the proof, thus completely solving the classification of spherical tilings by congruent triangles.

Ueno and Agaoka [8] gave several examples of spherical tilings by congruent quadrangles and showed that the classification of these would be considerably harder than the classification of spherical tilings by congruent triangles. Akama and Sakano [7] completed the classification of spherical tilings by congruent kites, darts and rhombi. Since these quadrangles can be subdivided into congruent triangles, they could rely on the classification by Ueno and Agaoka to solve this classification.

The spherical quadrangles can be subdivided into classes based on the cyclic list of edge lengths. Only four of these classes admit a spherical tiling by congruent quadrangles[8]:

1. $aaaa$
2. $aaab$
3. $aabb$
4. $aabc$

The cases handled by Akama and Sakano cover type 1 and type 3.

The remaining two cases are spherical tilings by congruent quadrangles which have three equal sides and one side different (type 2), and spherical tilings by congruent quadrangles which have three different sides and an adjacent pair of sides of the same length (type 4). Akama, Nakumara and Sakano [1, 2, 7] showed that if concave quadrangles are allowed, there exist several tilings which have non-congruent tiles but for which the inner angles and the underlying graph are the same.

In this paper we focus on convex quadrangles of type 2. We show that there exists no spherical tiling by congruent quadrangles of type 2 if the quadrangles are isosceles. Furthermore we show several forbidden substructures for the underlying graph of spherical tilings by congruent quadrangles of type 2.

This paper is organised as follows. We start by giving some general definitions and notations. Then we show the non-existence of spherical tilings by congruent quadrangles of type 2 with isosceles quadrangles. Next we look at the different possible configurations of angles around each vertex and finally we use this to show some forbidden substructures for the underlying graph.

2 Definitions

To simplify the notation we will always express angles in π radians.

A spherical tiling is a subdivision of the unit sphere into spherical polygons. Edges are always assumed to be parts of great circles. All tilings are edge-to-edge tilings.

A spherical quadrangle is of **type 2** if the cyclic list of edge-lengths is $aaab$ (with $a \neq b$). We use the naming convention shown in Figure 1. We only consider convex spherical quadrangles. This means that we always assume

$$0 < \alpha, \beta, \gamma, \delta < 1.$$

Throughout this paper G will always refer to a 2-connected, simple graph on the 2-sphere in which all faces are quadrangles. Let $A(G)$ be the set of ordered pairs (f, v) such that f is a face of G, v is a vertex of G and $v \in f$. A **chart** (G, ϕ), is an ordered pair consisting of a graph G and a function $\phi : A(G) \to \{\alpha, \beta, \gamma, \delta\}$ such that for each face

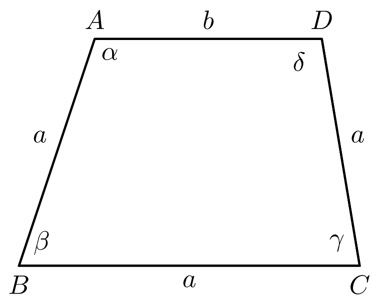

Figure 1: Naming conventions in a spherical quadrangle of type 2.

of the graph, the cyclic list of the inner angles is $(\alpha, \beta, \gamma, \delta)$ or the reverse. These four parameters, $\alpha, \beta, \gamma, \delta$, will take on the role of angles of tiles, so a chart can be seen as a combinatorial spherical tiling by congruent quadrangles. We say a vertex of the tiling has **vertex type** $n_1\alpha + n_2\beta + n_3\gamma + n_4\delta$, if there are n_1 pairs containing v that are mapped to α, n_2 pairs containing v that are mapped to β, n_3 pairs containing v that are mapped to γ, and n_4 pairs containing v that are mapped to δ.

It is clear how a chart can be obtained from a spherical tiling by congruent quadrangles. Vice versa, a chart (G, f) is **solvable**, if there exist values for the four angles such that there is a spherical tiling realising that graph and those values.

There are several conditions that need to be satisfied in order for a spherical tiling by congruent quadrangles to exist. If F is the number of tiles, then the following condition follows from the fact that the area of the tiles need to sum up to the area of the sphere.

$$\alpha + \beta + \gamma + \delta - 2 = \frac{4}{F} \tag{2.1}$$

Lemma 2.1. *In a convex spherical quadrangle of type 2, we have that*

$$\alpha + \delta < 1 + \beta, \tag{2.2}$$

$$\alpha + \beta < 1 + \delta, \tag{2.3}$$

$$\alpha + \delta < 1 + \gamma, \tag{2.4}$$

$$\gamma + \delta < 1 + \alpha. \tag{2.5}$$

Proof. Draw the diagonal as is shown in Figure 2. The area of the triangle ABD is given by

$$\alpha + \beta_1 + \delta_1 - 1.$$

The area of the spherical lune that is formed by the great circles AB and BD is given by $2\beta_1$. Since the area of the triangle is smaller than that of the lune, we have that

$$\alpha + \beta_1 + \delta_1 - 1 < 2\beta_1$$

which can be rewritten as

$$\alpha + \delta_1 < 1 + \beta_1 \tag{2.6}$$

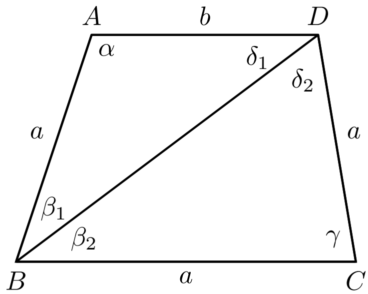

Figure 2: A diagonal in a spherical quadrangle of type 2.

