ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 8 (2015) 381–408

Quartic integral Cayley graphs*

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Received 19 June 2013, accepted 11 February 2015, published online 5 June 2015

Abstract

We give exhaustive lists of connected 4-regular integral Cayley graphs and connected 4-regular integral arc-transitive graphs. An *integral* graph is a graph for which all eigenvalues are integers. A *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ for a given group Γ and *connection set* $S \subset \Gamma$ is the graph with vertex set Γ and with a connected to b if and only if $ba^{-1} \in S$. Up to isomorphism, we find that there are 32 connected quartic integral Cayley graphs, 17 of which are bipartite. Many of these can be realized in a number of different ways by using non-isomorphic choices for Γ and/or different choices for S. A graph is *arc-transitive* if its automorphism group acts transitively on the ordered pairs of adjacent vertices. Up to isomorphism, there are 27 quartic integral graphs that are arc-transitive. Of these 27 graphs, 16 are bipartite and 16 are Cayley graphs. By taking quotients of our Cayley or arc-transitive graphs we also find a number of other quartic integral graphs. Overall, we find 9 new spectra that can be realised by bipartite quartic integral graphs.

Keywords: Graph spectrum, integral graph, Cayley graph, arc-transitive, vertex-transitive bipartite double cover, voltage assignment, graph homomorphism.

Math. Subj. Class.: 05C50, 05C25

1 Introduction

We give exhaustive lists of connected 4-regular integral Cayley graphs and connected 4-regular integral arc-transitive graphs. For reasons which will become apparent, we first restrict our attention to the bipartite case.

An *integral* graph is a graph for which all eigenvalues of the adjacency matrix are integers. The *spectrum* of a graph is the eigenvalues with their multiplicity. Bipartite graphs have eigenvalues that are symmetric with respect to 0 and r-regular graphs have largest eigenvalue r with multiplicity equal to the number of connected components. For

*Research supported by ARC grant DP120100197.

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details we refer to [10]. Therefore for connected 4-regular bipartite integral graphs, the spectrum has the form $\{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$; which we abbreviate by simply specifying the quadruple $[x, y, z, w]$.

There are only finitely many connected 4-regular bipartite integral graphs. Cvetković [6] proved that the diameter D of a connected graph satisfies $D \leqslant s - 1$, where s is the number of distinct eigenvalues. For connected r-regular integral graphs, it follows that $R \leqslant D \leqslant 2r$ where R is the radius of the graph. Cvetković *et al.* [7] showed that the number of vertices in a connected r-regular bipartite graph is bounded above by $(2(r-1)^R - 2)/(r-2)$ if $r \geqslant 3$. Therefore, connected 4-regular bipartite integral graphs have at most 6560 vertices.

All graphs in this paper are simple, undirected, and have n vertices. Since a 4-regular graph is integral if and only if each of its components is integral, from this point on we will assume that all graphs are connected. We use the acronym QIG as shorthand for a connected quartic integral graph. Cvetković *et al.* [7] found quadruples $[x, y, z, w]$ that are candidates for the spectrum of a bipartite QIG. They called these *possible spectra*. Research activities regarding the set of possible spectra fall into two streams: eliminate possible spectra based on new information and/or techniques, or find graphs that realize a possible spectrum. Useful tools include an identity by Hoffman [11] and equations relating the spectral moments to the closed walks of length $\ell \leqslant 6$. All QIGs that avoid eigenvalues of ± 3 and realize a possible spectrum are found in [24]. Stevanović [23] eliminates spectra using equations arising from graph angles. In the same paper he determines that the possible values for n are between 8 and 1260, except for 5 identified spectra.

Stevanović *et al.* [25] extend the equations for the ℓ-th spectral moment to an inequality for $\ell = 8$. They make use of a correspondence between closed walks in an r-regular graph and walks in an infinite r-regular tree and find recurrence relations for the number of closed walks. The upper bound for n is improved to give $8 \leqslant n \leqslant 560$. Equations for $\ell \geqslant 8$ are found in [16] by counting a certain type of closed walk in terms of the counts of small subgraphs of the graph. All of the bipartite QIGs with $n \leqslant 24$ that realize one of the possible spectra were found and are listed with drawings in [25]. We give 12 new graphs that realize possible spectra from the set given in [25]. Of these graphs, 3 are co-spectral to an integral graph listed in [7]. Their spectra are $[4, 6, 4, 5]$ and $[6, 16, 10, 3]$, and $[9, 16, 19, 0]$. The spectra not previously known to be realized by a graph are $[3, 4, 1, 6]$, $[3, 5, 9, 0]$, $[5, 4, 7, 4]$, $[6, 12, 2, 9]$, $[8, 10, 16, 1]$, $[10, 14, 18, 2]$, $[12, 28, 4, 15]$, $[22, 28, 34, 5]$, and $[27, 28, 49, 0]$. Of the 12 graphs, 3 appear in the census of Potočnik *et al.* [17, 18] but were not recognized as integral.

We also list 49 new non-bipartite QIGs that, to our knowledge, do not appear anywhere in the literature. Of these graphs, only 3 appear in the census of Potočnik *et al.* [17, 18] but were not tested for integrality.

A *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ for a group Γ and *connection set* $S \subset \Gamma$ is the graph with vertex set Γ and with a connected to b if and only if $ba^{-1} \in S$. Let $\mathbb{Z}_t$, D_t, and Q_t denote the cyclic, dihedral, and quaternion groups of order t respectively, and S_t, A_t the symmetric and alternating groups of degree t.

Klotz and Sander [12] showed that if every Cayley graph $\mathrm{Cay}(\Gamma, S)$ over a finite Abelian group Γ is integral then $\Gamma \in \{\mathbb{Z}_2^s, \mathbb{Z}_3^s, \mathbb{Z}_4^s, \mathbb{Z}_2^s \times \mathbb{Z}_3^t, \mathbb{Z}_2^s \times \mathbb{Z}_4^t\}$, where $s \geqslant 1$, $t \geqslant 1$. The analogous result for non-Abelian Γ was determined independently by Abdollahi and Jazaeri [1] and Ahmady *et al.* [4]: if every Cayley graph $\mathrm{Cay}(\Gamma, S)$ over a finite non-Abelian group Γ is integral then $\Gamma \in \{S_3, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, Q_8 \times \mathbb{Z}_2^r\}$, where $r \geqslant 0$.

Estélyi and Kovács [8] considered the groups for which all Cayley graphs $\mathrm{Cay}(\Gamma, S)$ over a group Γ are integral if $|S| \leqslant k$. The authors proved that for $k \geqslant 6$, Γ consists only of the groups above: $\{\mathbb{Z}_2^s, \mathbb{Z}_3^s, \mathbb{Z}_4^s, \mathbb{Z}_2^s \times \mathbb{Z}_3^t, \mathbb{Z}_2^s \times \mathbb{Z}_4^t\} \cup \{S_3, \mathbb{Z}_3 \rtimes \mathbb{Z}_4, Q_8 \times \mathbb{Z}_2^r\}$, $s \geqslant 1$, $t \geqslant 1$, $r \geqslant 0$. Moreover, for $k \in \{4, 5\}$ there is only one extra possibility, namely that Γ is the generalised dicyclic group with $\mathbb{Z}_{3^q} \times \mathbb{Z}_6$ as a subgroup of index 2, where $q \geqslant 1$.

Abdollahi and Vatandoost [2] showed that there are exactly 7 connected cubic integral Cayley graphs. They found that $\mathrm{Cay}(\Gamma, S)$ is integral for some S with $|S| = 3$ if and only if Γ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, S_3, $\mathbb{Z}_2^3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_8, $\mathbb{Z}_2 \times \mathbb{Z}_6$, D_{12}, A_4, S_4, $D_8 \times \mathbb{Z}_3$, $D_6 \times \mathbb{Z}_4$ or $A_4 \times \mathbb{Z}_2$.

A set of possible orders for Cayley QIGs on finite Abelian groups have been determined by Abdollahi *et al.* [3]. They showed that for an Abelian group, Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\},$$

but they did not establish whether Cayley QIGs of these orders exist. We find that the precise set of orders of Cayley QIGs on Abelian groups is $\{5, 6, 8, 9, 10, 12, 16, 18, 24, 36\}$. More generally, we consider all groups and find that many Cayley QIGs are on non-Abelian groups. Thus, we show that for any group Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 72, 120\}.$$

Furthermore, for each of these orders Cayley QIGs exist.

For a given Cayley graph G, there may exist many different pairs (Γ, S) of groups Γ and connection sets S such that $G \cong \mathrm{Cay}(\Gamma, S)$. We call isomorphic Cayley graphs on the same group Γ *equivalent* if their connection sets are from the same orbit of the automorphism group of Γ (see for example [13]):

Definition 1.1. Let Γ be a group and $\mathrm{Aut}(\Gamma)$ be the automorphism group of Γ. If Cayley graph $\mathrm{Cay}(\Gamma, S) \cong \mathrm{Cay}(\Gamma, T)$ and $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(\Gamma)$ then $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, T)$ are *equivalent*.

Any other connection sets give non-equivalent Cayley Graphs. Cayley graphs from different groups are non-equivalent. There are, up to isomorphism, only 32 connected quartic integral Cayley graphs; but each graph is realized in up to 18 non-equivalent ways. Of the 32 graphs, 17 are bipartite.

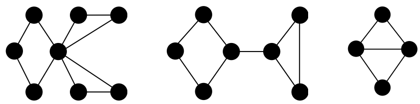
A graph is *arc-transitive* if its automorphism group acts transitively on the ordered pairs of adjacent vertices. There are, up to isomorphism, only 27 connected quartic integral graphs that are arc-transitive. Of the 27 graphs, 16 are bipartite, 5 of which are not Cayley graphs.

In Section 2 we find that most of the feasible spectra from [25] cannot be realized by vertex-transitive QIGs. Section 3 summarises the algorithm used for finding all of the bipartite Cayley QIGs. Section 4 gives our main results. It includes tables giving the details of the Cayley QIGs and the bipartite arc-transitive QIGs, some drawings, and some non-bipartite QIGs that result from finding quotients of our bipartite graphs.

2 Vertex-transitive quartic integral graphs

A graph is *vertex-transitive* if its automorphism group acts transitively on its vertices. In this section, our aim is to compile a set Ξ that includes all possible spectra that might be realized by a vertex-transitive QIG, but is otherwise as small as we can make it. Initially we take Ξ to be all possible spectra from [25], and candidates will be progressively removed from the set as we work through this section.

