ARS MATHEMATICA CONTEMPORANEA

Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 10 (2016) 79–83

An improvement of a result of Zverovich–Zverovich

Grant Cairns *, Stacey Mendan
Department of Mathematics and Statistics, La Trobe University, Melbourne, Australia 3086

Received 8 March 2013, accepted 20 October 2013, published online 27 May 2015

Abstract

We give an improvement of a result of Zverovich and Zverovich which gives a condition on the first and last elements in a decreasing sequence of positive integers for the sequence to be graphic, that is, the degree sequence of a finite graph.

Keywords: Graph, graphic sequence.

Math. Subj. Class.: 05C07

1 Statement of Results

A finite sequence of positive integers is *graphic* if it occurs as the sequence of vertex degrees of a graph. Here, graphs are understood to be *simple*, in that they have no loops or repeated edges. A result of Zverovich and Zverovich states:

Theorem 1.1 ([8, Theorem 6]). *Let a, b be reals. If $\underline{d} = (d_1, \dots, d_n)$ is a sequence of positive integers in decreasing order with $d_1 \le a, d_n \ge b$ and*

$$n \ge \frac{(1+a+b)^2}{4b},$$

then $\underline{d}$ is graphic.

Notice that here the term $\frac{(1+a+b)^2}{4b}$ is monotonic increasing in a, for $a \ge 1$ and fixed b, and it is also monotonic decreasing in b, for $a \ge b \ge 1$ and fixed a. Thus any sequence that satisfies the inequality $n \ge \frac{(1+a+b)^2}{4b}$, for any pair $a \ge d_1, b \le d_n$, will also satisfy the inequality $n \ge \frac{(1+d_1+d_n)^2}{4d_n}$. So Theorem 1.1 has the following equivalent expression.

*Corresponding author
E-mail addresses: G.Cairns@latrobe.edu.au (Grant Cairns), spmendan@students.latrobe.edu.au (Stacey Mendan)

Theorem 1.2. *Suppose that $\underline{d} = (d_1, \dots, d_n)$ is a decreasing sequence of positive integers with even sum. If*

$$n \geq \frac{(1 + d_1 + d_n)^2}{4d_n}, \tag{1.1}$$

then $\underline{d}$ is graphic.

The simplified form of Theorem 1.2 also affords a somewhat simpler proof, which we give in Section 2 below. Admittedly, the proof in [8] is already quite elementary, though it does use the strong index results of [4, 3].

The following corollary of Zverovich–Zverovich's is obtained by taking $a = d_1$ and $b = 1$ in Theorem 1.1.

Corollary 1.3 ([8, Corollary 2]). *Suppose that $\underline{d} = (d_1, \dots, d_n)$ is a decreasing sequence of positive integers with even sum. If $d_1 \leq 2n^{\frac{1}{2}} - 2$, then $\underline{d}$ is graphic.*

Note that this can be expressed in the following equivalent form.

Corollary 1.4. *Suppose that $\underline{d} = (d_1, \dots, d_n)$ is a decreasing sequence of positive integers with even sum. If $n \geq \frac{d_1^2}{4} + d_1 + 1$, then $\underline{d}$ is graphic.*

Zverovich–Zverovich state that the bound of Corollary 1.4 "cannot be improved", and they give examples to this effect. In fact, there is an improvement, as we will now describe. The subtlety here is that Zverovich–Zverovich formulated their result as an upper bound on d_1, and, as an upper bound on d_1, this upper bound on d_1 cannot be improved. However, the reformulation of their result as a lower bound on n can be slightly improved. We prove the following result in Section 2.