The triangle BDC is an isosceles triangle. This implies that

$$\delta_2 = \beta_2 \tag{2.7}$$

If we combine inequality 2.6 and equation 2.7, we find equation 2.2. Equation 2.3 can be proven using the spherical lune formed by AD and BD. Equation 2.4 and equation 2.5 can be proven by using the other diagonal. □

Lemma 2.2. *In a convex spherical quadrangle of type 2, we have that*

$$\alpha \neq \gamma \tag{2.8}$$

and

$$\delta \neq \beta. \tag{2.9}$$

Proof. Assume that $\delta = \beta$. Draw the diagonal as is shown in Figure 2. The triangle BDC is an isosceles triangle, so we have that $\delta_2 = \beta_2$. This implies that $\delta_1 = \beta_1$, so ABD is an isosceles triangle and $a = b$. This is however a contradiction, so we find that $\delta \neq \beta$. By using the other diagonal, we can prove that $\alpha \neq \gamma$. □

Lemma 2.3. *In a convex spherical quadrangle of type 2, we have that*

$$\alpha = \delta \Leftrightarrow \beta = \gamma. \tag{2.10}$$

Proof. Assume that $\alpha = \delta$. The great circles AB and DC in Figure 1 intersect in two points, N and S. Since $\alpha = \delta$, the triangle ADN is an isosceles triangle, but since the distance from A to B and from D to C is a, then also the triangle BCN is an isosceles triangle and $\beta = \gamma$. The other direction is completely analogous. □

Lemma 2.4. *Let (G, f) be a solvable chart. Let v be a vertex of G with vertex type $n_1\alpha + n_2\beta + n_3\gamma + n_4\delta$ in (G, f), then $n_1 + n_4$ is even.*

Proof. This follows immediately from the fact that each edge of length b that is incident to v contributes exactly two to $n_1 + n_4$ and each angle α and δ at the vertex v corresponds to exactly one edge of length b incident to v. □

The following lemma can easily be proved using Euler's formula.

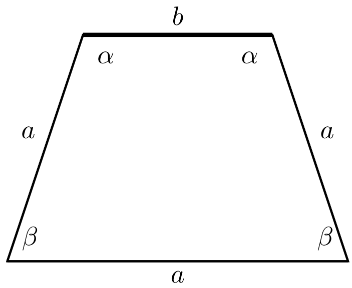

Figure 3: An isosceles quadrangle of type 2

Lemma 2.5. *Let G be a quadrangulation of the sphere. Let V_i (with $3 \leq i \leq \Delta$, where Δ is the largest degree of the G) be the number of vertices in G with degree i, then we have the following equality:*

$$V_3 = 8 + \sum_{i=5}^{\Delta} (i-4)V_i.$$

3 Spherical tilings by congruent isosceles quadrangles of type 2

An isosceles spherical quadrangle of type 2 is a convex spherical quadrangle having the cyclic list of edge-lengths $aaab$ (with $a \neq b$) and in which $\alpha = \delta$ and $\beta = \gamma$. Therefore the cyclic list of the inner angles in a isosceles quadrangle is $(\alpha, \beta, \beta, \alpha)$. An example of such a quadrangle is given in Figure 3.

We can rewrite several of the conditions for general spherical quadrangles of type 2. Equation 2.1 can be rewritten as

$$2\alpha + 2\beta - 2 = \frac{4}{F}. \tag{3.1}$$

The corresponding lemma for Lemma 2.1 is

Lemma 3.1. *In an isosceles spherical quadrangle of type 2, we have that*

$$2\alpha < 1 + \beta. \tag{3.2}$$

The corresponding lemma for Lemma 2.2 is

Lemma 3.2. *In an isosceles spherical quadrangle of type 2, we have that*

$$\alpha \neq \beta \tag{3.3}$$

We now have the tools to prove the main theorem of this section.

Theorem 3.3. *There is no isosceles spherical tiling by congruent quadrangles of type 2.*

Proof. From Lemma 2.5, we know that each quadrangulation contains at least 8 vertices of degree 3. The possible vertex types for a vertex of degree 3 in a spherical tiling by congruent isosceles spherical quadrangles of type 2 are $2\alpha+\beta$ and 3β. There is no isosceles spherical tiling by congruent quadrangles of type 2 with two vertices of degree 3 with a different vertex type, because in that case we would have $\alpha = \beta$, which does not correspond to a quadrangle of type 2 (cf. Lemma 3.2). So all vertices of degree 3 have the same type.

We will examine both possible vertex types.

vertex type $2\alpha + \beta$

We first assume that all vertices of degree 3 have vertex type $2\alpha + \beta$.

As a consequence all vertices of degree $d > 3$ have vertex type $d\beta$ or $d\alpha$. Otherwise there would be a vertex of degree $d > 3$ with vertex type

$$2i\alpha + (d - 2i)\beta$$

with $0 < i < \left\lfloor \frac{d}{2} \right\rfloor$. If we combine this with the vertex type for the vertices of degree 3, then we find that

$$(2i - 2)\alpha + (d - 2i - 1)\beta = 0.$$

Since $\alpha > 0$ and $\beta > 0$, this is equivalent with

$$\begin{cases} 2i - 2 = 0 \\ d - 2i - 1 = 0 \end{cases}$$

But since $d > 3$, this has no solution.

