Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 10 (2016) 211–222

Algorithmic enumeration of regular maps*

Thomas Connor †

Université Libre de Bruxelles, Département de Mathématiques - C.P.216, Boulevard du Triomphe, B-1050 Bruxelles

Dimitri Leemans ‡

University of Auckland, Department of Mathematics, Private Bag 92019, Auckland, New Zealand

Received 23 September 2013, accepted 22 July 2015, published online 24 September 2015

Abstract

Given a finite group G, we describe an algorithm that enumerates the regular maps having G as rotational subgroup, using the knowledge of its table of ordinary characters and its subgroup lattice. To show the efficiency of our algorithm, we use it to compute that, up to isomorphism, there are 796,772 regular maps whose rotational subgroup is the sporadic simple group of O'Nan and Sims.

Keywords: Regular map, O'Nan sporadic simple group, subgroup lattice, character table.

Math. Subj. Class.: 05E18, 52B10, 20D08

1 Introduction

According to Coxeter (see [9], Chapter 8), systematic enumeration of orientable regular maps began in the 1920s by fixing a genus g and enumerating all maps embeddable on surfaces of genus g. Genus 2 was the first case considered by Errera and finished by Threlfall. Since then, a lot of work has been done on the subject, culminating in the enumeration of all orientable maps on surfaces of genus up to 301 by Conder (see [5, 4] and Conder's website for the latest results[1]).

*Research is supported by Marsden Grant UOA1218 of the Royal Society of New Zealand.

†Boursier FRIA.

‡Corresponding author.

E-mail addresses: tconnor@ulb.ac.be (Thomas Connor), d.leemans@auckland.ac.nz (Dimitri Leemans)

[1] http://www.math.auckland.ac.nz/~conder/

Another way to enumerate orientable maps is to fix a group G (or a family of groups) and count how many regular maps have G acting as rotational subgroup of the full automorphism group. In other words, we want to determine, for a given group G, the number of pairs of elements $[R, S] \in G^2$ such that

$$o(RS) = 2, o(R) = p, o(S) = q \text{ and } \langle R, S\rangle = G \tag{1.1}$$

where p and q are arbitrary orders of elements in G. The second type of enumeration can be done using a formula due to Frobenius [12] (see Section 2.2) based on character theory. Frobenius' formula has been used by Sah (see [20], Section 2) to obtain some enumeration results for the first group of Janko and the small Ree groups ${}^2G_2(q)$ with $q = 3^{2e+1}$, among other things. Conder et al. [6] extracted an enumeration result for all regular hypermaps of a given type with automorphism group isomorphic to $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ from the latter reference. Their result does not make use of character theory.

Jones and Singerman [16] set up the theoretical framework that links the study of maps to that of Riemann surfaces, showing among others that every map $\mathcal{M}$ is isomorphic to some canonical map $\overline{\mathcal{M}}$ on a Riemann surface. In [11], Downs and Jones set up the theoretical framework to determine the number of orientable maps of type $\{3, p\}$ with automorphism group a group $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$. Jones and Silver showed in [15] that the Suzuki groups $\mathrm{Sz}(q)$ are automorphism groups of regular maps of type $\{4, 5\}$. They also enumerated these maps: they used character theory and techniques developed by Philip Hall in [13] using Möbius inversion to show that there is at least one pair $[R, S]$ as above in each $\mathrm{Sz}(q)$. Then they used the fact that each element of order 4 is not conjugate to its inverse in $\mathrm{Aut}(\mathrm{Sz}(q))$ to conclude that every such map has to be chiral. For more results of that kind, we refer to [15, 14]. Mazurov and Timofeenko also used similar techniques to find those sporadic groups that can be generated by triples of involutions, two of which commute (see [18, 21]), therefore determining which sporadic groups are full automorphism groups of non-orientable regular maps.

Given a pair $[R, S] \in G^2$ satisfying (1.1), we can construct a regular map $\mathcal{M}$ of type $\{p, q\}$ from it with G being the orientation-preserving subgroup of the full automorphism group of $\mathcal{M}$. Frobenius' formula therefore gives us the number of regular maps that have G as such subgroup. The idea of the present paper is to use this formula in a systematic way to determine for a given group G what are the possible types for a map $\mathcal{M}$ with G being either the orientation-preserving subgroup of $\mathrm{Aut}(\mathcal{M})$ or G being the full automorphism group of $\mathcal{M}$ in the non-orientable case.