We will need some notation for (unlabelled) subgraphs. We let C_i denote the i-cycle, $C_{i_1 \cdot i_2 \cdots i_h}$ denote i_j-cycles sharing a single vertex for $j = 1, \ldots, h$, $C_{i_1 - i_2}$ an i_1-cycle joined to an i_2-cycle by an edge, and $\Theta_{i_1, i_2, \ldots, i_h}$ two vertices joined by internally disjoint paths of lengths i_j for $j = 1, \ldots, h$. Examples of this notation for subgraphs appear in Figure 1.



(a) $C_{4 \cdot 3 \cdot 3}$ (b) C_{4-3} (c) $\Theta_{2,2,1}$

Figure 1: Subgraph notation

If at any point we encounter subgraphs that cannot be described by our notation, we draw a picture of the subgraph like those in Figure 1. For any graph H, let $[H]$ denote the number of subgraphs of G that are isomorphic to H, where the parent graph G will be implicitly specified by the context.

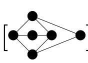
In [7], Equations (2.1) and (2.2) are used to determine $[C_4]$ and $[C_6]$ for a given $[x, y, z, w]$.

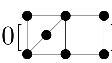
$$2(4^4 + 3^4 x + 2^4 y + z) = 28n + 8[C_4], \tag{2.1}$$

$$2(4^6 + 3^6 x + 2^6 y + z) = 232n + 144[C_4] + 12[C_6]. \tag{2.2}$$

In [16], these equations were extended to higher spectral moments of general regular graphs. By specialising to 4-regular bipartite graphs, we obtain the following equations:

$$\begin{aligned} 2(4^8 + 3^8 x + 2^8 y + z) &= 2092n + 2024[C_4] + 288[C_6] + 16[C_8] + 32[C_{4\cdot 4}] \\ &\quad + 96[\Theta_{2,2,2,2}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,3,1}], \\ 2(4^{10} + 3^{10} x + 2^{10} y + z) &= 19864n + 26160[C_4] + 4860[C_6] + 480[C_8] + 20[C_{10}] \\ &\quad + 960[C_{4\cdot 4}] + 40[C_{4-4}] + 40[C_{6\cdot 4}] + 1440[\Theta_{2,2,2}] \\ &\quad + 520[\Theta_{3,3,1}] + 2880[\Theta_{2,2,2,2}] + 40[\Theta_{4,2,2}] + 20[\Theta_{5,3,1}] \\ &\quad + 120[\Theta_{3,3,3,1}] + 120[\Theta_{4,2,2,2}] + 120[\;\cdot\;] + 80[\;\cdot\;]. \end{aligned} \tag{2.3}$$





The *girth* of a graph is the length of the shortest cycle contained in the graph. We use Equations (2.3) to determine the girth where $[C_4] = [C_6] = 0$ for a given $[x, y, z, w]$ and also to determine the values for $[C_8]$ and $[C_{10}]$ where possible. Vertex-transitive graphs have the same number of i-cycles incident with each vertex, so the number of vertices divides $i[C_i]$. We apply this observation for $i \in \{4, 6, 8, 10\}$ to the possible spectra for which the value of $[C_i]$ can be deduced. We eliminate those quadruples that cannot be realized by a vertex-transitive QIG from Ξ.

For example, if we consider $[5, 6, 11, 1]$ with $n = 48$, $[C_4] = 24$, and $[C_6] = 140$ then

$$\frac{4[C_4]}{n} = \frac{4(24)}{48} = 2 \in \mathbb{N} \text{ but } \frac{6[C_6]}{n} = \frac{6(140)}{48} = \frac{35}{2} \notin \mathbb{N},$$

where $\mathbb{N}$ denotes the set of non-negative integers. Thus $[5, 6, 11, 1]$ is eliminated from Ξ. In contrast, for $[12, 12, 20, 3]$ with $n = 96$, $[C_4] = 24$, $[C_6] = 128$, and $[C_8] = 528$. We are able to find $[C_8]$ from (2.3) by deducing that $[C_{4\cdot4}] = [\Theta_{2,2,2,2}] = [\Theta_{2,2,2}] = [\Theta_{3,3,1}] = 0$ because there is only one 4-cycle incident with each vertex. In fact,

$$\frac{4[C_4]}{n} = \frac{4(24)}{96} = 1 \in \mathbb{N}, \quad \frac{6[C_6]}{n} = \frac{6(128)}{96} = 8 \in \mathbb{N}, \text{ and } \frac{8[C_8]}{n} = \frac{8(528)}{96} = 44 \in \mathbb{N}.$$

In this case, $[C_{10}]$ cannot be determined from (2.3), so we consider it unknown. Thus $[12, 12, 20, 3]$ remains in Ξ.

It is also plausible to eliminate quadruples from Ξ using arguments specific to particular cases. We give one example to demonstrate the possibility. Consider $[24, 4, 40, 3]$ with $[C_4] = 72$ and $[C_6] = 0$. There are $4(72)/144 = 2$ copies of C_4 incident at each vertex. Since $[C_6] = 0$, we know $[\Theta_{3,3,1}] = 0$. Also, with only two 4-cycles at each vertex, $[\Theta_{2,2,2,2}] = [\Theta_{2,2,2}] = 0$. Since two 4-cycles meet at exactly one vertex of a $C_{4\cdot4}$, $[C_{4\cdot4}] = 144$. From Equation (2.3) we get that,

$$2(4^8 + 3^8(24) + 2^8(4) + 40) = 2092(144) + 2024(72) + 16[C_8] + 32(144),$$

which gives the contradiction $[C_8] = -216$. Thus we remove $[24, 4, 40, 3]$ from Ξ. This entry is underlined in Table 1.

We eliminate two quadruples from Ξ using the following Lemma [5, Prop. 16.6]:

Lemma 2.1. *Let G be a vertex-transitive graph which has degree r and an even number of vertices. If λ is a simple eigenvalue of G, then λ is one of the integers $2\alpha - r$ for $0 \leqslant \alpha \leqslant r$.*

The orders associated with the eliminated quadruples are 36 and 72. Both entries have 1 as a simple eigenvalue. These entries are underlined and highlighted in bold in Table 1.

Using the above methods, we reduced the set Ξ from the initial 828 possible spectra to 59 quadruples in the final version. Henceforth Ξ will refer to this final set of 59 quadruples (see Appendix A).

n	Girth	n	Girth	n	Girth	n	Girth
8	4	36	$\underline{\mathbf{4}}$,4,4,q^{7}	96	4,q^{27},h^{3}	240	6,8,q^{30},h^{2}
10	4	40	4,q^{10}	112	q^{34},h^{2}	252	q^{28},h^{3}
12	4,4	42	4,q^{14}	120	4,4,4,6,6,q^{28}	280	8,q^{23},h^{2}
14	q^{1}	48	4,q^{12},h^{2}	126	4,6,q^{38},h^{1}	288	6,q^{21},h^{1}
16	4,q^{1}	56	q^{16},h^{2}	140	q^{40},h^{2}	336	q^{14},h^{2}
18	4,q^{1}	60	4,4,4,4,6,q^{15}	144	$\underline{4}$,4,6,q^{31},h^{1}	360	6,6,8,q^{11}
20	4,q^{3}	70	6,q^{23}	160	q^{33},h^{2}	420	8,q^{5},h^{1}
24	4,4,4,q^{3}	72	$\underline{\mathbf{4}}$,4,4,4,6,q^{18}	168	6,q^{35},h^{2}	480	8,q^{2}
28	q^{8}	80	q^{22},h^{2}	180	4,6,6,6,q^{38}	504	h^{1}
30	4,4,6,q^{6}	84	q^{23},h^{7}	210	6,q^{35},h^{2}	560	10
32	6,q^{8}	90	4,4,6,6,q^{27}	224	q^{32},h^{3}		

Table 1: Finding the set Ξ

Table 1 summarizes the process of finding Ξ. For every order, we consider each $[x, y, z, w]$ and check whether we get integer counts at each vertex for each C_i where $[C_i]$ is known and $i \in \{4, 6, 8, 10\}$. A 'q^j' in the table denotes that for the given n there were j possible spectra eliminated because $4[C_4]/n \notin \mathbb{N}$. An '$h^j$' in the table denotes that for the given n there were j possible spectra which satisfied $4[C_4]/n \in \mathbb{N}$ that were eliminated because $6[C_6]/n \notin \mathbb{N}$. If $i[C_i]/n \in \mathbb{N}$ for all i where $[C_i]$ is known for a possible spectra, then the girth is recorded. Thus an entry of 4,4,6,q^6 indicates that there are three possible spectra in Ξ associated with that order. If those quadruples are all realized by graphs (where a graph in this case may actually be a set of cospectral graphs) then two graphs will have girth 4 and the other will have girth 6. It also indicates that 6 possible spectra with $4[C_4]/n \in \mathbb{N}$ were eliminated because $6[C_6]/n \notin \mathbb{N}$.

3 The algorithm

In this section we outline our method for finding bipartite Cayley QIGs, using the set Ξ compiled in Section 2.

Define Ω to be the set of orders associated with the spectra in Ξ. Cayley graphs are vertex-transitive, so we only consider groups Γ of order $n \in \Omega$. To reduce the number of groups to be considered, we use a result similar to one in [18]. Let Γ' denote the commutator subgroup of a group Γ.

Lemma 3.1. *Let Γ be a finite group and let $\mathrm{Cay}(\Gamma, S)$ be a connected Cayley graph of degree at most 4. Then Γ/Γ' is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$ with $a \geqslant 2$; $\mathbb{Z}_a \times \mathbb{Z}_b$ with $a, b \geqslant 2$; or $\mathbb{Z}_a$ with $a \geqslant 1$.*

Proof. Since $\mathrm{Cay}(\Gamma, S)$ is connected and has degree at most 4, Γ is generated by an inverse-closed set of at most 4 elements. This must also be true of the quotient group Γ/Γ'. Now since Γ/Γ' is Abelian, the result follows. □

By Lemma 3.1, we need only consider groups Γ with Γ/Γ' isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$, $\mathbb{Z}_a \times \mathbb{Z}_b$, or $\mathbb{Z}_a$. We denote the set of groups that satisfy this property by Φ.

To construct connected simple undirected 4-regular Cayley graphs $\mathrm{Cay}(\Gamma, S)$, we considered inverse-closed sets S of four non-identity elements of Γ that generate Γ. The search was pruned by placing additional restrictions on S. Let g denote the girth of the graph $\mathrm{Cay}(\Gamma, S)$.