It is also not the case that all vertices are of degree 3, since that would mean that there are more α's than β's.

This means that there are only a limited number of possibilities for different degrees in this situation:

- the quadrangulation has two types of vertices: vertices of degree 3 with vertex type $2\alpha + \beta$ and vertices of degree $d > 3$ with vertex type $d\beta$, or
- the quadrangulation has three types of vertices: vertices of degree 3 with vertex type $2\alpha + \beta$, vertices of degree $d > 3$ with vertex type $d\beta$, and vertices of even degree $d_e > 3$ with vertex type $d_e\alpha$ (d_e is even due to Lemma 2.4).

Assume first that there are only vertices of degree 3 with vertex type $2\alpha + \beta$ and vertices of degree $d > 3$ with vertex type $d\beta$. In this case we get two equations:

$$\begin{cases} 2\alpha + \beta = 2 \\ d\beta = 2 \end{cases}$$

This is equivalent to

$$\begin{cases} \alpha = 1 - \frac{1}{d} \\ \beta = \frac{2}{d} \end{cases}$$

If we substitute these values for α and β in inequality 3.2, we find

$$2 - \frac{2}{d} < 1 + \frac{2}{d}.$$

This is equivalent to

$$d < 4,$$

which contradicts $d > 3$.

Next we assume that there are only vertices of degree 3 with vertex type $2\alpha + \beta$, vertices of degree $d > 3$ with vertex type $d\beta$, and vertices of even degree $d_e > 3$ with vertex type $d_e\alpha$. In this case we get three equations:

$$\begin{cases} 2\alpha + \beta = 2 \\ d\beta = 2 \\ d_e\alpha = 2 \end{cases}$$

This is equivalent to

$$\begin{cases} \frac{4}{d_e} + \frac{2}{d} = 2 \\ \beta = \frac{2}{d} \\ \alpha = \frac{2}{d_e} \end{cases}$$

The first equation has no solution, since $d_e \geq 4$ and $d > 3$.

vertex type 3β

Next we assume that all vertices of degree 3 have vertex type 3β. This means that $\beta = \frac{2}{3}$ and from equation 3.1, we then find that

$$\alpha = \frac{1}{3} + \frac{2}{F} = \frac{F+6}{3F}. \tag{3.4}$$

As 3β is equal to 2, any vertex type that contains a β, has at most 2β. Since there has to be at least one vertex for which the vertex type contains an α, there is a vertex of degree d with one of the following three types:

- $d\alpha$,
- $(d-1)\alpha + \beta$,
- $(d-2)\alpha + 2\beta$.

We examine the three possibilities:

$d\alpha$

Combined with equation 3.4, this gives us

$$d\frac{F+6}{3F} = 2$$

which can be rewritten as

$$6d = (6-d)F.$$

Since d and F are both positive integers, and d is even and larger than 3, we find that this only holds if $d = 4$ and $F = 12$.

$(d-1)\alpha + \beta$

Combined with equation 3.4, this gives us

$$(d-1)\frac{F+6}{3F} = 2 - \frac{2}{3} = \frac{4}{3}$$

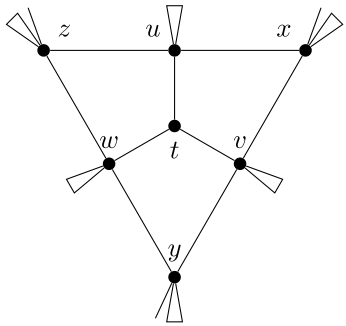

Figure 4: The quadrangles around a vertex t of degree 3.

which can be rewritten as

$$6(d-1) = (5-d)F.$$

Since d and F are both positive integers, and d is odd and larger than 3, we find that this never holds.

$(d-2)\alpha + 2\beta$
Combined with equation 3.4, this gives us

$$(d-2)\frac{F+6}{3F} = 2 - \frac{4}{3} = \frac{2}{3}$$

which can be rewritten as

$$6(d-2) = (4-d)F.$$

Since d and F are both positive integers, and d is even and larger than 3, we find that this never holds.

So the only possibility is a quadrangulation which 12 faces. Such a quadrangulation has 14 vertices, of which at least 8 have degree 3 and vertex type 3β. This already accounts for all of the 24β's, so all remaining 6 vertices have degree 4 and vertex type 4α.

Assume we have a vertex t of degree 3 as is shown in Figure 4. This vertex has vertex type 3β. This means that, in the quadrangle $tuzw$, the angle at vertex t is β and either the angle at the vertex w or the angle at the vertex u is α. Without loss of generality, we can assume that the angle at the vertex u is α. This implies that the vertex type of u is 4α, and we find that this means that the vertex type of both the vertices w and v is 3β. But then the quadrangle $twyv$ has three consecutive angles β. This is a contradiction, so there is no spherical tiling by congruent isosceles quadrangles of type 2 with vertex types 3β and 4α.