In this paper, we design an algorithm to compute up to isomorphism the number of regular maps (reflexible or chiral) having a given group G as group of orientation-preserving automorphisms, based on the character tables of G and its subgroups and on the subgroup lattice of G. To show the efficiency of our algorithm, we implemented it in MAGMA [2] and used it on the O'Nan sporadic simple group O′N. The choice of O′N is motivated by the fact that this is one of the most mysterious sporadic groups. Its smallest permutation representation is on 122,760 points and its subgroup lattice is relatively small.

The motivation of the paper first came from abstract regular polytopes. A recent paper by the authors and Mark Mixer [8] classifies all abstract regular polytopes of rank at least four for the O'Nan group. Hence rank three remains open. For a simple group G, a non-orientable regular map $\mathcal{M}$ whose full automorphism group is G is also an abstract regular polyhedron while a chiral map is a chiral polyhedron. Hence, getting to know which types are possible for G is also interesting in the study of abstract polyhedra whose automorphism group is G.

There is most likely a very large number of pairwise non-isomorphic abstract polyhedra having the O'Nan group as automorphism group. For instance, as shown in [17], the third Conway group, whose order is comparable, has $21,118$ abstract regular polyhedra up to isomorphism. Here, we derive the possible types $\{p,q\}$ for maps having $O'N$ as automorphism group. Our results for the O'Nan group may be summarized as follows.

Theorem 1.1. *Let G be the O'Nan sporadic simple group and let*

$$P := \{3,4,5,6,7,8,10,11,12,14,15,16,19,20,28,31\}.$$

1. *There exist two elements $R,S \in G$ such*

$$o(R) = p, o(S) = q, o(RS) = 2, \langle R,S\rangle = G$$

   for every $p \leq q \in P$ except for $\{p,q\} = \{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{3,6\}$, $\{3,7\}$, $\{3,12\}$ and $\{4,4\}$.

2. *There are 796,772 orbits of such pairs $\{R,S\}$ under the action of* $\mathrm{Aut}(\mathrm{O'N}) = \mathrm{O'N} : C_2$.
3. *Orientably-regular but chiral maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = G$ exist for all pairs $\{p,q\}$ of (1) except $\{3,15\}$ (that is 128 possible types).*
4. *Non-orientable regular maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = G$ exist for all pairs $\{p,q\}$ of (1) except $\{20,q\}$, $\{31,q\}$ (with $q \in P$), $\{3,10\}$, $\{4,5\}$ and $\{4,6\}$ (that is 95 possible types).*
5. *Reflexible maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(G)$ exist for all pairs $\{p,q\}$ of (1) except $\{8,q\}$, $\{16,q\}$ (with $q \in P$) (that is 98 possible types).*

The paper is organized as follows. In Section 2, we introduce the theoretical background needed to understand this paper. In Section 3, we describe our algorithm. In Section 4, we summarize the results obtained on the O'Nan sporadic simple group and obtain (1) and (2) of Theorem 1.1. In Section 5, we determine the types of maps that exist for the O'Nan group, deriving (3), (4) and (5) of Theorem 1.1. In Section 6, we give an algorithm to generate efficiently all maps of type $\{p,q\}$ for a fixed p. Finally, in Section 7, we conclude our paper with some remarks.

2 Theoretical background

2.1 Regular maps

In this paper, a map is a 2-cell embedding of a connected graph into a closed surface without boundary. Such a map $\mathcal{M}$ has a vertex-set $V := V(\mathcal{M})$, an edge-set $E := E(\mathcal{M})$ and a set of faces $F := F(\mathcal{M})$. We call $V \cup E \cup F$ the set of *elements* of $\mathcal{M}$. A triple $T := \{v,e,f\}$ where $v \in V$, $e \in E$ and $f \in F$ is called a *flag* if each element of T is incident with the other elements of T. The map is called *orientable* if the underlying surface on which the graph is embedded is orientable. Otherwise, it is called *non-orientable*. Faces of $\mathcal{M}$ are simply-connected components of the space obtained by removing the embedded graph from the surface. An *automorphism* of a map is a permutation of its elements preserving the sets V, E and F and incidence between the elements. Automorphisms form a group under composition called the *automorphism group* of the map and denoted by $\mathrm{Aut}(\mathcal{M})$.