- Since $\mathrm{Cay}(\Gamma, S)$ is bipartite, the order of s is even for each $s \in S$.
- If $s_1, s_2 \in S$ and $s_1 \neq s_2^{-1}$, then the order of $s_1 s_2$ is at least $g/2$ (in particular non-involutions have order no smaller than the girth).
- For any set of connection sets that result in equivalent Cayley graphs (in the sense of Definition 1.1), only one representative is chosen.

We note that the minimum girth possible for $\mathrm{Cay}(\Gamma, S)$ is given by Table 1.

We summarize the results of our computations in Table 2. The values for $n \in \Omega$ appear as the first column and in the second column the number of groups of order n is given. (We reiterate that Ω does not include orders eliminated by the vertex-transitive tests of Section 2). The number of groups in Φ of order n are listed in column three. Column 4 contains the number of connection sets S among the groups counted by column 3, subject to the restrictions on S given above. The graphs $\mathrm{Cay}(\Gamma, S)$ that are bipartite are counted in column 5. The number of isomorphism classes of these graphs appears in column 6. The number of isomorphism classes of integral graphs is recorded in column 7. The last column gives the number of isomorphism classes of arc-transitive integral graphs. A '-' indicates that there are no integral graphs to consider.

n	#Groups Γ	#$\Gamma \in \Phi$	#Sets S	#Bipartite $\mathrm{Cay}(\Gamma, S)$	#Isomorphism Classes	#Integral	#Arc-Transitive
8	5	5	13	7	1	1	1
10	2	2	2	2	1	1	1
12	5	5	19	11	3	2	1
16	14	14	66	44	5	1	1
18	5	5	12	12	5	1	1
20	5	5	34	20	8	0	-
24	15	15	151	98	23	3	1
30	4	4	31	31	17	1	1
32	51	48	58	51	16	1	1
36	14	14	149	105	48	1	1
40	14	14	201	146	54	1	0
42	6	6	55	55	36	0	-
48	52	51	840	616	177	1	0
60	13	13	385	281	161	0	-
70	4	4	96	96	73	0	-
72	50	49	1014	765	338	2	1

90	10	10	236	236	175	0	-
96	231	218	4434	3545	1292	0	-
120	47	47	2833	1968	1123	1	1
126	16	16	427	427	346	0	-
144	197	190	6563	5350	2722	0	-
168	57	57	2388	2212	1601	0	-
180	37	37	2927	2497	1883	0	-
210	12	12	1172	1172	1017	0	-
240	208	205	10884	9885	6791	0	-
280	40	40	4080	3929	3223	0	-
288	1045	968	26391	24815	15695	0	-
360	162	160	15928	14703	11524	0	-
420	41	41	10558	10204	9271	0	-
480	1213	1148	68179	63804	48322	0	-
560	180	177	21764	21433	18704	0	-

Table 2: Results at each algorithm step

4 Quartic integral graphs

In this section we present the graphs that our computations discovered, starting with the bipartite Cayley case.

4.1 Bipartite Cayley integral graphs

As a result of the computation described in Section 3, we have:

Theorem 4.1. *There are precisely* 17 *isomorphism classes of connected* 4*-regular bipartite integral Cayley graphs, as detailed in Table 3.*

For each bipartite Cayley QIG in Table 3 we give n and the spectrum $[x, y, z, w]$. Graphs appearing in the paper by Cvetković *et al.* [7] are labelled $I_{n,index}$ as in that paper. If the graph is in the census of Potočnik *et al.* [17, 18] then we give the index in their notation: AT4Val[n][index]. In two columns, we give the groups and connection sets that give rise to each Cayley graph. The first column contains the group, Γ, with a presentation of that group. We stick as close as possible to the convention of using generators in $\{a, b, c, d, e\}$ for cyclic groups, $\{s, t, u, v\}$ for symmetric or alternating groups, and $\{r, f\}$ for the quaternion group, the dihedral group, or the quasidihedral group. The last column contains the number of involutions in the connection set, S, followed by the connection set itself in terms of the generators from the previous column.

Group	Connection Sets (#involutions S)

$\mathbf{G_1: n = 8 \quad [0,0,0,3] \quad I_{8,1} \quad AT4Val[8][1]}$

$\mathbb{Z}_8$ $< a \mid a^8 >$	0 $\{a, a^3, a^5, a^7\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 >$	2 $\{a, b, a^3, a^2b\}$ 0 $\{a, ab, a^3, a^3b\}$
D_8 $< r, f \mid r^4, f^2, (rf)^2 >$	4 $\{f, fr, fr^2, rf\}$ 2 $\{f, r, fr^2, frf\}$
Q_8 $< r, f \mid r^4, f^4, r^2f^2, rfrf^{-1} >$	0 $\{r, f, r^3, r^2f\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 >$	4 $\{a, b, c, abc\}$

$\mathbf{G_2: n = 10 \quad [0,0,4,0] \quad I_{10,1} \quad AT4Val[10][2]}$

D_{10} $< r, f \mid r^4, f^2, (rf)^2 >$	4 $\{f, fr, fr^2, r^2f\}$
$\mathbb{Z}_{10}$ $< a \mid a^{10} >$	0 $\{a, a^3, a^7, a^9\}$

$\mathbf{G_3: n = 12 \quad [0,2,0,3] \quad I_{12,4} \quad AT4Val[12][2]}$

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1} >$	0 $\{b, b^3, ba, b^3a\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	0 $\{a, a^5, a^7, a^{11}\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	2 $\{r^2f, f, r^5, r\}$ 4 $\{r^4f, rf, r^2f, r^5f\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	0 $\{a^5, a^2b, a, a^4b\}$

$\mathbf{G_4: n = 12 \quad [0,1,4,0] \quad I_{12,2}}$

D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	2 $\{rf, r^3, r, r^5\}$ 4 $\{rf, r^3, r^5f, r^3f\}$ 4 $\{rf, r^4f, r^5f, r^3f\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	2 $\{a^3, b, a^5, a\}$

$\mathbf{G_5} : \mathbf{n = 16} \quad [\mathbf{0, 4, 0, 3}] \quad \mathbf{I_{16,1}} \quad$ AT4Val[16][1]

$\mathbb{Z}_4 \times \mathbb{Z}_4$ $< a \mid a^4 > \times < b \mid b^4 >$	$0\ \{a, b, a^3, b^3\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, (bc)^2, baca^{-1}c >$	$2\ \{ac, a^2bc, a^3bc, a^2c\}$ $2\ \{bc, a^3b, a^2c, ab\}$ $0\ \{a, a^3c, a^3, abc\}$
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ $< a, b \mid a^4, b^4, aba^{-1}b >$	$0\ \{a, a^3ba, a^3, b\}$
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ $< a, b \mid a^8, b^2, aba^3b >$	$0\ \{a, ab, a^3b, a^7\}$
QD_{16} $< r, f \mid r^8, f^2, rfr^5f >$	$2\ \{r, r^4f, r^6f, r^7\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 > \times < c \mid c^2 >$	$2\ \{a, b, c, a^3\}$
$\mathbb{Z}_2 \times D_8$ $< a \mid a^2 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$4\ \{a, f, r^3f, r^2f\}$ $4\ \{f, r^3f, af, rf\}$ $4\ \{f, r^3f, af, arf\}$ $2\ \{a, r, f, r^3\}$ $2\ \{a, r, af, r^3\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 > \times < d \mid d^2 >$	$4\ \{a, b, c, d\}$

$\mathbf{G_6} : \mathbf{n = 18} \quad [\mathbf{0, 4, 4, 0}] \quad \mathbf{I_{18,1}} \quad$ AT4Val[18][2]

$\mathbb{Z}_3 \times S_3$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{s, st, ats, a^2ts\}$ $0\ \{sa, sa^2, sat, sa^2t\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^3, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >$	$4\ \{c, ca, cb, cab\}$
$\mathbb{Z}_6 \times \mathbb{Z}_3$ $< a \mid a^6 > \times < b \mid b^3 >$	$0\ \{a, a^5, a^3b, a^3b^2\}$

$\mathbf{G_7} : \mathbf{n = 24} \quad [\mathbf{0, 8, 0, 3}] \quad \mathbf{I_{24,2}} \quad$ AT4Val[24][1]

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{s, st, at, a^3sts\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, a^3(cb)^2 >$	$2\ \{a^3c, a^2c, ab, a^5b\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$0\ \{ar^3f, a^2r^3f, ar^3, a^2r\}$
S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$4\ \{st^2sts, t^2st, stst^2s, tst^2\}$ $0\ \{ts, st^2, ststs, tst\}$ $2\ \{st^2sts, tst^2, ts, st^2\}$

$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$0\ \{ast, astst, asts, atst\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4\ \{s, bs, st, ast\}$

$\mathbf{G_8}$: $\mathbf{n = 24}$ $\mathbf{[2, 2, 6, 1]}$ $\mathbf{I_{24,3}}$

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{a, a^3, s, st\}$ $2\ \{s, st, ats, a^3ts\}$
D_{24} $< r, f \mid r^{12}, f^2, (rf)^2 >$	$4\ \{f, rf, r^5f, r^6f\}$ $2\ \{f, r^3, r^9, r^8f\}$
$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0\ \{c, c^3, ab, ac^3bc\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^4 >$	$4\ \{c, b, ca, cbc\}$ $2\ \{c, bcb, ba, bcac\}$ $2\ \{c, ca, cbcac, bac\}$ $0\ \{cb, bc, ba, bcac\}$
$\mathbb{Z}_{12} \times \mathbb{Z}_2$ $< a \mid a^{12} > \times < b \mid b^2 >$	$0\ \{a^3, a^9, a^4b, a^8b\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$2\ \{r^3f, rf, a^2f, af\}$ $0\ \{r, r^3, ar^3f, a^2r^3f\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4\ \{s, b, a, st\}$ $4\ \{s, b, st, ats\}$ $4\ \{s, sb, ast, ats\}$ $2\ \{s, b, at, asts\}$ $2\ \{s, sb, at, asts\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 > \times < c \mid c^2 >$	$2\ \{a^3, b, a^2c, a^4c\}$

$\mathbf{G_9}$: $\mathbf{n = 24}$ $\mathbf{[3, 0, 5, 3]}$ $\mathbf{I_{24,4}}$

S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$4\ \{s, t^2st, st^2sts, stst^2st\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$2\ \{a, as, at^2s, ast\}$