This proves that there is no spherical tiling by congruent isosceles spherical quadrangles of type 2. $\square$

	1	2	3	4	5	6	7	8	9	10
1		4.1a				4.1a	4.1a		4.1b	
2	4.1a					4.1a	4.1a	4.1c		4.1b
3					4.1a		4.1c	4.1a	4.1a	
4					4.1a	4.1b		4.1a		4.1a
5			4.1a	4.1a			4.1b		4.1a	
6	4.1a	4.1a		4.1b			4.1a			
7	4.1a	4.1a	4.1c		4.1b	4.1a				
8		4.1c	4.1a	4.1a						4.1a
9	4.1b		4.1a		4.1a					4.1a
10		4.1b		4.1a				4.1a	4.1a	

Table 1: Overview of the combinations of two vertex types for vertices of degree 3. For each impossible combination of vertex type, the corresponding case is given.

4 Vertex types in spherical tilings by arbitrary congruent quadrangles of type 2

Since there are at least 8 vertices of degree 3 (and in most cases even more), it can be interesting to look at the possible vertex types for these vertices, and examine whether certain combination are not possible. Owing to Lemma 2.4, there are ten possible vertex types for vertices of degree 3 in a spherical tiling by congruent quadrangles of type 2:

1) 3β
2) $2\beta+\gamma$
3) $\alpha+\delta+\beta$
4) $2\alpha+\gamma$
5) $2\alpha+\beta$
6) 3γ
7) $2\gamma+\beta$
8) $\alpha+\delta+\gamma$
9) $2\delta+\beta$
10) $2\delta+\gamma$

The last five of these types can be obtained from the first five by interchanging α with δ, and β with γ.

The following lemma shows that several combinations of vertex types for vertices of degree 3 are not possible in a spherical tiling by congruent quadrangles of type 2. Table 1 gives an overview of all combinations.

Lemma 4.1. *There is no spherical tiling by congruent quadrangles of type 2 which has any of the following combinations of vertex types:*

a) 3β *and* $2\beta+\gamma$, 3β *and* 3γ, 3β *and* $\beta+2\gamma$, $2\beta+\gamma$ *and* 3γ, $2\beta+\gamma$ *and* $\beta+2\gamma$, $\alpha+\delta+\beta$ *and* $2\alpha+\beta$, $\alpha+\delta+\beta$ *and* $\alpha+\delta+\gamma$, $\alpha+\delta+\beta$ *and* $2\delta+\beta$, $2\alpha+\gamma$ *and* $2\alpha+\beta$, $2\alpha+\gamma$ *and* $\alpha+\delta+\gamma$, $2\alpha+\gamma$ *and* $2\delta+\gamma$, $2\alpha+\beta$ *and* $2\delta+\beta$*;*

b) 3β *and* $2\delta+\beta$, $2\beta+\gamma$ *and* $2\delta+\gamma$, $2\alpha+\gamma$ *and* 3γ, $2\alpha+\beta$ *and* $2\gamma+\beta$*;*

c) $2\beta+\gamma$ *and* $\alpha+\delta+\gamma$, $2\gamma+\beta$ *and* $\alpha+\delta+\beta$.

Proof.

a) Each of these combinations either implies that $\alpha = \delta$, or that $\beta = \gamma$. This means that the quadrangle is a isosceles quadrangle of type 2. Due to Theorem 3.3, there are no spherical tilings by congruent quadrangles with such a tile.

b) The first two combinations imply that $\beta = \delta$, but this contradicts inequality 2.9. The last two combinations imply that $\alpha = \gamma$, but this contradicts inequality 2.8.

c) We will only give the proof for $2\gamma + \beta$ and $\alpha + \delta + \beta$. The other case can be obtained by interchanging α with δ, and β with γ.

   When we combine
$$\alpha + \delta + \beta = 2$$
   with equation 2.1, we get
$$\gamma = \frac{4}{F}.$$
   When we combine this with
$$2\gamma + \beta = 2,$$
   we get
$$\beta = 2 - \frac{8}{F}.$$
   Since $\beta < 1$, this implies that $F < 8$. However, if $F = 6$, we have that $\beta = \gamma$. This is not possible due to Lemma 2.3 and Theorem 3.3. So we find that this combination is not possible.

□

Lemma 4.2. *In each spherical tiling by congruent quadrangles of type 2 we have the restrictions on the number of faces that are given in Table 2.*

Proof. We will examine case by case. First we note that for all quadrangulations, we have that $F \geq 6$, so $F \neq 6$ is equivalent to $F > 6$.

- **Vertex type 1 and vertex type 3**
  In this case we have the following system of equations:
$$\begin{cases} 3\beta = 2 \\ \alpha + \beta + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$
  The last equation in this system corresponds to equation 2.1. If we subtract the second equation from this last equation, we find that
$$\gamma = \frac{4}{F}.$$
  Owing to Lemma 2.3 and Theorem 3.3, we have that $F \neq 6$, because otherwise $\beta = \gamma$. By combining the first two equations in the system, we find that
$$\alpha + \delta = \frac{4}{3}.$$

	1	2	3	4	5	6	7	8	9	10
1			$6 < F < 12$					$F = 6$		$6 < F$
2			$6 < F$							
3	$6 < F < 12$	$6 < F$		$6 < F$		$F = 6$				
4			$6 < F$						$F = 6$	
5						$6 < F$				$F = 6$
6			$F = 6$		$6 < F$			$6 < F < 12$		
7								$6 < F$		
8	$F = 6$					$6 < F < 12$	$6 < F$		$6 < F$	
9				$F = 6$				$6 < F$		
10	$6 < F$				$F = 6$					

Table 2: Combinations of two vertex types for vertices of degree 3. For each combination of vertex type with known restrictions on the number of faces, that restriction is given. The red cells are the impossible combinations which where already given in Table 1.

