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ARS MATHEMATICA CONTEMPORANEA 1 (2008) 38–43

On the Cayley Isomorphism Problem for a Digraph with 24 Vertices

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Received 9 January, accepted 15 May 2007, published online 19 June 2008

Abstract

In this paper we are mainly concerned with the Cayley isomorphism problem for groups containing Q_8. We prove that the group $Q_8 \times C_3$ is not a CI-group with respect to colour ternary relational structures. Further, we prove that the non-nilpotent group $C_3 \ltimes Q_8$ is not a CI-group with respect to graphs.

Keywords: Regular subgroup, Cayley isomorphism.

Math. Subj. Class.: 05E20, 05E99

1 Introduction

A k*-ary relational structure* X is an ordered pair (Ω, E), where E is a subset of the set Ω^k. A 3-ary relational structure is also referred to as a ternary relational structure. Further, if we assign a colour to each "edge" of E, then the resulting structure X is said to be a *colour* k*-ary relational structure*. Let $X = (\Omega, E)$ be a colour k-ary relational structure. We denote by $\operatorname{Aut} X$ the permutation group on Ω defined by $\{\sigma \in \operatorname{Sym}(\Omega) \mid e^\sigma \in E$ for any $e \in E$ and e, e^σ have the same colour$\}$.

Let G be a permutation group on Ω and X be a (colour) k-ary relational structure on Ω. We say that X is a *Cayley (colour)* k*-ary relational structure* on the group G if the right regular representation of G is contained in $\operatorname{Aut} X$. We note that in this case there is a natural bijection between Ω and G. Therefore, X is isomorphic to the (colour) k-ary relational structure (G, F), for some subset F of G^k. In particular, without loss of generality, we can assume that the underlying "vertex-set" of a Cayley (colour) k-ary relational structure on G is the group G itself.

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Recall that if $X = (G, E)$ and $X' = (G, E')$ are Cayley (colour) k-ary relational structures on G, then X and X' are said to be *Cayley isomorphic* if there exists an automorphism of G that takes E to E'.

The group G is said to be a CI-*group with respect to (colour) k-ary relational structures* if, for all Cayley (colour) k-ary relational structures X and X' on G, the structures X and X' are isomorphic if and only if they are Cayley isomorphic.

See [2] for an account of Cayley colour k-ary relational structures and CI-groups. We note that if $k = 2$, then we get the usual definition of digraph, Cayley graph and CI-group. Furthermore, it is clear that if G is a CI-group with respect to Cayley colour k-ary relational structures, then G is CI-group with respect to Cayley k-ary relational structures

We recall that G is a CI-group with respect to (colour) k-ary relational structures if and only if, for any Cayley (colour) k-ary relational structure X on G, any two regular subgroups of $\mathrm{Aut}\, X$ isomorphic to G are conjugate in $\mathrm{Aut}\, X$, see [1].

It is fairly interesting to note that, if $k \geq 4$, then the classification of CI-groups with respect to (colour) k-ary relational structures was achieved in [5].

Note that the classification of CI-groups with respect to (colour) graphs is a wide open and very interesting problem, see [4] for an overview of the main results.

We point out that the classification of CI-groups with respect to (colour) ternary relational structures is also wide open. We refer to [2] for an account of this problem.

In Theorem 6, we prove that $\mathrm{SL}(2, 3)$ is not a CI-group with respect to graphs. In particular this result gives further restrictions on the structure of a CI-group and it narrows the list of possible CI-groups given in [4]. We note that $\mathrm{SL}(2, 3)$ is isomorphic to $C_3 \ltimes Q_8$, where the action of C_3 on Q_8 is non-trivial.

Also, in Theorem 8, we prove that $Q_8 \times C_3$ is not a CI-group with respect to colour ternary relational structures. So, this result improves the list of possible CI-groups with respect to colour ternary relational structures given in [2].