If there exist a face f and two automorphisms R and S such that R cyclically permutes the consecutive edges of f and S cyclically permutes the consecutive edges incident to some vertex v of f, then $\mathcal{M}$ is called a *regular map* in the sense of Brahana [3]. In this case, the group $\mathrm{Aut}(\mathcal{M})$ acts transitively on the vertices, on the edges and on the faces. All faces are thus bordered by the same number of edges, say p and all the vertices have same degree, say q. The pair $\{p, q\}$ is known as the *type* of $\mathcal{M}$. Observe that the topological *dual* of $\mathcal{M}$, denoted by $\mathcal{M}^*$ is obtained by switching vertices and faces (that is $V(\mathcal{M}^*) := F(\mathcal{M})$, $E(\mathcal{M}^*) := E(\mathcal{M})$, $F(\mathcal{M}^*) := V(\mathcal{M})$). It is also regular and its type is $\{q, p\}$.

Note that R and S may be assumed to be such that RS interchanges v with one of its neighbors along an edge e on the border of f, interchanging f with the other face containing e. The three automorphisms R, S and RS then satisfy the following relations.

$$R^p = S^q = (RS)^2 = 1 \tag{2.1}$$

If a regular map $\mathcal{M}$ also has an automorphism a which flips the edge e but preserves f, then we say that $\mathcal{M}$ is *reflexible*. In that case, $\mathrm{Aut}(\mathcal{M})$ has a unique orbit on the set of flags. Moreover, $\mathrm{Aut}(\mathcal{M})$ is generated by the three automorphisms a, $b := aR$ and $c := bS$ that satisfy the following relations: $a^2 = b^2 = c^2 = (ab)^p = (ac)^2 = (bc)^q$.

If the map $\mathcal{M}$ is orientable, then the elements $R = ab$ and $S = bc$ generate a normal subgroup of $\mathrm{Aut}(\mathcal{M})$ of index 2, consisting of all elements expressible as words of even length in $\{a, b, c\}$. This subgroup is called the *rotational subgroup* and denoted by $\mathrm{Aut}^+(\mathcal{M})$. All elements of $\mathrm{Aut}^+(\mathcal{M})$ are precisely those preserving the orientation of the underlying surface while all other elements of $\mathrm{Aut}(\mathcal{M})$ reverse the orientation. In the non-orientable case, each of a, b and c can be expressed as a word in $\{R, S\}$ and hence, $\mathrm{Aut}(\mathcal{M}) = \langle R, S\rangle$.

If there is no automorphism a which flips the edge e but preserves f, then we say that the map $\mathcal{M}$ is *chiral*. Its automorphism group can be generated by the rotations R and S and $\mathcal{M}$ is necessarily orientable. Moreover, chiral maps occur in opposite pairs, each member of which is obtainable from the other by reflection.

2.2 Frobenius' formula

The search for maps having $G := \langle R, S\rangle$ as an automorphism group is equivalent to the search for triples of elements $x, y, z \in G$ satisfying (1.1) by posing $x = (RS)^{-1} = RS$, $y = R$ and $z = S$. Let G be a finite group and let

$$\Pi_G(\{p, q\}) := \{[x, y, z] \in G^3 | o(x) = 2, o(y) = p, o(z) = q, o(xyz) = 1\}.$$

In order to determine the cardinality $\pi_G(\{p, q\})$ of $\Pi_G(\{p, q\})$, we use the following result, due to Frobenius (see [12], section 4, equation 2).

Theorem 2.1. *If C_i, C_j and C_k denote conjugacy classes of elements in a finite group G, the number of solutions of $g_i g_j g_k = 1$ in G, with each $g_x \in C_x$ is*

$$\lambda_{i,j,k} = \frac{|C_i| \cdot |C_j| \cdot |C_k|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_k)}{\chi(1)} \tag{2.2}$$

where $\mathrm{Irr}(G)$ *is the set of irreducible characters of G.*

This theorem gives us an easy way to compute $\pi_G(\{p, q\})$.