When we substitute these previous two equalities into inequality 2.4, we find that

$$\frac{4}{3} = \alpha + \delta < 1 + \gamma = 1 + \frac{4}{F},$$

which can be rewritten as

$$F < 12.$$

- **Vertex type 1 and vertex type 8**
In this case we have the following system of equations:

$$\begin{cases} 3\beta = 2 \\ \alpha + \gamma + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

By combining the last two equations, we find that $\beta = \frac{4}{F}$, but together with the first equation of the system, this implies that $F = 6$.

- **Vertex type 1 and vertex type 10**
In this case we have the following system of equations:

$$\begin{cases} 3\beta = 2 \\ \gamma + 2\delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

By substituting the first two equations in the third, we get:

$$\alpha - \delta = \frac{4}{F} - \frac{2}{3}.$$

In combination with Lemma 2.3 and Theorem 3.3, this implies that $F \neq 6$.

- **Vertex type 2 and vertex type 3**
In this case we have the following system of equations:

$$\begin{cases} 2\beta + \gamma = 2 \\ \alpha + \beta + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

The last two equations give us that

$$\gamma = \frac{4}{F},$$

and using the first equation from the system, this then implies that

$$\beta = 1 - \frac{2}{F}.$$

In combination with Lemma 2.3 and Theorem 3.3, these last two equations imply that $F \neq 6$.

- **Vertex type 2 and vertex type 8**
  In this case we have the following system of equations:
$$\begin{cases} 2\beta + \gamma = 2 \\ \alpha + \gamma + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$
  The last two equations give us that
$$\beta = \frac{4}{F},$$
  and using the first equation from the system, this then implies that
$$\gamma = 2 - \frac{8}{F}.$$
  In combination with Lemma 2.3 and Theorem 3.3, these last two equations imply that $F \neq 6$.

- **Vertex type 3 and vertex type 4**
  In this case we have the following system of equations:
$$\begin{cases} 2\alpha + \gamma = 2 \\ \alpha + \beta + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$
  Once again, the last two equations give us that
$$\gamma = \frac{4}{F},$$
  and using the first equation from the system, this then implies that
$$\alpha = 1 - \frac{2}{F}.$$
  In combination with inequality 2.8, these last two equations imply that $F \neq 6$.

- **Vertex type 4 and vertex type 9**
  In this case we have the following system of equations:
$$\begin{cases} 2\alpha + \gamma = 2 \\ 2\delta + \beta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$
  This is equivalent to the following system:
$$\begin{cases} \gamma = 2 - 2\alpha \\ \beta = 2 - 2\delta \\ \alpha + \delta = 2 - \frac{4}{F} \end{cases}$$
  By combining the last equation in this system with inequalities 2.2 and 2.4, we find that
$$\beta > 1 - \frac{4}{F},$$

and

$$\gamma > 1 - \frac{4}{F}.$$

By combining these inequalities with the first two equations in the system, we find that

$$\alpha < \frac{1}{2} + \frac{2}{F},$$

and

$$\delta < \frac{1}{2} + \frac{2}{F}.$$

If we add up these two inequalities, we get

$$\alpha + \delta < 1 + \frac{4}{F}.$$

If we then combine this last inequality with the last equation of the system, we find the following inequality:

$$2 - \frac{4}{F} < 1 + \frac{4}{F},$$

which is equivalent to

$$F < 8.$$

- **Vertex type 5 and vertex type 10**
  In this case we have the following system of equations:

$$\begin{cases} 2\alpha + \beta = 2 \\ 2\delta + \gamma = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

  This is equivalent to the following system:

$$\begin{cases} \beta = 2 - 2\alpha \\ \gamma = 2 - 2\delta \\ \alpha + \delta = 2 - \frac{4}{F} \end{cases}$$

  Similar to the previous case, we find that

$$\alpha + \delta < 1 + \frac{4}{F}.$$

  Together with the last equation of the system, this implies

$$F < 8.$$

The remaining cases are equivalent to one of these cases by interchanging α with δ and β with γ. □

A question that pops up naturally at this point is which combinations of three vertex types for vertices of degree 3 are possible. There are 12 combinations of three vertex types for vertices of degree 3 which we can not exclude at this point. The remaining combinations can be excluded because they contain one of the combinations of two vertex types for vertices of degree 3 that are not allowed by Table 1. The 12 combinations come in pairs, since interchanging α with δ, and β with γ gives a different combination with the same properties. From these 12 combinations we can also discard combinations (1,5,10) and (5,6,10), since (1,10), resp. (6,5), implies that $6 < F$, and (5,10) implies that $6 = F$. The remaining 10 combinations are

- (1,3,4) and (6,8,9);
- (1,3,10) and (5,6,8);
- (2,3,4) and (7,8,9);
- (1,5,8) and (3,6,10);
- (2,4,9) and (4,7,9).