It is worth noticing that Q_8 and C_3 are CI-groups with respect to colour ternary relational structures. In particular, $Q_8 \times C_3$ is the only example known to the author of this paper, of a non CI-group with respect to colour ternary relational structures that is the direct product of CI-groups with respect to colour ternary relational structures of *coprime* order. We would like to point out that no example of this behaviour is known for CI-groups with respect to graphs.

2 The construction

Let Q_H and Q_K be isomorphic to Q_8, with generators i_H, j_H and i_K, j_K (respectively). So, $i_H^2 = j_H^2 = [i_H, j_H] \in \xi(Q_H)$ and $i_H^4 = 1$, and similar relations hold for the group Q_K. In this paper, $\xi(G)$ denotes the centre of a group G.

We denote by E the extraspecial group $Q_H \circ Q_K$, i.e. the central product of Q_H and Q_K, see [3]. In other words E is the direct product of Q_H and Q_K with their centres identified, i.e. $E = (Q_H \times Q_K)/\langle i_H^2 i_K^2 \rangle$. Then E is the 2-group of order 32 with generators i_H, j_H, i_K, j_K and with relations

$$\begin{aligned} i_H^4 = [i_H, i_K] = [i_H, j_K] = [j_H, i_K] = [j_H, j_K] &= 1, \\ i_H^2 = i_K^2 = j_H^2 = j_K^2 = [i_H, j_H] = [i_K, j_K] &\in \xi(E). \end{aligned}$$

We recall that if G is a finite group and $\varphi : G \to G$ is a function mapping a generating set of G to another generating set of G and preserving the defining relations of G, then φ is an automorphism of G.

Now we denote by C the subgroup of $\mathrm{Aut}(E)$ generated by x, y, t, where x, y and t are defined as follows:

$$\begin{array}{cccc} i_H^x = j_H, & j_H^x = i_H j_H, & i_K^x = i_K j_K, & j_K^x = i_K \\ i_H^y = i_H j_H, & j_H^y = i_H, & i_K^y = i_K j_K, & j_K^y = i_K \\ i_H^t = i_K, & j_H^t = j_K, & i_K^t = i_H, & j_K^t = j_H. \end{array}$$

Using the previous paragraph and the relations of E given above, the reader may check that x, y and t define automorphisms of E.

Lemma 1. (i) $t^2 = x^3 = y^3 = 1$,
(ii) $x^t = x^{-1}$, $[x, y] = 1$ *and* $[t, y] = 1$ *(so y is in the centre of C),*
(iii) *C is isomorphic to* $\mathrm{Sym}(3) \times C_3$*, and*
(iv) *xy centralizes i_H, j_H and xy^{-1} centralizes i_K, j_K.*

Proof. (i) By definition of t, we have that t^2 fixes i_H, j_H, i_K, j_K. Therefore, t^2 fixes every element of E, thus $t^2 = 1$. Now,

$$i_H^{x^3} = (i_H^x)^{x^2} = (j_H^x)^x = (i_H j_H)^x = j_H i_H j_H = i_H.$$

This yields that x^3 centralizes i_H. Similarly, the reader can check that x^3, y^3 centralize the generators i_H, j_H, i_K, j_K of E. Therefore $x^3 = y^3 = 1$.

(ii) We note that $i_H^{x^t} = i_H^{txt} = i_K^{xt} = (i_K j_K)^t = i_H j_H = i_H^{x^{-1}}$. Similarly, the reader can check that $j_H^{x^t} = j_H^{x^{-1}}$, $i_K^{x^t} = i_K^{x^{-1}}$, $j_K^{x^t} = j_K^{x^{-1}}$. This says that $x^t = x^{-1}$. The proofs that $[y, t] = 1$ and $[x, y] = 1$ are analogous.

(iii) It follows from $(i), (ii)$.

(iv) By definition of x and y, we have $i_H^{xy} = j_H^y = i_H$ and $j_H^{xy} = (i_H j_H)^y = i_H j_H i_H = j_H$. Thus xy centralizes i_H, j_H. Similarly, the reader can check that xy^{-1} centralizes i_K, j_K. □

If H is a subgroup of a group G, then we say that H is a *core-free subgroup of G* if the only normal subgroup of G contained in H is 1, i.e. $\cap_{g \in G} H^g = 1$. Recall that if H is a core-free subgroup of G, then the action of G on the right cosets of H in G is faithful.