Corollary 2.2. *Let G be a group. Let $C_1, \ldots, C_r$ be the conjugacy classes of elements of G. Let $K_n := \{i \in \{1, \ldots, r\} \mid o(x) = n$ for some $x \in C_i\}$. Then*

$$\pi_G(\{p,q\}) = \sum_{i \in K_p} \sum_{j \in K_q} \sum_{k \in K_2} \frac{|C_i| \cdot |C_j| \cdot |C_k|}{|G|} \sum_{\chi \in Irr(G)} \frac{\chi(g_i)\chi(g_j)\chi(g_k)}{\chi(1)}. \tag{2.3}$$

Proof. Straightforward. □

Let $\Gamma_G(\{p,q\}) := \{[x,y,z] \in \Pi_G(\{p,q\}) \mid \langle x,y,z \rangle = G\}$ and let $\gamma_G(\{p,q\}) := |\Gamma_G(\{p,q\})|$. The following lemma is the basis of our algorithm.

Lemma 2.3. *For a given group G and two integers $p, q > 1$, we have*

$$\gamma_G(\{p,q\}) = \pi_G(\{p,q\}) - \sum_{H<G} \gamma_H(\{p,q\})$$

or equivalently

$$\gamma_G(\{p,q\}) = \pi_G(\{p,q\}) - \sum_{H \in \mathcal{C}} \gamma_H(\{p,q\}) \times [G : N_G(H)]$$

where $\mathcal{C}$ is a set containing one representative of each conjugacy class of nontrivial proper subgroups of G.

Proof. Straightforward. □

As observed by Hall in [13], page 135, the number $n_G(\{p,q\})$ of pairwise non-isomorphic triples satisfying (1.1) is then obtained by dividing $\gamma_G(\{p,q\})$ by the order of the automorphism group of G. In other words,

$$n_G(\{p,q\}) = \frac{\gamma_G(\{p,q\})}{|\mathrm{Aut}(G)|}. \tag{2.4}$$

Following Lemma 2.3, we readily see that, in order to compute $\gamma_G(\{p,q\})$, it suffices to get one representative H of each conjugacy class of subgroups of G, and for each such H, to compute its normalizer and $\gamma_H(\{p,q\})$.

3 An algorithm to compute $\gamma_G(\{p,q\})$

Let G be a finite group. We detail an algorithm that determines $\pi_G(\{p,q\})$ and $\gamma_G(\{p,q\})$ for given values of p and q. In view of the developments of Section 2.2, $\pi_G(\{p,q\})$ can be computed using only the table of ordinary characters of G. Assuming that the character table of G is available (as it is the case for many simple groups in MAGMA [2] for instance) or easily computable, this is straightforward. Trickier is the computation of $\gamma_G(\{p,q\})$. Since

$$\gamma_G(\{p,q\}) = \pi_G(\{p,q\}) - \sum_{H<G} \gamma_H(\{p,q\})$$

we observe that there is a natural recursive way to compute $\gamma_G(\{p,q\})$. It only requires the knowledge of the subgroup lattice of G.

```
Input : G a permutation group
        p, q two positive integers

Output : γ_G := γ_G({p,q})

Compute the subgroup lattice Λ(G) of G.
  The subgroup lattice Λ(G) can be seen as an ordered list
  with least element G and greatest element the trivial
  group. If a subgroup of class i contains a subgroup of
  class j, then i < j.

For each conjugacy class C of subgroups of G,
  Take a representative H of C.
  If the order of H is divisible by pq/GCD(p,q)
    Compute δ : H → H̃ an isomorphism that reduces
      the permutation degree of H.
    Compute the subgroup lattice Λ(H̃) of H̃ and
      compute π_H̃({p,q}) using equation (2.3).

Now, read through Λ(G) starting from the trivial subgroup.
  At each step i, let H be a subgroup of the i^th conjugacy
  class that is considered and compute γ_H̃({p,q}).
    This computation requires the knowledge of Λ_H̃ and
    γ_Ĩ({p,q}) for all I < H. Note however that it is
    guaranteed that γ_Ĩ({p,q}) has been already computed at
    this stage since the lattice Λ(G) is endowed
    with a suitable ordering as mentioned earlier.

When all steps above have been done, γ_G has been computed.
Return γ_G
```

Figure 1: An algorithm to compute $\gamma_G(\{p,q\})$

The algorithm given in Figure 1 makes use of the obvious recursive way of computing $\gamma_G(\{p,q\})$ but it is not a recursive algorithm. Indeed, it carefully avoids multiple computations, for instance by computing only once the subgroup lattice and character table of one representative of each conjugacy class of subgroups of G. It also tries to reduce the permutation degree of each subgroup before dealing with it which speeds up computations of the subgroup lattice and the character table of the subgroups.