Lemma 4.3. *There is no spherical tiling by congruent quadrangles of type 2 on more than 8 vertices that contains 3 vertices of degree 3 with pairwise different vertex types.*

Proof. We need to examine the remaining 5 cases stated above.

- (1,3,4): In this case we have the following system of equations

$$\begin{cases} 3\beta = 2 \\ \alpha+\beta+\delta = 2 \\ 2\alpha+\gamma = 2 \\ \alpha+\beta+\gamma+\delta = 2+\frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = 1-\frac{4}{F}\beta = \frac{2}{3} \\ \gamma = \frac{8}{F} \\ \delta = \frac{1}{3}+\frac{4}{F} \end{cases}$$

If we combine this with inequality 2.4, we get

$$\frac{4}{3} < \frac{F+8}{F}$$

which is equivalent to

$$F < 8.$$

This is a contradiction because the combination (1,3) implies that $6 < F$.

- (1,3,10): In this case we have the following system of equations

$$\begin{cases} 3\beta = 2 \\ \alpha+\beta+\delta = 2 \\ 2\delta+\gamma = 2 \\ \alpha+\beta+\gamma+\delta = 2+\frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = \frac{1}{3}+\frac{4}{F}\beta = \frac{2}{3} \\ \gamma = \frac{8}{F} \\ \delta = 1-\frac{4}{F} \end{cases}$$

If we combine this with inequality 2.4, we get

$$\frac{4}{3} < \frac{F+8}{F}$$

which is equivalent to

$$F < 8.$$

This is a contradiction because the combination (1,3) implies that $6 < F$.

- (2,3,4): In this case we have the following system of equations

$$\begin{cases} 2\beta + \gamma = 2 \\ \alpha + \beta + \delta = 2 \\ 2\alpha + \gamma = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = \beta = 1 - \frac{2}{F} \\ \gamma = \delta = 1 + \frac{2}{F} \end{cases}$$

This is a contradiction with inequality 2.5.

- (1,5,8): In this case we have the following system of equations

$$\begin{cases} 3\beta = 2 \\ 2\alpha + \beta = 2 \\ \alpha + \gamma + \delta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = \beta = \frac{2}{3} \\ \delta = \frac{4}{3} - \gamma \\ F = 6 \end{cases}$$

So we find that a quadrangulation which has this combination, has 8 vertices.

- (2,4,9): In this case we have the following system of equations

$$\begin{cases} 2\beta + \gamma = 2 \\ 2\alpha + \gamma = 2 \\ 2\delta + \beta = 2 \\ \alpha + \beta + \gamma + \delta = 2 + \frac{4}{F} \end{cases}$$

which is equivalent to

$$\begin{cases} \alpha = 1 - \frac{2}{F} \\ \beta = \alpha \\ \gamma = 2 - 2\alpha \\ \delta = 2 - 2\alpha \end{cases}$$

If we combine this system with inequality 2.3, we get

$$2 - \frac{4}{F} < 1 + \frac{4}{F},$$

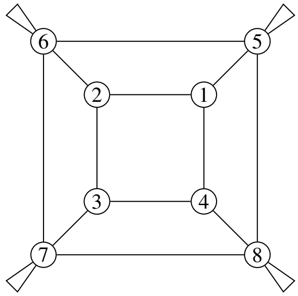

Figure 5: An example of a cubic quadrangle

which is equivalent to

$$F < 8.$$

So we find that a quadrangulation which has this combination, has $F = 6$, which implies that it has 8 vertices.

□

Theorem 4.4. *In a spherical tiling by congruent quadrangles of type 2 there are at most two different vertex types for cubic vertices.*

Proof. An enumeration of all possible angle assignments for the cube shows that, up to equivalence, only one angle assignment admits a spherical tiling by congruent quadrangles of type 2, and this angle assignment has two vertex types: 3β and $\alpha + \gamma + \delta$. Together with Lemma 4.3 this proves the theorem. □

5 Forbidden substructures in spherical tilings by arbitrary congruent quadrangles of type 2

5.1 Cubic quadrangles

A **cubic quadrangle** in a quadrangulation is a quadrangle such that all four vertices have degree 3. Figure 5 shows an example of a cubic quadrangle. In Table 3 an overview of the number of quadrangulations which contain a cubic quadrangle is given. Note that the percentage of quadrangulations which contain a cubic quadrangle increases as the size of the quadrangulations increases.

We prove the following theorem.

Theorem 5.1. *A quadrangulation on more than 8 vertices that contains a cubic quadrangle does not admit a realisation as a spherical tiling by congruent quadrangles of type 2.*

Proof. There are two ways of assigning the edges of length a and b to the edges of the cubic quadrangle and its neighbouring faces. These two ways are shown in Figure 6. If we take the complete quadrangulation into account, then these two ways will of course be realised in different ways, but this is not important for this proof.