Now we denote by A the group $C \ltimes E$. Consider $B = \langle i_H i_K^{-1}, t, x\rangle$. We denote by v_1 the element $i_H i_K^{-1}$, and set $v_2 = v_1^x$, $v_3 = v_2^x$ and $B_E = B \cap E$.

Lemma 2. *The group B_E is an elementary abelian 2-group of order 4 and consists of* $\mathrm{id}, v_1, v_2, v_3$*. The group B is isomorphic to* $\mathrm{Sym}(4)$ *and $B \cap B^y = \langle x, t\rangle$. The group B is a core-free subgroup of A.*

Proof. The elements v_1, v_2, v_3 have order 2 and

$$v_1 v_2 = i_H i_K^{-1} j_H (i_K j_K)^{-1} = i_H j_H j_K^{-1} = v_3.$$

So, $v_1, v_2, v_1 v_2$ are involutions and hence v_1 and v_2 commute. Therefore $\langle v_1, v_2, v_3\rangle$ is an elementary abelian 2-group of order 4. Further, $v_1^t = v_1$ and $v_2^t = v_3$. This shows that $B_E = \langle v_1, v_2\rangle$. In particular, $B_E = \{\mathrm{id}, v_1, v_2, v_3\}$.

Now, $B = \langle x, t\rangle \ltimes B_E$ and Lemma $1(i), (ii)$ yields $B \cong \mathrm{Sym}(4)$.

By Lemma $1(ii)$, we have $B \cap B^y \geq \langle x, t\rangle$. Now, as B is isomorphic to $\mathrm{Sym}(4)$ and $\langle x, t\rangle$ is isomorphic to $\mathrm{Sym}(3)$, we have that either $B \cap B^y = \langle x, t\rangle$ or $B = B^y$. Since $v_1^y = i_H j_H (i_K j_K)^{-1} \in B^y \setminus B$, we get $B \cap B^y = \langle x, t\rangle$.

Thus, if B contains a normal subgroup N of A, we must have $N \leq \langle x, t\rangle$. But $\langle x, t\rangle \cong \mathrm{Sym}(3)$, and the only subgroup of $\mathrm{Sym}(3)$ normal in $\mathrm{Sym}(4)$ is 1. So $N = 1$. Thus B is a core-free subgroup of A. □

Define $H = \langle i_H, j_H, xy\rangle$, $K = \langle i_K, j_K, xy^{-1}\rangle$, $U = \langle i_H, j_H, y\rangle$ and $V = \langle i_H, j_H, xy^{-1}\rangle$.

Note that, by the definition of x, y and by Lemma $1(iv)$, the groups H, K are isomorphic to $Q_8 \times C_3$ and U, V are isomorphic to $\mathrm{SL}(2,3)$ (we recall that $\mathrm{SL}(2,3)$ is isomorphic to $C_3 \ltimes Q_8$, where the action of C_3 on Q_8 is non-trivial).

Lemma 3. $B \cap H = B \cap K = B \cap U = B \cap V = 1$ *and* $BH = BK = BU = BV = A$.

Proof. We first prove that $B \cap H = B \cap K = 1$ and $BH = BK = A$. Since t lies in B and $H^t = K$, it is enough to prove that $B \cap H = 1$ and $BH = A$. We have $B \cap H \subseteq E\langle x, t\rangle \cap E\langle xy\rangle = E$. So, $B \cap H = B \cap (H \cap E) = B \cap \langle i_H, j_H\rangle = 1$. Since $|A| = |E||C| = 32 \cdot 18 = 24 \cdot 24 = |H||B|$ and $B \cap H = 1$, we have $BH = HB = A$.