Indeed, our algorithm computes (or at least yields) the Möbius function of G; this could be useful in many other contexts, e.g. in enumerating quotients isomorphic to G in other finitely generated groups.

4 An application: the O'Nan sporadic simple group

In order to illustrate the efficiency of our algorithm, we implemented it in MAGMA [2] and we ran it on O′N, the sporadic simple group of O'Nan, of order $460,815,505,920$ and smallest permutation representation degree $122,760$. Observe that $|\mathrm{Out}(\mathrm{O'N})| = 2$. In MAGMA, the function `SubgroupLattice` computes the subgroup lattice of a given finite group. In the case of O′N however, `SubgroupLattice` is not able to compute this lattice[2]. Fortunately, an algorithm to compute subgroup lattices of groups like O′N is made available in [7].

We computed the numbers $\pi_{\mathrm{O'N}}(\{p,q\})$, $\gamma_{\mathrm{O'N}}\{p,q\})$ and $n_{\mathrm{O'N}}(\{p,q\})$ for all possible values of p and q. Recall that, by Formula (2.4), $n_{\mathrm{O'N}}(\{p,q\})$ is obtained by dividing $\gamma_{\mathrm{O'N}}(\{p,q\})$ by the order of $\mathrm{Aut}(\mathrm{O'N})$ which is $2 \cdot |\mathrm{O'N}|$. There are 17 distinct orders of elements in O′N. One of them is 2 and if p or q is 2, then $\gamma_G(\{p,q\})$ is null as O′N is a simple group. Hence, in total, there are 16*15/2 + 16 = 136 possible unordered pairs $\{p,q\}$. Out of these, five give obviously 0, namely those pairs that give groups which are solvable or isomorphic to A_5, that is $\{3,3\}$, $\{3,4\}$, $\{3,5\}$, $\{3,6\}$ and $\{4,4\}$. Therefore, there remain 131 of them to compute. We give in Table 1 the values $n_G(\{p,q\})$ for O′N. The 131 cases have been spread on several processors. Each case took on average 5 days to finish. Point (1) of Theorem 1.1 is then obtained by collecting the nonzero entries of Table 1. The sum of all the numbers appearing in that table gives point (2) of Theorem 1.1.

Note that Woldar had already shown in [22] that O′N is not a Hurwitz group, meaning that $\gamma_{\{3,7\}} = 0$. He also showed that $\gamma_{\{3,11\}} \neq 0$. Moreover, in [10], Darafsheh, Ashrafi and Moghani showed that $\gamma_{\{p,q\}} \neq 0$ for the following twelve pairs: $\{3,19\}$, $\{3,31\}$, $\{5,7\}$, $\{5,11\}$, $\{5,19\}$, $\{5,31\}$, $\{7,11\}$, $\{7,19\}$, $\{7,31\}$, $\{11,19\}$, $\{11,31\}$ and $\{19,31\}$. Very recently, Al-Khadi [1] showed that $\gamma_{\{3,12\}} = 0$ and $\gamma_{\{3,q\}} \neq 0$ for $q \in \{8,10,12,14,16,20,28\}$.

5 Regular maps for $O'N$

By Table 1, we know exactly how many pairs of generating elements $\{R,S\}$ satisfying (2.1) exist up to isomorphism for any given pair $\{o(R),o(S)\}$. For instance, there are 7 such pairs $\{R,S\}$ with $\{o(R),o(S)\} = \{3,10\}$.

If there is no automorphism of $G := \langle R,S\rangle$ that inverts R and S, G is the full automorphism group of an orientably-regular but chiral map of type $\{p,q\}$ (and its dual of type $\{q,p\}$). The pair $\{R,S\}$ is then called *chiral*.