In both edge assignments there is at least one cubic vertex that is incident to an edge of length b, and one cubic vertex that is not. Owing to Theorem 4.4, all cubic vertices

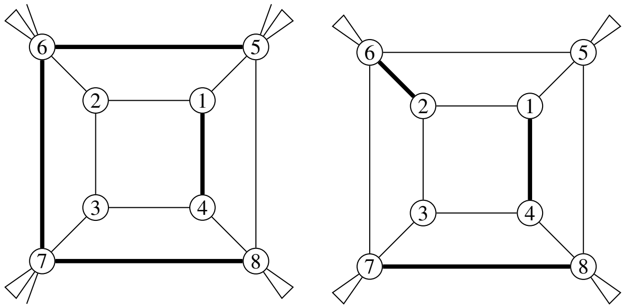

Figure 6: Possible edge assignments for a cubic quadrangle and its neighbouring faces. The bold edge corresponds to an edge of length b.

incident to an edge of length b have the same type, and all cubic vertices not incident to an edge of length b have the same type. This means that we can already fix some angle assignments for both cases in Figure 6. This partial angle assignment is shown in Figure 7. The angles in the cubic quadrangle can be fixed, since interchanging α with δ and β with γ gives the same results. The angles in the face that shares an edge of length b with the cubic quadrangle can be fixed, since the other possible assignment implies that there are two different vertex types for cubic vertices incident to an edge of length b: one containing 2α and one containing 2δ.

We first consider the edge assignment on the left side in Figure 6. The angle assignment for the quadrangle 1562 fixes all remaining angle assignments for the faces neighbouring the cubic quadrangle: either the vertex type of 1 and 4 is $\alpha + \delta + \beta$ and the vertex type of 2 and 3 is $2\gamma + \beta$, or the vertex type of 1 and 4 is $\alpha + \delta + \gamma$ and the vertex type of 2 and 3 is $2\beta + \gamma$. Owing to Lemma 4.1, these combinations are not possible, so this edge assignment is not possible.

Next we consider the edge assignment on the right side in Figure 6. The angle assignment for the quadrangle 1562 fixes all remaining angle assignments for the faces neighbouring the cubic quadrangle: either the vertex type of 1, 2 and 4 is $\alpha + \delta + \beta$ and the vertex type of 3 is 3γ, or the vertex type of 1, 2 and 4 is $\alpha + \delta + \gamma$ and the vertex type of 3 is 3β. Owing to Lemma 4.2, this implies that the quadrangulation has 6 faces, and thus 8 vertices. □

5.2 Cubic tristars

A **cubic tristar** in a quadrangulation is a cubic vertex v such that all three neighbouring vertices have degree 3. The vertex v is called the **central vertex** of the cubic tristar. Figure 8 shows an example of a cubic tristar in a quadrangulation.

Theorem 5.2. *In a spherical tiling by congruent quadrangles of type 2, there is no cubic tristar for which the central vertex is incident to an edge of length b.*

Proof. We use the vertex labels as given in Figure 8. Assume that the edge 12 has length b. This implies that either edge 36 or edge 46 has length b. Both cases are completely analogous, so we will assume without loss of generality that edge 36 has length b.

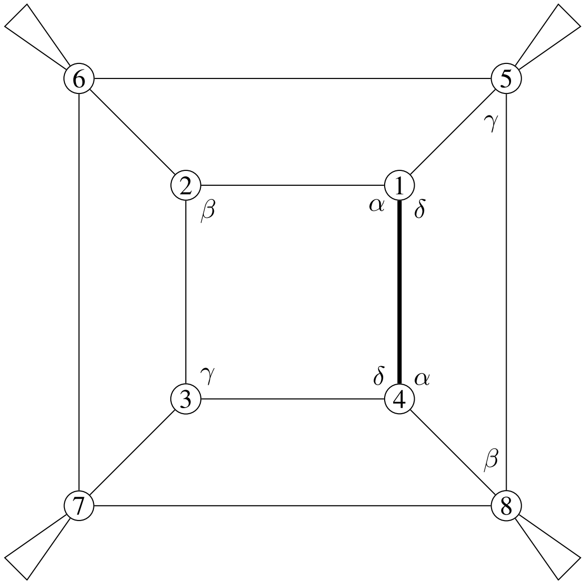

Figure 7: Partial angle assignment for a cubic quadrangle.

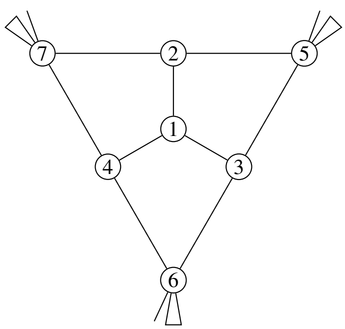

Figure 8: An example of a cubic tristar in a quadrangulation.