Theorem 1.5. *Suppose that $\underline{d} = (d_1, \dots, d_n)$ is a decreasing sequence of positive integers with even sum. If $n \geq \left\lfloor \frac{d_1^2}{4} + d_1 \right\rfloor$, then $\underline{d}$ is graphic.*

Example 1.6. There are many examples of sequences that verify the hypotheses of Theorem 1.5 but not those of Corollary 1.4. In fact, there are 81 such sequences of length $n \leq 8$. Figure 1 shows three graphs whose degree sequences have this property; they have degree sequences $(2, 2, 2)$, $(3, 3, 2, 2, 2)$ and $(3, 3, 3, 3, 3, 3)$ respectively. For infinite families of examples, for every positive odd integer x, consider the sequence $(2x, 1^{x^2+2x-1})$, and for x even, consider the sequence $(2x, 2x, 1^{x^2+2x-2})$. Here, and in sequences throughout this paper, the superscripts indicate the number of repetitions of the entry.

Example 1.7. The following examples show that the bound of Theorem 1.5 is sharp when $d_n = 1$. For d even, say $d = 2x$ with $x \geq 1$, let $\underline{d} = (d^{x+1}, 1^{x^2+x-2})$. For d odd, say $d = 2x + 1$ with $x \geq 1$, let $\underline{d} = (d^{x+1}, 1^{x^2+2x-1})$. In each case g has even sum, $n = \left\lfloor \frac{d^2}{4} + d \right\rfloor - 1$, but $\underline{d}$ is not graphic, as one can see from the Erdős–Gallai Theorem [6].

Remark 1.8. The fact that Theorem 1.2 is not sharp has also been remarked in [1], in the abstract of which the authors state that Theorem 1.2 is "sharp within 1". They give the bound

$$n \geq \frac{(1 + d_1 + d_n)^2 - \epsilon'}{4d_n}, \tag{1.2}$$

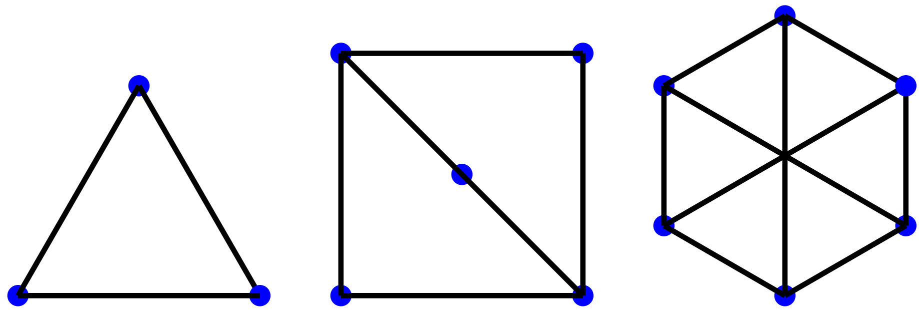

Figure 1: Three examples

where $\epsilon' = 0$ if $d_1 + d_n$ is odd, and $\epsilon' = 1$ otherwise. Consider any decreasing sequence with $d_1 = 2x+1$ and $d_n = 1$. Note that the bound given by Theorem 1.2 is $n \geq x^2+3x+3$, the bound given by (1.2) is $n \geq x^2 + 3x + 2$, while Theorem 1.5 gives the stronger bound $n \geq x^2 + 3x + 1$. The paper [1] gives more precise bounds, as a function of d_1, d_n, and the maximal gap in the sequence.

Remark 1.9. There are many other recent papers on graphic sequences; see for example [5, 7, 1, 2].

2 Proofs of Theorems 1.2 and 1.5

We will require the Erdős–Gallai Theorem, which we recall for convenience.