On the other hand, if there exists an automorphism $\theta \in \mathrm{Aut}(G)$ such that $\theta([R,S]) = [R^{-1},S^{-1}]$, then θ is an involution and the pair $\{R,S\}$ is called *reflexible*. In this case, the group generated by R, S and θ is the full automorphism group of a reflexible map $\mathcal{M}$ of type $\{p,q\}$, with $G \cong \mathrm{Aut}^+(\mathcal{M})$, the orientation-preserving subgroup (of index 2) and $\mathrm{Aut}(\mathcal{M}) \cong G : C_2$ where ':' denotes a semi-direct product. This semi-direct product is sometimes a direct product, namely when the automorphism θ is an inner automorphism of G. In that case, G is also the full automorphism group of a non-orientable map $\mathcal{N}$ of type $\{p,q\}$ (and its dual of type $\{q,p\}$). Moreover, $\mathcal{M}$ is then an orientable double cover of $\mathcal{N}$, and $\mathrm{Aut}(\mathcal{M}) \cong G \times C_2$.

The following lemma gives point (3) of Theorem 1.1.

[2] at least up to version 2.19-3 of MAGMA

Lemma 5.1. *Let G be the O'Nan sporadic simple group. For every pair $\{p, q\}$ of Theorem 1.1.(1) except $\{3, 15\}$, there exists at least one chiral map $\mathcal{M}$ of type $\{p, q\}$ with $\mathrm{Aut}(\mathcal{M}) \cong G$.*

Proof. A non-exhaustive computer search with MAGMA produced chiral maps of all possible types except $\{3, 15\}$ in a few days. By Table 1, there are 6 non-isomorphic pairs of type $\{3, 15\}$. Using MAGMA, we produced the 6 non-isomorphic pairs $\{R, S\}$ and checked that for each of them, there exists $\theta \in \mathrm{Aut}(G)$ that inverts R and S. For four of them, $\theta \in \mathrm{Inn}(G)$ and for two of them, θ is an outer automorphism. □

Since $\mathrm{O'N}$ is simple, a non-chiral regular map $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{O'N}$ is necessarily non-orientable.

Lemma 5.2. *Let G be the O'Nan sporadic simple group. Non-orientable regular maps of type $\{p, q\}$ with G as full automorphism group do not exist for pairs $\{p, q\}$ with p or q equal to* 20 *or* 31.

Proof. It suffices to observe that all elements of order 20 and 31 are not conjugate to their inverse. Hence, an automorphism that would reverse R and S in this case is necessarily an outer automorphism. □

The above lemma combined with those values $\gamma_{\{p,q\}}$ equal to 0 gives at most 98 possible types for non-orientable maps having $\mathrm{O'N}$ as full automorphism group. A non-exhaustive brute force search gave in a few days examples of such maps for 92 types. For the remaining 6 types, we did exhaustive searches and here is a summary of what we found.

- $\{3, 10\}$: an exhaustive search found 6 chiral maps and 1 pair $\{R, S\}$ with θ an outer automorphism;
- $\{4, 5\}$: an exhaustive search found 16 chiral maps and 2 pairs $\{R, S\}$ with θ an outer automorphism;
- $\{4, 6\}$: an exhaustive search found 42 chiral maps and 1 pair $\{R, S\}$ with θ an outer automorphism;
- $\{5, 5\}$: an exhaustive search found 22 chiral maps, 2 non-orientable maps and 2 pairs $\{R, S\}$ with θ an outer automorphism;
- $\{5, 7\}$ and $\{7, 7\}$: we found at least one non-orientable map for each type.

The above results are summarized in point (4) of Theorem 1.1.

Lemma 5.3. *Let G be the O'Nan sporadic simple group. There is no reflexible map $\mathcal{M}$ of type $\{p, q\}$ such that $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(G)$ for any pair $\{p, q\}$ with p or q equal to* 8 *or* 16.

Proof. All elements of order 8 and 16 are conjugate to their inverse. Moreover, there is no outer automorphism mapping such an element to its inverse. □

The above lemma combined with those values $\gamma_{\{p,q\}}$ equal to 0 give at most 98 possible types for reflexible maps $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(G)$. A brute force search gave us 95 types for which such pairs exist in a couple of days. We dealt separately with the three types that the search did not find. Below is a summary of what we found.

- $\{4, 5\}$: an exhaustive search found 16 chiral maps and 2 pairs $[R, S]$ with θ an outer automorphism;
- $\{5, 5\}$: an exhaustive search found 22 chiral maps, 2 non-orientable regular maps and 2 pairs $[R, S]$ with θ an outer automorphism;
- $\{6, 12\}$: at least one pairs $[R, S]$ with θ an outer automorphism was found.

The above results are summarized in point (5) of Theorem 1.1.