$\mathbf{G_{10}}$: $\mathbf{n = 30}$ $\mathbf{[0, 10, 4, 0]}$ $\mathbf{I_{30,1}}$ $\mathbf{AT4Val[30][4]}$

$\mathbb{Z}_5 \times S_3$ $< a \mid a^5 > \times < s, t \mid s^2, t^3, (st)^2 >$	$0\ \{as, a^2st, a^4s, a^3st\}$
D_{30} $< r, f \mid r^{15}, f^2, (rf)^2 >$	$4\ \{f, r^2f, r^3f, r^{11}f\}$

$\mathbf{G_{11}}$: $\mathbf{n = 32}$ $\mathbf{[0, 12, 0, 3]}$ $\mathbf{I_{32,1}}$ AT4Val[32][4]

$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$ $< a, b \mid a^8, b^4, ab^2a^{-1}b^2, aba^3b^{-1} >$	$0\ \{a, a^7, ab, a^3b^3\}$
$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, a^2ba^6b, (aac)^2, (bc)^2, ba^{-1}cac >$	$2\ \{a^4c, a^2c, a^7bc, a^5c\}$
$\mathbb{Z}_2.((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) = (\mathbb{Z}_2 \times \mathbb{Z}_2).(\mathbb{Z}_4 \times \mathbb{Z}_2)$ $< a, b \mid a^8, b^4, ab^2a^{-1}b^2, a^4b^2, aba^{-1}b^{-1}ab^{-1}a^{-1}b, aba^6ba >$	$0\ \{ba, a^3b, a^3, a^5\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^4, c^2, aba^{-1}b^{-1}, aca^3c, (bbc)^2(bc)^4, a^3(bc)^2 >$	$2\ \{ab^2c, c, bc, a^3bc\}$
$\mathbb{Z}_4.D_8 = \mathbb{Z}_4.(\mathbb{Z}_4 \times \mathbb{Z}_2)$ $< a, b \mid a^8, b^8, aba^3b, ab^{-1}a^3b^{-1}, ab^{-1}a^{-1}b^3 >$	$0\ \{a, a^7, a^7ba, a^4b\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^4, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >$	$4\ \{c, cb, ca, abc\}$
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, aba^{-1}b, aca^{-1}c, a^4bcbc, (bc)^4 >$	$2\ \{b, c, abc, a^3bc\}$
$\mathbb{Z}_2 \times QD_{16}$ $< a \mid a^2 > \times < r, f \mid r^8, f^2, rfr^5f >$	$2\ \{ar, r^3, r^5, r^2f\}$
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, a^4(bc)^2 >$	$4\ \{a^7c, a^2c, ac, a^4b\}$
$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^2, b^2, aba^{-1}b^{-1}, r(fa)^2, r^2(bf)^2 >$	$4\ \{r^2f, ar^2, rf, rab\}$
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^4, b^2, ara^{-1}r^{-1}, afa^{-1}f^{-1}, r^2f^2, rfrf^{-1},$ $(rrb)^2, r^2(ab)^2, brbr^{-1}f >$	$2\ \{r^2b, a, ar^3fb, ar^3b\}$
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^4, b^2, ara^{-1}r^{-1}, afa^{-1}f^{-1}, r^2f^2, rfrf^{-1},$ $(rb)^2, fbf^{-1}b^{-1}, arbar^{-1}b >$	$4\ \{b, a, br, brf\}$

$\mathbf{G_{12}}$: $\mathbf{n = 36}$ $\mathbf{[4, 4, 4, 5]}$ $\mathbf{I_{36,3}}$ AT4Val[36][3]

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^3 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0\ \{ac, a^2c^3, a^2cb, ac^3b\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, (acc)^2, acac^{-1}b^{-1} >$	$0\ \{a^2b^2c^3, b^2c, a^2bc^3, ac\}$
$S_3 \times S_3$ $< s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^3, (uv)^2 >$	$4\ \{u, s, uv, st\}$ $4\ \{u, uv, svu, stvu\}$ $2\ \{su, stu, tsv, tsuvu\}$ $0\ \{tu, stsu, sv, suvu\}$

$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a, a^5, a^3s, a^3st\}$ $2\ \{a^3s, a^3st, a^2ts, a^4ts\}$ $0\ \{as, a^3t, a^5s, a^3sts\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$ $< a \mid a^2 > \times < b,c,d \mid b^3, c^3, d^2, bcb^{-1}c^{-1}, (bd)^2, (cd)^2 >$	$4\ \{d, dc, adb, adcdbd\}$ $2\ \{d, dc, ab, adbd\}$
$\mathbb{Z}_6 \times \mathbb{Z}_6$ $< a \mid a^6 > \times < b \mid b^6 >$	$0\ \{a, a^5, b, b^5\}$

$\mathbf{G_{13}: \ n = 40 \quad [4, 6, 4, 5]}$

$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b,c \mid b^5, c^4, cbc^{-1}b^2, cb^2c^{-1}b^{-1} >$	$2\ \{ac^2, ac^2b, c, c^3\}$

$\mathbf{G_{14}: \ n = 48 \quad [6, 4, 10, 3] \quad I_{48,1}}$

$\mathbb{Z}_2 \times \mathbb{Z}_4 \times S_3$ $< a \mid a^2 > \times < b \mid b^4 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{s, a, bt, b^3sts\}$
$D_8 \times S_3$ $< r,f \mid r^4, f^2, (rf)^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	$4\ \{s, rfs, fts, r^2fst\}$ $4\ \{rf, rfs, fts, r^2fst\}$ $2\ \{s, rfs, r^3t, rsts\}$ $2\ \{rf, rfs, r^3t, rsts\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $< a \mid a^2 > \times < b,c,d \mid b^6, c^2, d^2, bcb^{-1}c^{-1}, (bbd)^2, b^3(dc)^2 >$	$4\ \{a, c, b^4d, b^3d\}$
$\mathbb{Z}_6 \times D_8$ $< a \mid a^6 > \times < r,f \mid r^4, f^2, (rf)^2 >$	$2\ \{a^3, a^3r^3f, ar, a^5r^3\}$
$\mathbb{Z}_2 \times S_4$ $< a \mid a^2 > \times < s,t \mid s^2, t^3, (st)^4 >$	$4\ \{a, s, stst^2s, st^2sts\}$ $4\ \{as, at^2st, atst^2, stst^2st\}$ $2\ \{s, astst^2, atst^2s, atst^2st\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < b \mid b^2 > \times < s,t \mid s^2, t^3, (st)^3 >$	$2\ \{a, abtst^2, abtst, abt^2st^2\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	$4\ \{s, b, cst, ats\}$

$\mathbf{G_{15}: \ n = 72 \quad [6, 16, 10, 3] \quad AT4Val[72][12]}$

$\mathbb{Z}_3 \times S_4$ $< a \mid a^3 > \times < s,t \mid s^2, t^3, (st)^4 >$	$0\ \{ast, a^2tst, a^2t^2s, aststs\}$
$(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ $< a,s,t,b \mid a^3, s^2, t^3, b^2, asa^{-1}s^{-1}, ata^{-1}t^{-1}, stbsbt^{-1}, (ab)^2,$ $(tb)^2, (st)^3 >$	$4\ \{atb, ab, tsbt, tbs\}$
$A_4 \times S_3$ $< s,t \mid s^2, t^3, (st)^3 > \times < u,v \mid u^2, v^3, (uv)^2 >$	$0\ \{tu, t^2u, tsuv, st^2uv\}$

$\mathbb{Z}_6 \times A_4$ $< a \mid a^6 > \times < s, t \mid s^2, t^3, (st)^3 >$	$0\ \{ast, a^3t, a^3t^2, a^5stst\}$

$\mathbf{G_{16}}$: $\mathbf{n = 72}$ $[\mathbf{8, 10, 16, 1}]$

$(\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)) \rtimes \mathbb{Z}_2$ $< a, b, c, d \mid a^3, b^3, c^4, d^2, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bdb^{-1}d^{-1},$ $adad^{-1}, bcbc^{-1}, c^2d^2 >$	$2\ \{dc, dacb, ad, d^2ad\}$
$(\mathbb{Z}_6 \times S_3) \rtimes \mathbb{Z}_2$ $< a, b, c, d \mid a^2, b^4, c^3, d^3, (ab^{-1})^2, acac^{-1}, (ad)^2, cbcb^{-1},$ $bdb^{-1}d^{-1}, cdc^{-1}d^{-1} >$	$4\ \{a, ab^2, abd, abacda\}$ $2\ \{ab, abcd, cb, b^2cb\}$
$\mathbb{Z}_6 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^6 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0\ \{ab^2, a^5b, a^3b^2c, a^3b^2c^3\}$
$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $< a \mid a^3 > \times < b, c, d \mid$ $b^6, c^2, d^2, bcb^{-1}c^{-1}, (bbd)^2, b^3(dc)^2 >$	$2\ \{b^5d, b^2d, a^2b^4c, ab^2c\}$ $0\ \{b^2cd, b^5cd, a^2b^4c, ab^2c\}$
$(S_3 \times S_3) \rtimes \mathbb{Z}_2$ $< s, t, u, v, a \mid s^2, t^3, u^2, v^3, a^2, tvt^{-1}v^{-1}, (uv)^2, (av)^2,$ $svst^{-1}, asasu >$	$4\ \{a, sastsat, s, sast\}$ $2\ \{sas, stsa, atsa, asat^2\}$ $2\ \{asa, atsat^2, saststst, asastsat\}$ $0\ \{sa, as, atsat, asastst^2\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b, c, d \mid$ $b^3, c^3, d^4, bcb^{-1}c^{-1}, (bdd)^2, bdbd^{-1}c^{-1} >$	$2\ \{ad^2, ab^2c^2d^2, ab^2d^3, ab^2cd\}$ $0\ \{ab^2cd, ab^2d^3, abc^2, ab^2c\}$
$\mathbb{Z}_2 \times S_3 \times S_3$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid$ $u^2, v^3, (uv)^2 >$	$4\ \{u, s, auvst, atsvu\}$ $4\ \{u, au, suv, stvu\}$ $2\ \{u, s, atv, astsuvu\}$
$\mathbb{Z}_2 \times \mathbb{Z}_6 \times S_3$ $< a \mid a^2 > \times < b \mid b^6 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{ts, ab^3ts, bsts, b^5t\}$

$\mathbf{G_{17}}$: $\mathbf{n = 120}$ $[\mathbf{12, 28, 4, 15}]$ AT4Val[120][4]