Now, we prove that $B \cap U = 1$ and $BU = A$. We have $B \cap U \subseteq E\langle x, t\rangle \cap E\langle y\rangle = E$. So, $B \cap U = B \cap (U \cap E) = B \cap \langle i_H, j_H\rangle = 1$. Since $|A| = |U||B|$ and $B \cap H = 1$, we have $BU = A$. Similarly, the reader can check that $B \cap V = 1$ and $BV = A$. □

Next, we consider the action of A on the right cosets $\Omega = A/B$. Lemma 2 yields that A is a permutation group of degree 24 with point stabilizer isomorphic to $\mathrm{Sym}(4)$. Lemma 3 yields that H, K, U and V are regular subgroups of A.

Lemma 4. *Let Δ be the B-orbit of the point By of Ω. We have*

$$\Delta = \{By, Byv_1, Byv_2, Byv_3\}$$

and the action of B on Δ is equivalent to the action of $\mathrm{Sym}(4)$ *on four points.*

Proof. By Lemma 2, we have $B \cap B^y = \langle x, t\rangle$. Therefore, the group B_E acts regularly on Δ. Since y centralizes $\langle x, t\rangle$, we have that every element of $\langle x, t\rangle$ fixes By. So, $\Delta = \{By, Byv_1, Byv_2, Byv_3\}$. Moreover,

$$Byv_1x = Bx(yv_1)^x = Byv_2, Byv_2x = Bx(yv_2)^x = Byv_3.$$

This shows that x acts on Δ as a 3-cycle. Similarly,

$$Byv_1t = Btyv_1 = Byv_1, Byv_2t = Bt(yv_2)^t = Byv_3.$$

This says that t acts on Δ as a 2-cycle. Thus the lemma is proved. □

Let $\mathcal{S}$ be the orbital corresponding to the suborbit Δ of A, i.e. $\mathcal{S} = \{(Ba, Bya) \mid a \in A\}$. Let Γ be the orbital digraph of A corresponding to the orbital $\mathcal{S}$, we recall that Γ has vertex set Ω and edge set $\mathcal{S}$. We note that, by construction, A is a subgroup of $\mathrm{Aut}\,\Gamma$.

Proposition 5. $A = \mathrm{Aut}\,\Gamma$.

Proof. Since B acts 4-transitively on Δ, it is enough to prove that if σ is an automorphism of Γ fixing the vertex B and the out-neighbours of B, then $\sigma = \mathrm{id}$. We leave this routine exercise to the conscientious reader. □

3 $\mathrm{SL}(2,3)$

In this section, we prove that $\mathrm{SL}(2,3)$ is not a CI-group (with respect to graphs). We recall that $U \cong V \cong \mathrm{SL}(2,3)$.

Theorem 6. *The group* $\mathrm{SL}(2,3)$ *is not a* CI-*group with respect to graphs.*

Proof. The groups U, V are regular subgroups of A isomorphic to $\mathrm{SL}(2,3)$. Furthermore, by Proposition 5, the group A is the automorphism group of the digraph Γ. In particular, Γ is a Cayley graph on $\mathrm{SL}(2,3)$. Therefore, it is enough to prove that U, V are not conjugate in A. We argue by contradiction. Let g be in A such that $U^g = V$. Since $\langle i_H, j_H\rangle$ is a characteristic subgroup of U and V, we have $\langle i_H, j_H\rangle^g = \langle i_H, j_H\rangle$, i.e. $g \in N_A(\langle i_H, j_H\rangle) = \langle x, y\rangle E$. Now, $g = ze$ for some $z \in \langle x, y\rangle$ and $e \in E$. The element y lies in U, therefore $y^g = (y^z)^e = y^e = y[y,e]$ lies in V. But $V \subseteq \langle xy^{-1}\rangle E$ and $[y,e] \in E$, so $y \in \langle xy^{-1}\rangle$, a contradiction. □

4 $Q_8 \times C_3$

In this section, we prove that $Q_8 \times C_3$ is not a CI-group with respect to colour ternary relational structures. We recall that $H \cong K \cong Q_8 \times C_3$.