6 Generating all maps for the O'Nan group

The O'Nan group has a unique conjugacy class of involutions and the centralizer of an involution is a group $4 \cdot L_3(4) : 2$ of order $161,280$. It is the largest centralizer of an element of order at least 2. This suggests an algorithm to construct all of the 796,772 pairs (R, S) for the O'Nan group to study the prevalence of chirality over regularity for this group.

To generate all pairs $\{R, S\}$ with S an element of order p and R an element of order $\geq p$, we construct a permutation representation of O$'$N on its involutions. This is done by constructing the coset space of O$'$N on $C_{\mathrm{O'N}}(\rho)$ for an arbitrary involution $\rho \in \mathrm{O'N}$.

Let P be a sequence. We will use P to store pairs of elements of O$'$N. Let G be the permutation representation on the cosets of $C_{\mathrm{O'N}}(\rho)$ and let $\phi : \mathrm{O'N} \to G$ be an isomorphism between O$'$N in its natural permutation representation and G. Let S be a sequence containing one representative of each conjugacy class of elements of order p in O$'$N. For $s \in S$, let $\mathcal{O}$ be the set of orbits of $\phi(s)$. For each $o \in \mathcal{O}$, let x be a representative of o and let $\phi^{-1}(G_x)$ be the centralizer of an involution in O$'$N that correspond to the fixed point x. Let τ be the involution centralized by $\phi^{-1}(G_x)$. Let $R := \tau * S^{-1}$. Then $\{R, S\}$ is a pair with $RS = \tau$ an involution. If $\langle R, S\rangle = \mathrm{O'N}$ and there is no pair $\{R', S'\}$ in P isomorphic to $\{R, S\}$, append $\{R, S\}$ to P. When a new pair $\{R, S\}$ is found, we can determine whether it gives an orientably-regular but chiral map or a non-orientable map whose full automorphism group is O$'$N. In the process, we use the results of Section 5 to shorten the computations: we keep track of how many pairs of each type have been generated so that, once we get the total number for a given type, we do not have to consider that type anymore.

Each chiral map (respectively non-orientable map) whose full automorphism group is O$'$N is also an abstract chiral polyhedron (respectively abstract regular polyhedron). Therefore, the algorithm described above permits in theory to construct all chiral and regular polyhedra for the O'Nan group.

7 Concluding remarks

In practice, to generate all the 284 pairs of type $\{3, q\}$, it took less than 4 hours on a computer with a processor running at 2.9Ghz. We needed 11 days to generate all 5176 pairs of type $\{4, q\}$ and 28 days for the 7738 pairs of type $\{5, q\}$. Experiments with other types gave an average time of more than five minutes per map.

Out of the 284 pairs of type $\{3, q\}$, 230 give a chiral map and 39 a non-orientable map with full automorphism group O$'$N. Out of the 5176 pairs of type $\{4, q\}$, 4906 give a chiral map and 114 a non-orientable map with full automorphism group O$'$N. Out of the 7738 pairs of type $\{5, q\}$, 7340 give a chiral map and 188 a non-orientable map with full automorphism group $\mathrm{O'N}$. The tendency of maps of chiral type being more prevalent seems confirmed by the partial results we obtained on maps of type $\{p, q\}$ with $q \geq p \geq 6$.

For all these maps, answering questions like "what are the exponents[3] of $\mathcal{M}$, is it self-dual, etc." is possible.

8 Acknowledgements

Part of this research was done while the first author was visiting the second author at the University of Auckland. He therefore gratefully acknowledges the University of Auckland for its hospitality. He also acknowledges the *Fonds pour la formation à la Recherche dans l'Industrie et l'Agriculture*, (F.R.I.A.) and the *Fonds National pour la Recherche Scientifique* (F.N.R.S.) for financial support. The second author acknowledges financial support of the Royal Society of New Zealand Marsden Fund (Grant UOA1218). The authors also thank Marston Conder, Gareth Jones and Jozef Širáň for interesting discussions while writing this paper.

[3]See [19] for a definition.