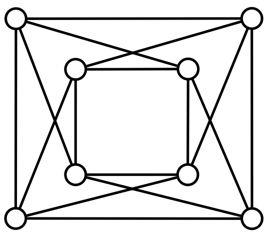
S_5 $< s, t \mid s^2, t^5, (st)^4, (st^2st^3)^2 >$	$0\ \{t^2st^3, tst^2st^2st, st^2stst, tst^4\}$ $4\ \{t(st)^2tst^4, st^2(st)^2t, (t^2s)^2ts,$ $(st^2)^2st\}$
$\mathbb{Z}_2 \times A_5$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^5 >$	$2\ \{a(tst^2s)^2t, ast(ts)^2, ast^2(st)^2,$ $a(st)^3ts\}$
$S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ $< s, t \mid s^2, t^3, (st)^2 > \times < a, b \mid$ $a^5, b^4, ab^{-1}a^2b, a^2b^{-1}a^{-1}b >$	$2\ \{sb^2, stb^2a, tb^3, stsb\}$
$\mathbb{Z}_5 \times S_4$ $< a \mid a^5 > \times < s, t \mid s^2, t^3, (st)^4 >$	$0\ \{a^2st, a^3t^2s, aststs, a^4tst\}$

$(\mathbb{Z}_5 \times A_4) \rtimes \mathbb{Z}_2$ $< a, s, t, b \mid a^5, s^2, t^3, b^2, asa^{-1}s^{-1}, ata^{-1}t^{-1}, bsbt^{-1}st,$ $(st)^3, (tb)^2, (ab)^2 >$	4	$\{tba^2, bta, btbsb, tabs\}$

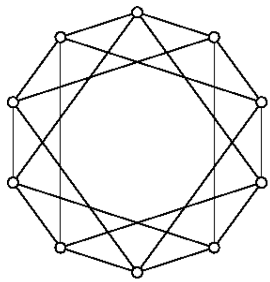
Table 3: Bipartite Cayley QIGs

Drawings for all but the three largest bipartite Cayley QIGs appear below. With over 70 vertices, it is difficult to present G_{15}, G_{16}, and G_{17} clearly.

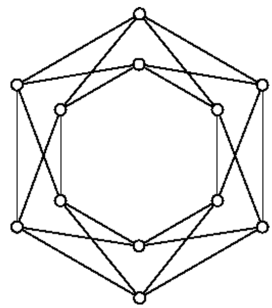
$\mathbf{G_1}$



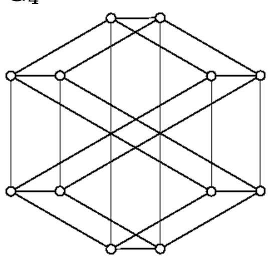
$\mathbf{G_2}$



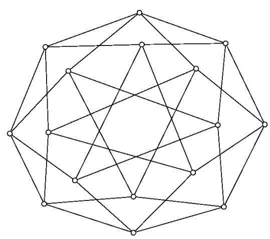
$\mathbf{G_3}$



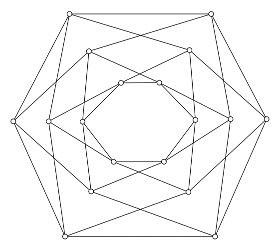
$\mathbf{G_4}$



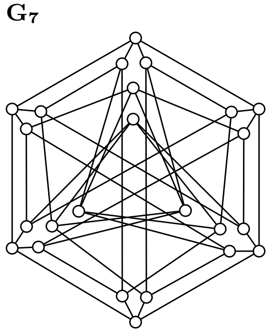
$\mathbf{G_5}$



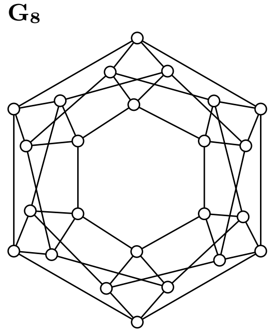
$\mathbf{G_6}$



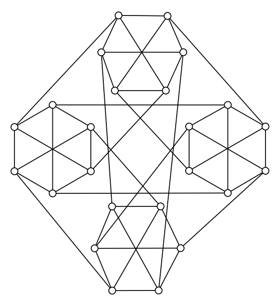
$\mathbf{G_7}$



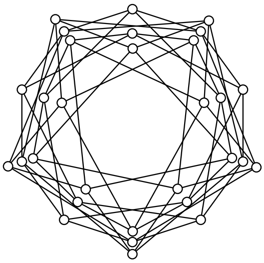
$\mathbf{G_8}$



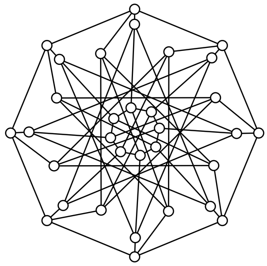
$\mathbf{G_9}$



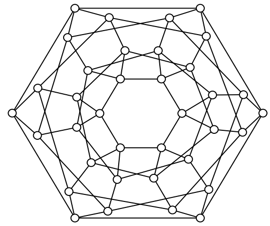
$\mathbf{G_{10}}$



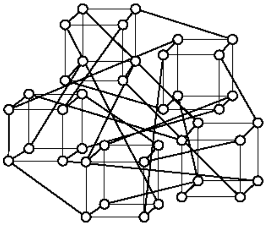
$\mathbf{G_{11}}$



$\mathbf{G_{12}}$



$\mathbf{G_{13}}$



$\mathbf{G_{14}}$

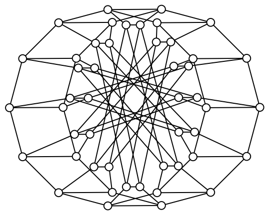


Table 4: Drawings of quartic bipartite integral Cayley graphs G_1 to G_{14}

4.2 Bipartite arc-transitive integral graphs

We considered all arc-transitive 4-regular graphs from the census of Potočnik *et al.* [17, 18] and tested them for integrality. The only arc-transitive bipartite QIGs that are not Cayley and thus not accounted for in Table 3 are the five that appear in Table 5. We let $[\Gamma : H] = \{Ha : a \in \Gamma\}$ denote the set of right cosets of $H \in \Gamma$. A *Schreier coset graph* $\mathrm{Sch}(\Gamma, H, HSH)$ for a group Γ, subgroup $H \leqslant \Gamma$, and connection set $S \subset \Gamma$ is the graph with vertex set $[\Gamma : H]$ and with Ha connected to Hb if and only if $ba^{-1} \in HSH$. We represent these 5 graphs as Schreier coset graphs. We give the order n and the spectrum $[x, y, z, w]$ followed by the graph index from [17, 18]. Graphs appearing in the paper by Cvetković *et al.* are labelled with the notation of [7]: $I_{n,index}$. The first line consists of the group Γ, with a presentation of that group. The second line consists of the subgroup H and its generators in terms of the generators of Γ followed by the connection set S in terms of the generators of Γ.

Group Subgroup, Subset
$\mathbf{F_1: \ n = 60 \quad [4, 16, 4, 5] \quad I_{60,1} \quad AT4Val[60][4]}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_5 : \quad < a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^5, (st)^4, (st^2st^3)^2 >$ $D_8 : \quad < bstst^2st^{-1}, abstst >, \quad \{s, bt^2, s^t, bt^{-2}\}$

$\mathbf{F_2: \ n = 70 \quad [6, 14, 14, 0] \quad I_{70,1} \quad AT4Val[70][4]}$

$\mathbb{Z}_2 \times S_7: \ < a \mid a^2 > \times < s, t \mid s^2, t^7, (st)^6, (st^2st^5)^2, (stst^{-1})^3 >$
$S_3 \times S_4: \ < t^2st^{-2}, t^{-2}st^{-1}(st)^2t, t^2(st)^2(ts)^3t^{-1}, t(st)^2(ts)^3, stst^{-1}s >,$
$\{ast^4, atstst^{-1}, at, at^{-1}\}$

$\mathbf{F_3: \ n = 90 \quad [9, 16, 19, 0] \quad I_{90,1} \quad AT4Val[90][1]}$

$\mathbb{Z}_2 \times \mathrm{P\Gamma L}(2, 9): \ < a \mid a^2 > \times < x, y, z \mid$
$x^8, y^3, z^2, xzx^5z, yzy^{-1}z^{-1}, xyxy^{-1}x^6yx^6y^{-1},$
$(xyx^2y)^2, xyx^{-2}y^{-1}x^4yx^{-1}y^{-1} >$
$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2: \ < yzxy^{-1}x, x^{-1}yxzx^{-1}y, x^2zy^{-1}x^{-1}y^{-1}xy >,$
$\{ayx^{-1}y^{-1}x, az, ayxy^{-1}x,$
$axy^{-1}x^{-1}y\}$

$\mathbf{F_4: \ n = 180 \quad [22, 28, 34, 5] \quad AT4Val[180][12]}$

$\mathbb{Z}_2 \times S_3 \times S_5: \ < a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^5, (uv)^4, (uv^2uv^3)^2 >$
$D_8: \ < v^{-2}uv^2, vuv^2uv^2, astuvuv^2uv^{-1} >, \quad \{at^{-1}v^{-2}, atuv^2, su, atv^2\}$

$\mathbf{F_5: \ n = 210 \quad [27, 28, 49, 0] \quad AT4Val[210][10]}$

$S_7: \ < s, t \mid s^2, t^7, (st)^6, (st^2st^5)^2, (stst^{-1})^3 >$
$S_4: \ < tst^3st^3, tst^{-2}st, (st)^2t^2st^{-1}(st)^2tst >, \quad \{t^3s, st^4, (st)^3, (ts)^3\}$

Table 5: Bipartite arc-transitive non-Cayley QIGs

The census [17, 18] of arc-transitive graphs contains all arc-transitive graphs with at most 640 vertices. Thus, the upper bound of 560 given in [25] for the order of a bipartite QIG, ensures that Table 3 and Table 5 contain all bipartite arc-transitive QIGs. The non-bipartite arc-transitive QIGs will be given in Sections 4.4 and 4.5. However, first we describe our method for finding all Cayley QIGs.