Set $G = E\langle x, y\rangle$. Let $\mathcal{T}_1$ be the subset of Ω^3 defined by $\{(Bg, Bg, Byg) \mid g \in G\}$. Also, let $\mathcal{T}_2$ be the subset of Ω^3 given by $\{(Bg, Bi_Hg, Bj_H^{-1}g) \mid g \in G\}$. The sets $\mathcal{T}_1, \mathcal{T}_2$ define two ternary relational structures on Ω. We recall that $\mathrm{Aut}\,\mathcal{T}_i = \{\sigma \in \mathrm{Sym}(\Omega) \mid t^\sigma \in \mathcal{T}_i \text{ for any } t \in \mathcal{T}_i\}$, for $i = 1, 2$.

Proposition 7. $G = \mathrm{Aut}\,\mathcal{T}_1 \cap \mathrm{Aut}\,\mathcal{T}_2$.

Proof. We claim that $A = \mathrm{Aut}\,\mathcal{T}_1$. The group G is a transitive subgroup of A and, by Lemma 4, the stabilizer in G of the point B of Ω is isomorphic to $\mathrm{Alt}(4)$ and acts transitively on Δ. This says that $\{(Bg, Byg) \mid g \in G\} = \{(Ba, Bya) \mid a \in A\} = \mathcal{S}$. Let σ be in $\mathrm{Aut}\,\mathcal{T}_1$ and e be in $\mathcal{S}$. Now, $e = (Bg, Byg)$, for some $g \in G$. Set $\tau = (Bg, Bg, Byg)$. Now, $\tau \in \mathcal{T}_1$, therefore $\tau^\sigma \in \mathcal{T}_1$. So, $\tau^\sigma = (Bg', Bg', Byg')$, for some $g' \in G$. In particular $e^\sigma = (Bg', Byg') \in \mathcal{S}$. Since e is an arbitrary element of $\mathcal{S}$, we get $\sigma \in \mathrm{Aut}\,\Gamma = A$. Since σ is an arbitrary element of $\mathrm{Aut}\,\mathcal{T}_1$, we get $\mathrm{Aut}\,\mathcal{T}_1 \subseteq A$. By a similar argument, $A \subseteq \mathrm{Aut}\,\mathcal{T}_1$. Therefore, $A = \mathrm{Aut}\,\mathcal{T}_1$.

By construction, the group G is a subgroup of $\mathrm{Aut}\,\mathcal{T}_i$, for $i = 1, 2$. The group G has index 2 in A and $A = G\langle t\rangle$. Therefore $G = \mathrm{Aut}\,\mathcal{T}_1 \cap \mathrm{Aut}\,\mathcal{T}_2$ if and only if $t \notin \mathrm{Aut}\,\mathcal{T}_2$. Now, by Lemma 1, $(B, Bi_H, Bj_H^{-1})^t = (Bt, Bi_Ht, Bj_H^{-1}t) = (B, Bi_K, Bj_K^{-1})$. Since there is no element $g \in G$ such that $(Bg, Bi_Hg, Bj_H^{-1}g) = (B, Bi_K, Bj_K^{-1})$, we have $t \notin \mathrm{Aut}\,\mathcal{T}_2$. Thus the proposition is proved. □

Theorem 8. *The group* $Q_8 \times C_3$ *is not a* CI-*group with respect to colour ternary relational structures.*

Proof. The groups H, K are regular subgroups of G. Furthermore, by Proposition 7, the group G is the automorphism group of a colour ternary relational structure (indeed, with just two colours: $\mathcal{T}_1, \mathcal{T}_2$). In particular, $\mathcal{T}_i$ is a Cayley ternary relational structure on $Q_8 \times C_3$, for $i = 1, 2$. So, it is enough to prove that H, K are not conjugate in G. We argue by contradiction. Let g be in G such that $H^g = K$. Clearly, $\langle i_H, j_H\rangle^g = \langle i_K, j_K\rangle$. Since $\langle i_H, j_H\rangle$ is a normal subgroup of G, we get a contradiction. □

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