	3	4	5	6	7	8	10	11	12	14	15	16	19	20	28	31
3	0	0	0	0	0	10	7	37	0	10	6	68	57	44	20	25
4		0	18	43	102	284	285	503	120	166	234	1292	846	554	370	359
5			26	98	150	470	365	718	211	290	340	1966	1242	800	560	502
6				165	354	1122	953	1776	474	597	874	4700	2943	1948	1370	1268
7					648	1848	1506	2687	815	1054	1284	7448	4725	2916	2214	1995
8						5424	4460	8096	2370	3056	3960	22040	13926	8880	6276	5752
10							3613	6532	1969	2526	3072	17822	11262	7224	4992	4632
11								11839	3583	4601	5814	32488	20493	13094	9202	8325
12									1072	1391	1710	9764	6126	3984	2796	2577
14										1796	2330	12504	7899	5024	3616	3266
15											2834	15808	10020	6424	4496	4062
16												88784	56052	35644	25316	23048
19													35442	22572	15978	14553
20														14238	10292	7246
28															7246	6658
31																5999

Table 1: Values of $n_G(\{p,q\})$ with $G \cong \mathrm{O'N}$

References

[1] M. A. Al-Kadhi. On the generation of sporadic simple group $O'N$ by $(2,3,t)$ generators. *International J. Alg.*, **7** (2013), 167–176.

[2] W. Bosma, J. Cannon, and C. Playoust. The Magma Algebra System. I. The User Language. *J. Symbolic Comput.* **24** (1997), 235–265.

[3] H. R. Brahana. Regular maps and their groups. *Amer. J. Math* **49** (1927), 268–284.

[4] M. Conder. Regular maps and hypermaps of Euler characteristic -1 to -200. *J. Combin. Theory Ser. B* **99** (2009), 455–459.

[5] M. Conder and P. Dobcsányi. Determination of all regular maps of small genus. *J. Combin. Theory Ser. B* **81** (2001), 224–242.

[6] M. Conder, P. Potočnik, and J. Širáň. Regular hypermaps over projective linear groups. *J. Aust. Math. Soc.* **85** (2008), 155–175.

[7] T. Connor and D. Leemans. An atlas of subgroup lattices of finite almost simple groups. *Ars Math. Contemp.* **8** (2015), 259–266.

[8] T. Connor, D. Leemans, and M. Mixer. Abstract regular polytopes for the O'Nan group. *Int. J. Alg. Comput.* **24** (2014), 59–68.

[9] H. S. M. Coxeter and W. O. J. Moser. *Generators and relations for discrete groups*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, fourth edition, 1980.

[10] M. R. Darafsheh, A. R. Ashrafi, and G. A. Moghani. (p,q,r)-generations of the sporadic group O'N. In *Groups St. Andrews 2001 in Oxford. Vol. I*, volume 304 of *London Math. Soc. Lecture Note Ser.*, pages 101–109. Cambridge Univ. Press, Cambridge, 2003.

[11] M. L. N. Downs and G. A. Jones. Enumerating regular objects with a given automorphism group. *Discrete Math.* **64** (1987), 299–302.

[12] G. Frobenius. Ueber Gruppencharaktere. *Berl. Ber.* **1896**, 985–1021.

[13] P. Hall. The Eulerian functions of a group. *Q. J. Math., Oxf. Ser.* **7** (1936), 134–151.

[14] G. A. Jones. Ree groups and Riemann surfaces. *J. Algebra* **165** (1994), 41–62.

[15] G. A. Jones and S. A. Silver. Suzuki groups and surfaces. *J. London Math. Soc. (2)* **48** (1993), 117–125.

[16] G. A. Jones and D. Singerman. Theory of maps on orientable surfaces. *Proc. London Math. Soc. (3)* **37** (1978), 273–307.

[17] D. Leemans and M. Mixer. Algorithms for classifying regular polytopes with a fixed automorphism group. *Contr. Discrete Math.* **7** (2012), 105–118.

[18] V. D. Mazurov. On the generation of sporadic simple groups by three involutions, two of which commute. *Sibirsk. Mat. Zh.* **44** (2003), 193–198.

[19] R. Nedela and M. Škoviera. Exponents of orientable maps. *Proc.London Math. Soc.* **75** (1997), 1–31.

[20] C.-H. Sah. Groups related to compact Riemann surfaces. *Acta Math.* **123** (1969), 13–42.

[21] A. V. Timofeenko. On generating triples of involutions of large sporadic groups. *Diskret. Mat.* **15** (2003), 103–112.

[22] A. J. Woldar. On Hurwitz generation and genus actions of sporadic groups. *Illinois J. Math.* **33** (1989), 416–437.