4.3 Integral graphs as quotients

Let $V(G)$ denote the vertices of a graph G, and $E(G)$ the unordered pairs of vertices which are edges of G. A homomorphism from a graph G to a graph H is a map $V(G) \to V(H)$ which preserves adjacency. Each homomorphism induces an edge map $E(G) \to E(H)$. If the vertex and edge maps of the homomorphism are both surjective then we say that H is a *quotient* of G. In this section we find new integral graphs that are quotients of the integral graphs found in Table 3 and Table 5. To specify a quotient of a graph G it suffices to know G and the vertex map (the edges of the quotient are implied by the surjectivity of the edge map).

We start by considering special classes of possible homomorphisms. A *voltage assignment* α for a graph G is a function from the arcs of G to a group Γ such that $\alpha((u, v)) = \alpha((v, u))^{-1}$ for all $\{u, v\} \in E(G)$. The *derived graph* $\mathrm{Vol}(G, \Gamma, \alpha)$ is the graph with vertex set the Cartesian product $V(G) \times \Gamma$ with (u, x) connected to (v, y) whenever $\{u, v\} \in E(G)$ and $y = x * \alpha((u, v))$, where $*$ is the group operation of Γ. Projection onto the first coordinate, by definition, maps the derived graph of $\mathrm{Vol}(G, \Gamma, \alpha)$ onto G, and this map is a surjective homomorphism. Hence G is a quotient of the derived graph.

As an interesting example for quartic integral graphs, we found a voltage assignment α for which the derived graph $\mathrm{Vol}(F_1, \mathbb{Z}_3, \alpha)$ is isomorphic to F_4. Thus, F_1 is a quotient of F_4.

Given two graphs G_1, G_2 with vertex sets $V(G_1), V(G_2)$, let $G_1 \times G_2$ be the graph with vertex set the Cartesian product $V(G_1) \times V(G_2)$ with (u_1, u_2) adjacent to (w_1, w_2) whenever both u_1 is adjacent to w_1 in G_1 and u_2 is adjacent to w_2 in G_2. The *bipartite double cover* of G is the bipartite graph $G \times K_2$ where K_2 denotes the complete graph on two vertices. Equivalently, $G \times K_2$ is the derived graph $\mathrm{Vol}(G, \mathbb{Z}_2, \alpha)$, where α is the constant function assigning 1 to every arc of G.

We give an example for quartic integral graphs that was also noted in [7]. An *odd graph* O_i is the graph with one vertex for each of the $(i-1)$-element subsets of a $(2i-1)$-element set and with edges joining disjoint subsets. The graph F_2 is the bipartite double cover of the integral graph O_4.

Similar to the result by Schwenk [20] used in [24] and [25], we have that if G is a QIG, then the bipartite double cover of G is a bipartite QIG. If G is a bipartite QIG then the bipartite double cover consists of two disjoint copies of G. For this reason, we have restricted our search to integral graphs that are bipartite up to this point. However, we now want to find all graphs which have their bipartite double cover among the bipartite graphs that we have discovered. This requires us to find quotients of our bipartite graphs. Since it is computationally easy to do, we will actually consider a more general class of homomorphisms than what is required for the task just described. This will increase the number of quartic integral graphs that we find. However, we make no effort to be exhaustive in finding all possible quotients.

A graph automorphism is *k-semiregular* if all its orbits have the same size, k. Note that if $G = H \times K_2$ then the natural homomorphism from G onto H maps orbits of a 2-semiregular automorphism of G to single vertices of H. With this as motivation, the class of homomorphisms that we consider is the following. We identify any k-semiregular automorphism, ϑ of a target graph G. Our homomorphism is to collapse each orbit of ϑ to a single point.

We wrote a routine in Magma [21] to find such quotients of a target graph G, as follows. For one representative, ϑ, of each conjugacy class of (nontrivial) semiregular automorphisms of G, we collapsed the orbits of ϑ to single vertices to obtain a quotient H. If H was a 4-regular graph we checked to see if it was integral. If it was, then we printed it out and called the routine recursively on H.

In some cases we were only interested in finding those H for which G is a bipartite double cover. In such instances, it suffices to only consider 2-semiregular automorphisms and we do not need to make recursive calls to the routine.

We applied our Magma routine to all target graphs G_i for $i \in 1, \ldots, 17$ and to most of the arc-transitive graphs from the census of Potočnik *et al.* [17, 18]. There are graphs in the census with extremely large automorphism groups, and they were impractical for our simple routine. So we decided to only include target graphs from the census if their automorphism group had order no more than 2^{20}. The results of our Magma routine will be given in the following subsections.

4.4 Non-bipartite Cayley integral graphs

In this section we report all quartic Cayley integral graphs that are not bipartite. We rely on this Lemma:

Lemma 4.2. *If G is a 4-regular Cayley graph then $G \times K_2$, the bipartite double cover of G, is isomorphic to a 4-regular Cayley graph.*

Proof. If $G = \mathrm{Cay}(\Gamma, S)$ then we define $G' = \mathrm{Cay}(\Gamma \times \mathbb{Z}_2, \{(s,1) : s \in S\})$. This graph G' is an undirected Cayley graph. It is not hard to verify that G' is isomorphic to $G \times K_2$ which gives the desired result. □

Hence we can find all the graphs we seek by applying the Magma routine of Section 4.3 to our graphs G_i where $i = 1, \ldots, 17$. We use the following result by Sabidussi [19] to decide which of the graphs that we find are Cayley graphs:

Lemma 4.3. *A graph G is a Cayley graph if and only if $\mathrm{Aut}(G)$ contains a regular subgroup.*

Initial Graph	#Non-bipartite	#Cayley	#Vertex-transitive	#Arc-transitive
G_1	0	0	0	0
G_2	1	1	1	1
G_3	1	1	1	1
G_4	0	0	0	0
G_5	1	1	1	0
G_6	2	1	1	1
G_7	2	1	1	1
G_8	4	2	2	0
G_9	0	0	0	0
G_{10}	1	0	1	1
G_{11}	0	0	0	0
G_{12}	2	1	1	0
G_{13}	1	1	1	0
G_{14}	2	2	2	0
G_{15}	5	1	1	1
G_{16}	13	2	2	0
G_{17}	2	1	1	0

Table 6: Non-bipartite graphs found for G_i

Table 6 summarizes our results using the Magma routine of Section 4.3 when restricted to the 2-semiregular automorphisms for each given G_i. We give the number of non-bipartite graphs found, followed by the numbers of those that are Cayley, vertex-transitive, and arc-transitive.

The non-bipartite graphs counted in column 2 up to row 8 of Table 6 were previously found by Stevanović *et al.* [25]. All graphs counted by column 2 from rows 9 to 17 were previously unknown with the exception of the graph with bipartite double cover G_{10} and one of the two graphs with bipartite double cover G_{14}. In Table 7, we expand upon the counts of non-bipartite Cayley graphs in column three of Table 6 by producing a breakdown of the groups and the connection sets of the underlying graphs. We follow the same conventions as in Table 3 except that we use different notation for the spectrum, since there is no longer symmetry about the origin.

Group	Connection Sets (#involutions S)
$\mathbf{H_1 : n = 5 \quad -1^4, 4^1 \quad I_{5,1} \quad AT4Val[5][1]}$	
$\mathbb{Z}_5$ $< a \mid a^5 >$	$0 \; \{a^3, a^2, a^4, a\}$
$\mathbf{H_2 : n = 6 \quad -2^2, 0^3, 4^1 \quad I_{6,1} \quad AT4Val[6][1]}$	
S_3 $< s, t \mid s^2, t^3, (st)^2 >$	$2 \; \{st, t, t^2, s\}$
$\mathbb{Z}_6$ $< a \mid a^6 >$	$0 \; \{a^5, a^2, a, a^4\}$
$\mathbf{H_3 : n = 8 \quad -2^3, 0^3, 2^1, 4^1 \quad I_{8,2}}$	
$\mathbb{Z}_4 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 >$	$2 \; \{a, a^2, a^3, b\}$
D_8 $< r, f \mid r^4, f^2, (rf)^2 >$	$2 \; \{r, r^2, r^3, fr^2\}$ $4 \; \{f, r^2, rf, fr\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 >$	$4 \; \{b, a, abc, ac\}$
$\mathbf{H_4 : n = 9 \quad -2^4, 1^4, 4^1 \quad I_{9,2} \quad AT4Val[9][1]}$	
$\mathbb{Z}_3 \times \mathbb{Z}_3$ $< a \mid a^3 > \times < b \mid b^3 >$	$0 \; \{a^2b, ab^2, a^2, a\}$
$\mathbf{H_5 : n = 12 \quad -2^5, 0^3, 2^3, 4^1 \quad I_{12,7} \quad AT4Val[12][1]}$	
A_4 $< s, t \mid s^2, t^3, (st)^3 >$	$0 \; \{t^2s, ts, st^2, st\}$

$\mathbf{H_6:\ n=12 \quad -3^2, -1^4, 0^1, 1^2, 2^2, 4^1 \quad I_{12,5}}$

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b^2 >$	$0\ \{a, ba^2, b^3a^2, a^2\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0\ \{a^3, a^4, a^8, a^9\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{r^4, fr^4, r^2, fr\}$ $2\ \{r^3, f, r^4, r^2\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2\ \{a^2, b, a^3, a^4\}$

$\mathbf{H_7:\ n=12 \quad -3^2, -2^2, 0^1, 1^6, 4^1 \quad I_{12,1}}$

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b^2 >$	$0\ \{b^3, b, b^2a, b^2a^2\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0\ \{a^3, a^{10}, a^2, a^9\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{f, fr^3, r, r^5\}$ $4\ \{r^3, f, fr, fr^5\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2\ \{a^3b, a^3, a^2b, a^4b\}$

$\mathbf{H_8:\ n=18 \quad -3^2, -2^4, 0^5, 1^4, 3^2, 4^1 \quad I_{18,4}}$

$\mathbb{Z}_3 \times S_3$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{a, s, a^2, st^2\}$ $0\ \{t, t^2, sat^2, sa^2t^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^3, c^2, cac^{-1}a^{-2}, bc^{-1}b^{-2}c, aba^{-1}b^{-1} >$	$2\ \{a, c, a^2, cb^2\}$
$\mathbb{Z}_6 \times \mathbb{Z}_3$ $< a \mid a^6 > \times < b \mid b^3 >$	$0\ \{a^2b, a^5b^2, ab, a^4b^2\}$

$\mathbf{H_9:\ n=20 \quad -2^6, -1^4, 0^5, 3^4, 4^1}$

$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ $< a, b \mid a^5, b^4, ab^3a^3b, ab^2ab^2 >$	$2\ \{a^2b^2, ab^2, a^2b, a^4b^3\}$