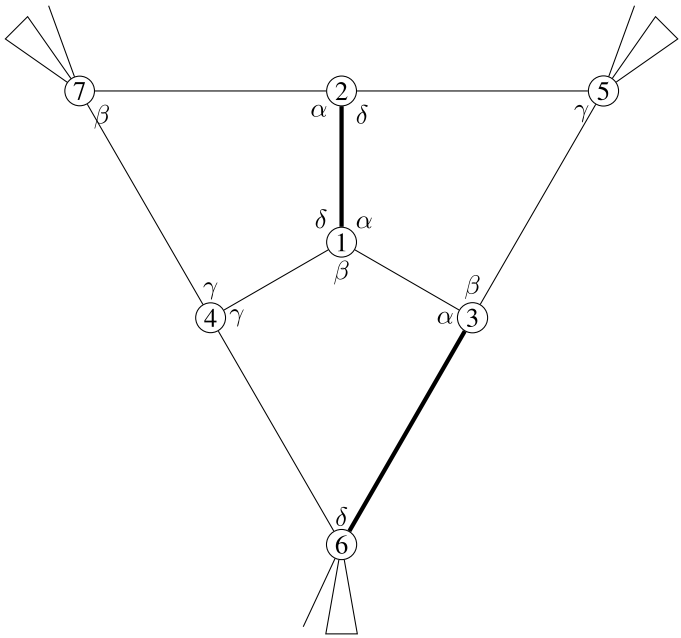

Figure 9: Partial angle assignment for a cubic tristar

We can fix the angle assignment in the quadrangle 1274, since interchanging α with δ and β with γ gives the same results. This also fixes the angle assignment in the quadrangle 1253, since the vertex 1 and the vertex 2 have the same type owing to Theorem 4.4. Since the vertex 3 is incident to an angle β, also the vertex 1 has to be incident to an angle β, and so the angle assignment in the quadrangle 1364 is also fixed. This gives the partial angle assignment shown in Figure 9.

The third angle at vertex 4 is either β or γ. Owing to Lemma 4.1, β is not possible. Owing to Lemma 4.2, γ implies that the quadrangulation has 6 faces, and thus 8 vertices. This proves the theorem. □

6 Conclusion

For the classification of spherical tilings by congruent quadranglesthere remain two open cases: spherical tilings by congruent quadrangles of type 2 and those of type 4. We show that the most symmetric of type 2 quadrangles, i.e., the isosceles quadrangles of type 2, cannot be used to tile the sphere. This might seem surprising, since spherical tilings by congruent quadrangles of type 2 do exist, but it can be explained because being isosceles and tiling the sphere forces the quadrangle to be of type 1.

Next we gave an overview of which vertex types of degree 3 can be used and showed that at most two different types can be used. We also showed that there is no spherical tiling by congruent quadrangles of type 2 for which the underlying graph contains a cubic quadrangle or a cubic tristar containing an edge of length b. As can be seen from Table 3 and Table 4, this excludes already a reasonable percentage of the quadrangulations that can

n	Quadrangulations	contain cubic quadrangle	Percentage
8	1	1	100.00%
10	1	0	0.00%
12	3	1	33.33%
14	12	3	25.00%
16	64	24	37.50%
18	510	210	41.18%
20	5 146	2 208	42.91%
22	58 782	25 792	43.88%
24	716 607	319 553	44.59%
26	9 062 402	4 110 016	45.35%
28	117 498 072	54 277 671	46.19%
30	1 553 048 548	731 637 255	47.11%

Table 3: Overview of quadrangulations on n vertices that contain a cubic quadrangle.

	10	12	14	16	18	20	22	24	26
0	1	2	7	31	217	2 065	22 869	272 106	3 355 499
1				6	68	747	8 804	108 738	1 383 419
2			2	3	15	119	1 249	15 363	201 586
3						5	66	832	11 619
4						2	2	15	259
5									4

Table 4: Overview of the number of cubic tristars in quadrangulations that do not contain a cubic quadrangle. The top row gives the number of vertices, the first column gives the number of cubic tristars and the remaining numbers give how many quadrangulations have that many vertices and that many cubic tristars.

appear as the underlying graph of a spherical tiling by congruent quadrangles of type 2, and also limits the possible charts that can correspond to a spherical tiling by congruent quadrangles of type 2. This is why these results can contribute to the completion of the classification of spherical tilings by congruent quadrangles. Table 3 and Table 4 were constructed using `plantri`[6, 4].

References

[1] Y. Akama, Classification of spherical tilings by congruent quadrangles over pseudo-double wheels (i) - a special tiling by congruent concave quadrangles., *Hiroshima Math. J.* **43**, 285–304.

[2] Y. Akama and K. Nakamura, Spherical tilings by congruent quadrangles over pseudo-double wheels (ii) the ambiguity of the inner angles, submitted, 2013.

[3] Y. Akama and Y. Sakano, Spherical tilings by congruent quadrangles over pseudo-double wheels (iii) the essential uniqueness in case of convex tiles, preprint, 2013.

[4] G. Brinkmann, S. Greenberg, C. Greenhill, B. D. McKay, R. Thomas and P. Wollan, Generation

of simple quadrangulations of the sphere, *Discrete Mathematics* **305** (2005), 33–54, http://dx.doi.org/10.1016/j.disc.2005.10.005.

[5] H. Davies, Packings of spherical triangles and tetrahedra, in: *Proc. Colloquium on Convexity (Copenhagen, 1965)*, Kobenhavns Univ. Mat. Inst., 1967, 42–51.

[6] plantri website, http://cs.anu.edu.au/~bdm/plantri.

[7] Y. Sakano and Y. Akama, Classification of spherical tilings by congruent kites, darts, and rhombi – spherical Hilbert's eighteenth problem, submitted, 2013.

[8] Y. Ueno and Y. Agaoka, Examples of spherical tilings by congruent quadrangles, *Mem. Fac. Integrated Arts and Sci.* **Ser. IV** (2001), 135–144.

[9] Y. Ueno and Y. Agaoka, Classification of tilings of the 2-dimensional sphere by congruent triangles, *Hiroshima Math. J.* **32** (2002), 463–540.