$\mathbf{H_{10}:\ n=24 \quad -3^3, -2^3, -1^5, 0^3, 1^5, 2^1, 3^3, 4^1 \quad I_{24,5}}$

S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$2\ \{s, st^2st, (tst)^2, t(ts)^2\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$2\ \{s, st^2, a, ts\}$

$\mathbf{H_{11}:\ n = 24} \quad \mathbf{-3^4, -2^3, -1^2, 0^3, 1^8, 2^1, 3^2, 4^1}$

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a^3t, at^2, sa^2, a^2\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a,b,c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (a^3c)^2b, cbc^{-1}b^{-1}, a^2ca^2c >$	$4\ \{ca^2, cb, b, a^3\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r,f \mid r^4, f^2, (rf)^2 >$	$2\ \{f, r^2, ar, a^2r^{-1}\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s,t \mid s^2, t^3, (st)^2 >$	$4\ \{sabt, sbt^2, ab, sa\}$

$\mathbf{H_{12}:\ n = 36} \quad \mathbf{-2^{13}, -1^6, 0^3, 1^4, 2^3, 3^6, 4^1} \quad \mathbf{AT4Val[36][6]}$

$\mathbb{Z}_3 \times A_4$ $< a \mid a^3 > \times < s,t \mid s^2, t^3, (st)^3 >$	$0\ \{ta, tst, t^2a^2, t^2st^2\}$

$\mathbf{H_{13}:\ n = 36} \quad \mathbf{-3^4, -2^{10}, 0^1, 1^{16}, 3^4, 4^1}$

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bc^{-1}b^{-2}c >$	$0\ \{ac^2b, cb^2, a^2c^2b^2, c^3b^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, ac^{-1}a^{-1}bc, ac^2ac^2, acbc^{-1}b, c^2bc^2b >$	$2\ \{a^2bc^3, a^2c, ac^2, ab^2c^2\}$
$S_3 \times S_3$ $< s,t \mid s^2, t^3, (st)^2 > \times < u,v \mid u^2, v^3, (uv)^2 >$	$4\ \{suvt, s, sut^2, uv^2\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a^5t^2, at, sa^3t, st\}$

$\mathbf{H_{14}:\ n = 36} \quad \mathbf{-3^4, -2^4, -1^{12}, 0^1, 1^4, 2^6, 3^4, 4^1}$

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bc^{-1}b^{-2}c >$	$0\ \{c^3b, cb, a^2b, ab^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a,b,c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, ac^{-1}a^{-1}bc, ac^2ac^2, acbc^{-1}b, c^2bc^2b >$	$0\ \{a^2bc^3, a^2c, b^2, b\}$
$S_3 \times S_3$ $< s,t \mid s^2, t^3, (st)^2 > \times < u,v \mid u^2, v^3, (uv)^2 >$	$2\ \{v^2t, s, uv, vt^2\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s,t \mid s^2, t^3, (st)^2 >$	$2\ \{a^2t, sa^3t, a^4t^2, st\}$

$\mathbf{H_{15}:\ n = 60} \quad \mathbf{-3^4, -2^{17}, -1^4, 0^{15}, 2^{11}, 3^8, 4^1}$

A_5 $< s,t \mid s^2, t^3, (st)^5 >$	$2\ \{st^2(st)^2, tst^2(st)^2t,$ $(st)^3ts, ((st)^2t)^2st\}$

Table 7: Non-bipartite Cayley QIGs

Thus, by Theorem 4.1 and Lemma 4.2 we have that $\{G_i : 1 \leqslant i \leqslant 17\} \cup \{H_j : 1 \leqslant j \leqslant 15\}$ is the complete set of Cayley QIGs.

4.5 Non-bipartite arc-transitive integral graphs

In Section 4.2, we listed all bipartite arc-transitive QIGs from the census of Potočnik *et al.* [17, 18]. When searching this census for integral graphs, we also found arc-transitive QIGs that are not bipartite. There are 6 such graphs that are not Cayley and thus not already accounted for in Table 7. By [25], the bipartite double cover of any QIG has order at most 560, so we can be sure that the census contains all the arc-transitive QIGs. In fact, the following folklore result tells us more:

Lemma 4.4. *The bipartite double cover of an arc-transitive graph is arc-transitive.*

Proof. Let G be an arc-transitive graph. Then $H = G \times K_2$ has vertices (a, x) for all $a \in G$ and $x \in \mathbb{Z}_2$ and arcs $((a, x), (b, x + 1))$ and $((b, x), (a, x + 1))$ whenever a is adjacent to b in G. It is not hard to show that the following maps are automorphisms of H:

- $(a, x) \to (\sigma(a), x + 1)$ for all $a \in G$ and $x \in \mathbb{Z}_2$ where $\sigma \in \mathrm{Aut}(G)$.
- $(a, x) \to (a, x + 1)$ for all $a \in G$ and $x \in \mathbb{Z}_2$.

Given these automorphisms, it is routine to check that H is arc-transitive. □

This last result provides a useful cross-check of our results and of the Magma routine from Section 4.3. It tells us that by applying the routine (restricted to 2-semiregular automorphisms) to the bipartite arc-transitive QIGs from Tables 3 and 5, we should find all the arc-transitive integral non-bipartite graphs. This list should tally with the list obtained by directly screening the census for integral graphs, which is what happened in practice.

We now list the spectrum of the non-bipartite arc-transitive QIGs that are not Cayley and whose bipartite double cover is one of the G_i for $i = 1, \dots, 17$ or F_i for $i = 1, \dots, 5$. We denote these graphs by J_i for $i = 1, \dots, 6$. Graphs appearing in the paper by Cvetković *et al.* are included using the notation of [7]: $I_{n,index}$. We give the graph index from the census of Potočnik *et al.* [17, 18] in their notation: AT4Val[n][index].

- From G_{10}, $J_1 \cong I_{15,2} \cong$ AT4Val[15][1] : $[-2^5, -1^4, 2^5, 4^1]$,
- From F_1, $J_2 \cong$ AT4Val[30][2] : $[-3^4, -2^5, -1^4, 0^5, 2^{11}, 4^1]$,
  $J_3 \cong I_{30,4} \cong$ AT4Val[30][3] : $[-2^{11}, -1^4, 0^5, 2^5, 3^4, 4^1]$,
- From F_2, $J_4 \cong I_{35,1} \cong$ AT4Val[35][2] $\cong O_4$: $[-3^6, -1^{14}, 2^{14}, 4^1]$,
- From F_3, $J_5 \cong I_{45,1} \cong$ AT4Val[45][1] : $[-2^{16}, -1^9, 1^{10}, 3^9, 4^1]$,
- From F_4, $J_6 \cong$ AT4Val[90][8] : $[-3^{14}, -2^7, -1^{24}, 0^5, 1^{10}, 2^{21}, 3^8, 4^1]$.

Of the arc-transitive non-bipartite non-Cayley graphs, only J_2 and J_6 were not previously known to be integral. Thus, the arc-transitive QIGs from the census are as follows: G_1, G_2, G_3, G_5, G_6, G_7, G_{10}, G_{11}, G_{12}, G_{15}, G_{17}, F_1, F_2, F_3, F_4, F_5, H_1, H_2, H_4, H_5, H_{12}, J_1, J_2, J_3, J_4, J_5, and J_6. We summarize these results by the following Lemma:

Lemma 4.5. *There are exactly* 27 *quartic integral graphs that are arc-transitive;* 16 *of which are bipartite.*

4.6 Other quartic integral graphs

Finally, we list the spectra of the remaining QIGs which we found using the Magma routine of Section 4.3 in its full generality. These are graphs that are neither Cayley nor arc-transitive, but are quotients of the graphs G_i for $i = 1, \dots, 17$ and/or of the graphs AT4Val[n][index] for $n \leqslant 640$ with automorphism groups of order less than 2^{20}. We note that many of these graphs were obtained from multiple starting graphs, but we only list each graph once.

We list the spectrum of the bipartite QIGs first. We denote these graphs by M_i for $i = 1, \dots, 9$ and follow the same conventions as in the list for J_i where $i = 1, \dots, 6$ except that we use the quadruple form for the spectrum of a bipartite graph.

- From AT4Val[60][4] we have $M_1 \cong I_{30,3} : [1, 8, 3, 2]$,
- From $G_{15} \cong$ AT4Val[72][12] we have $M_2 \cong I_{36,1} : [2, 8, 6, 1]$, $M_3 \cong I_{36,2} : [3, 6, 5, 3]$,
- From $G_{17} \cong$ AT4Val[120][4] we have $M_4 : [3, 4, 1, 6]$, $M_5 : [6, 12, 2, 9]$,
- From AT4Val[180][12] we have $M_6 : [9, 16, 19, 0]$, $M_7 : [10, 14, 18, 2]$,
- From AT4Val[216][12] we have $M_8 : [3, 5, 9, 0]$,
- From AT4Val[546][48] we have $M_9 : [5, 4, 7, 4]$.

We do not list graphs with at most 24 vertices since all bipartite QIGs on 24 or fewer vertices are known [25]. The 6 graphs $M_4, \dots, M_9$ were not previously known to be bipartite QIGs. We find that M_6 is co-spectral to F_3, but 5 of the above spectra were not previously known to be realized by any graph.

Next, we list the spectrum of the non-bipartite QIGs. We denote these graphs by L_i where $i \in 1, \dots, 44$.

- From AT4Val[30][3], $L_1 \cong I_{15,4} : [-2^5, -1^3, 0^2, 2^3, 3^1, 4^1]$.
- From $G_{12} \cong$ AT4Val[36][3], $L_2 : [-3^3, -2^2, -1^1, 0^5, 1^3, 2^2, 3^1, 4^1]$.
- From AT4Val[36][6], $L_3 \cong I_{18,5} : [-2^7, -1^2, 0^1, 1^4, 2^1, 3^2, 4^1]$,
  $L_4 \cong I_{18,6} : [-2^6, -1^3, 0^3, 1^2, 3^3, 4^1]$.
- From AT4Val[60][4], $L_5 : [-3^3, -2^7, -1^3, 0^5, 1^1, 2^9, 3^1, 4^1]$, and
  $L_6 : [-3^2, -2^9, -1^2, 0^5, 1^2, 2^7, 3^2, 4^1]$.
- From AT4Val[70][4], $L_7 : [-3^5, -2^4, -1^9, 1^5, 2^{10}, 3^1, 4^1]$, and
  $L_8 : [-3^4, -2^6, -1^8, 1^6, 2^8, 3^2, 4^1]$.
- From $G_{15} \cong$ AT4Val[72][12], $L_9 : [-3^1, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1]$,
  $L_{10} : [-3^2, -2^3, -1^4, 0^1, 1^2, 2^5, 4^1]$, $L_{11} : [-3^2, -2^5, 0^1, 1^6, 2^3, 4^1]$,
  $L_{12} : [-3^2, -2^4, -1^2, 0^1, 1^4, 2^4, 4^1]$, $L_{13} : [-3^1, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1]$,
  $L_{14} : [-3^2, -2^4, -1^1, 0^3, 1^4, 2^2, 3^1, 4^1]$, $L_{15} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1]$,
  $L_{16} : [-3^4, 2^9, -1^2, 0^3, 1^8, 2^7, 3^2, 4^1]$, $L_{17} : [-3^2, -2^{11}, -1^4, 0^3, 1^6, 2^5, 3^4, 4^1]$,
  and
  $L_{18} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1]$.
- From G_{16}, $L_{19} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1]$,
  $L_{20} : [-3^4, -2^9, -1^2, 0^1, 1^{14}, 2^1, 3^4, 4^1]$,
  $L_{21} : [-3^4, -2^5, -1^{10}, 0^1, 1^6, 2^5, 3^4, 4^1]$, $L_{22} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$,

$L_{23}: [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1]$, $L_{24}: [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$,
$L_{25}: [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1]$, $L_{26}: [-3^3, -2^7, -1^9, 0^1, 1^7, 2^3, 3^5, 4^1]$,
$L_{27}: [-3^3, -2^8, -1^7, 0^1, 1^9, 2^2, 3^5, 4^1]$, $L_{28}: [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1]$,
and
$L_{29}: [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$.

- From AT4Val[90][1], $L_{30}: [-3^4, -2^{10}, -1^9, 1^{10}, 2^6, 3^5, 4^1]$.
- From AT4Val[90][8], $L_{31}: [-3^5, -2^6, -1^{14}, 1^5, 2^{10}, 3^4, 4^1]$.
- From $G_{17} \cong$ AT4Val[120][4], $L_{32}: [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1]$,
  $L_{33}: [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1]$, $L_{34}: [-3^4, -2^5, -1^2, 0^9, 2^7, 3^2, 4^1]$,
  $L_{35}: [-3^7, -2^{13}, -1^3, 0^{15}, 1^1, 2^{15}, 3^5, 4^1]$.
- From AT4Val[180][12], $L_{36}: [-3^4, -2^8, -1^{12}, 0^2, 1^6, 2^6, 3^6, 4^1]$,
  $L_{37}: [-3^9, -2^{17}, -1^{19}, 0^5, 1^{15}, 2^{11}, 3^{13}, 4^1]$,
  $L_{38}: [-3^{11}, -2^{13}, -1^{21}, 0^5, 1^{13}, 2^{15}, 3^{11}, 4^1]$, and
  $L_{39}: [-3^{12}, -2^{15}, -1^{14}, 0^5, 1^{20}, 2^{13}, 3^{10}, 4^1]$.
- From AT4Val[210][10], $L_{40}: [-3^{16}, -2^9, -1^{29}, 1^{20}, 2^{19}, 3^{11}, 4^1]$.
- From AT4Val[273][4], $L_{41}: [-3^1, -2^4, -1^6, 0^4, 1^1, 3^4, 4^1]$.
- From AT4Val[546][48], $L_{42}: [-3^2, -2^3, -1^5, 0^4, 1^2, 2^1, 3^3, 4^1]$,
  $L_{43}: [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1]$, $L_{44}: [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1]$.

We do not list graphs with at most 12 vertices since all non-bipartite QIGs on 12 or fewer vertices are known [25]. Of the 44 graphs given above, only L_1, L_3 and L_4 previously appear in the literature about integral graphs. The remaining 41 non-bipartite QIGs are new.

5 Concluding remarks

There are precisely 32 connected 4-regular integral Cayley graphs up to isomorphism. Table 3 lists the 17 graphs of the 32 which are bipartite and Table 7 gives the details of the 15 non-bipartite graphs.

There are exactly 27 quartic integral graphs that are arc-transitive. We found that 16 of the 27 graphs are bipartite; these appear in Table 3 and Table 5. We found that 16 of the 27 graphs are Cayley graphs; these appear in Table 3 and Table 7.

There are integral Cayley bipartite graphs that can be decomposed into $H \times K_2$ where H is Cayley and arc-transitive, Cayley but not arc-transitive, or arc-transitive but not Cayley. The graph G_{10} is our only example of this last possibility; refer to Table 6.

The new 4-regular integral graphs that we found that are co-spectral to other graphs are as follows: G_{13} co-spectral to $I_{40,1}$ and $I_{40,2}$, G_{15} to $I_{72,1}$, H_9 to $I_{20,8}$, H_{12} to $I_{36,4}$, and F_3 to M_6. We also mention the co-spectral graphs among the known integral graphs: G_5 is co-spectral to $I_{16,2}$ and another graph appearing in [25], G_6 to $I_{18,2}$ and $I_{18,3}$, G_7 to $I_{24,1}$, F_1 to $I_{60,2}$, H_5 to $I_{12,6}$, and J_3 to $I_{30,5}$.

We find that some integral Cayley graphs are co-spectral to integral non-Cayley graphs and that some integral arc-transitive graphs are co-spectral to integral graphs that are not arc-transitive. For example, the arc-transitive Cayley graph H_5 has a co-spectral mate $I_{12,6}$, that is neither arc-transitive nor Cayley.

As can also be seen in Table 3, there are isomorphic integral graphs that are non-equivalent Cayley graphs $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma^*, S^*)$ in the sense of Definition 1.1. This can occur for $\Gamma \neq \Gamma^*$ as well as $\Gamma = \Gamma^*$ with $S \neq S^*$. Consider G_{12}, which has 12 non-equivalent Cayley Graphs on 6 different groups. For $\Gamma = S_3 \times S_3$, there are 4 non-equivalent Cayley graphs with connection sets occurring for each of the three possible numbers of involutions. There is only one Cayley graph up to equivalence for the graph of order 40. For all other orders the bipartite integral Cayley graphs are not unique up to equivalence. In the non-bipartite case; $H_1, H_4, H_5, H_9, H_{12}$, and H_{15} are all unique up to equivalence.

There are non-isomorphic integral Cayley graphs with the same number of vertices. As can be seen in Table 3 for the bipartite case, there are two graphs on 12 vertices, three graphs on 24 vertices, and two graphs on 72 vertices up to isomorphism. For all other orders there is at most one graph up to isomorphism. There are many more examples in the non-bipartite case (refer to Table 7).

There exist non-isomorphic integral Cayley graphs for the same group Γ. Consider G_i for $i = 7, 8, 9$ in Table 3. The following 6 groups are examples of this: $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_3 \times D_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$, S_4, $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, and $\mathbb{Z}_4 \times S_3$.

We began with the 828 possible spectra from [25], and then narrowed our focus to a set Ξ of 59 candidates for vertex transitive graphs; refer to Table 1 and Appendix A. Of these, we found 22 which are realised by Cayley graphs or arc-transitive graphs. In Section 4.6, by taking quotients, we found 6 new bipartite integral graphs that are neither arc-transitive nor Cayley, but realize a possible spectrum.

Overall, we found 9 bipartite quartic integral graphs (namely, G_{16}, G_{17}, F_4, F_5, M_4, M_5, M_7, M_8, M_9) that realise spectra not previously known to be achieved. It remains open whether the remaining possible spectra are realized by any 4-regular bipartite integral graphs.

All integral graphs discovered in this paper are available in Magma format from:

`http://users.monash.edu.au/~iwanless/data/graphs/IntegralGraphs.html`.

Acknowledgements

A special thanks to Marston Condor, Heiko Dietrich, and Csaba Schneider for their helpful advice and for confirming some of our computational results. The authors used a mixture of the computer algebra packages Magma [21] and GAP [9] (including the GRAPE package [22] for GAP), as well as nauty [14].

This research formed part of the first author's PhD thesis [15].

A Feasible vertex-transitive spectra

The following is the set Ξ of possible spectra that might be realized by a connected 4-regular bipartite integral graph G that is vertex-transitive. This set was determined in Section 2. The entries are given as $n\ x\ y\ z\ w\ [C_4]\ [C_6]$ where $|V(G)| = n$ and $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$.

8	0	0	0	3	36	96	90	13	7	19	5	45	60
10	0	0	4	0	30	130	90	8	22	4	10	0	150
12	0	1	4	0	27	138	90	9	16	19	0	0	210
12	0	2	0	3	30	112	96	12	12	20	3	24	128
16	0	4	0	3	24	128	120	12	28	4	15	0	120
18	0	4	4	0	18	162	120	13	22	19	5	0	180
20	0	5	4	0	15	170	120	15	20	9	15	30	40
24	0	8	0	3	12	160	120	16	14	24	5	30	100
24	2	2	6	1	30	124	120	19	6	29	5	60	20
24	3	0	5	3	42	80	126	13	28	7	14	0	126
30	0	10	4	0	0	210	126	20	7	28	7	63	0
30	3	2	9	0	30	130	144	16	28	16	11	0	144
30	4	1	4	5	45	60	144	20	16	28	7	36	72
32	0	12	0	3	0	192	168	20	28	28	7	0	168
36	4	4	4	5	36	84	180	20	40	4	25	0	60
36	5	1	7	4	45	66	180	21	34	19	15	0	120
40	4	6	4	5	30	100	180	22	28	34	5	0	180
42	6	0	14	0	42	98	180	26	19	34	10	45	30
48	6	4	10	3	36	96	210	27	28	49	0	0	210
60	4	16	4	5	0	180	240	28	52	4	35	0	0
60	6	9	14	0	15	170	240	30	40	34	15	0	120
60	7	8	9	5	30	100	280	34	56	14	35	0	0
60	8	7	4	10	45	30	288	36	52	28	27	0	48
60	9	1	19	0	45	90	360	46	64	34	35	0	0
70	6	14	14	0	0	210	360	47	58	49	25	0	60
72	11	4	13	7	54	24	360	48	52	64	15	0	120
72	6	16	10	3	0	192	420	55	70	49	35	0	0
72	8	10	16	1	18	156	480	64	76	64	35	0	0
72	9	10	7	9	36	60	560	76	84	84	35	0	0
90	12	13	4	15	45	0							

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