Erdős–Gallai Theorem. A sequence $\underline{d} = (d_1, \dots, d_n)$ of nonnegative integers in decreasing order is graphic if and only if its sum is even and, for each integer k with $1 \leq k \leq n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{EG}$$

Proof of Theorem 1.2. Suppose that $\underline{d} = (d_1, \dots, d_n)$ is a decreasing sequence with even sum, satisfying (1.1), and which is not graphic. By the Erdős–Gallai Theorem, there exists k with $1 \leq k \leq n$, such that

$$\sum_{i=1}^{k} d_i > k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{2.1}$$

For each i with $1 \leq i \leq k$, replace d_i by d_1; the left hand side of (2.1) is not decreased, while the right hand side of (2.1) is unchanged, so (2.1) still holds. Now for each i with $k+1 \leq i \leq n$, replace d_i by d_n; the left hand side of (2.1) is unchanged, while the right hand side of (2.1) has not increased, so (2.1) again holds. Notice that if $k < d_n$, then (2.1) gives $kd_1 > k(k-1) + (n-k)k = k(n-1)$, and so $d_1 \geq n$. Then (1.1) would give $4nd_n \geq (1+d_n+n)^2$, that is, $(n-(d_n-1))^2 - (d_n-1)^2 + (1+d_n)^2 \leq 0$. But this inequality clearly has no solutions. Hence $k \geq d_n$. Thus (2.1) now reads $kd_1 > k(k-1) + (n-k)d_n$, or equivalently

$$(k - \frac{1}{2}(1 + d_1 + d_n))^2 - \frac{1}{4}(1 + d_1 + d_n)^2 + nd_n < 0.$$

But this contradicts the hypothesis. □

The following proof uses the same general strategy as the preceding proof, but requires a somewhat more careful argument.

Proof of Theorem 1.5. Suppose that $\underline{d}$ satisfies the hypotheses of the theorem. First suppose that d_1 is even, say $d_1 = 2x$. If $d_n \geq 2$, then since $\frac{(1+d_n+d_1)^2}{4d_n}$ is a strictly monotonic decreasing function of d_n for $1 \leq d_n \leq d_1$, we have

$$n \geq \frac{d_1^2}{4} + d_1 = \frac{(2+d_1)^2}{4} - 1 > \frac{(1+d_n+d_1)^2}{4d_n} - 1,$$

so $n \geq \frac{(1+d_n+d_1)^2}{4d_n}$ and hence $\underline{d}$ is graphic by Theorem 1.2. So, assuming that $\underline{d}$ is not graphic, we may suppose that $d_n = 1$. Furthermore, by Corollary 1.4, we may assume that $n = \frac{d_1^2}{4} + d_1$, so $n = x^2 + 2x$.

Now, as in the proof of Theorem 1.2, by the Erdős–Gallai Theorem, there exists k with $1 \leq k \leq n$, such that

$$\sum_{i=1}^{k} d_i > k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{2.2}$$

For each i with $1 \leq i \leq k$, replace d_i by d_1; the left hand side of (2.2) is not decreased, while the right hand side of (2.2) is unchanged, so (2.2) still holds. For each i with $k+1 \leq i \leq n$, replace d_i by 1; the left hand side of (2.2) is unchanged, while the right hand side of (2.2) has not increased, so (2.2) again holds. Then (2.2) reads $kd_1 > k(k-1) + (n-k)$, and consequently, rearranging terms, $(k-x-1)^2 - 1 < 0$. Thus $k = x+1$. Notice that for $1 \leq i \leq k$, if any of the original terms d_i had been less than d_1, we would have obtained $(k-x-1)^2 < 0$, which is impossible. Similarly, for $k+1 \leq i \leq n$, all the original terms d_i must have been all equal to one. Thus $\underline{d} = (d_1^k, 1^{n-k}) = ((2x)^{x+1}, 1^{x^2+x-1})$. So $\underline{d}$ has sum $2x(x+1) + x^2 + x - 1 = 3x^2 + 3x - 1$, which is odd, regardless of whether x is even or odd. This contradicts the hypothesis.

Now consider the case where d_1 is odd, say $d_1 = 2x - 1$. The theorem is trivial for $\underline{d} = (1^n)$, so we may assume that $x > 1$. We use essentially the same approach as we used in the even case, but the odd case is somewhat more complicated. By Corollary 1.4, assuming $\underline{d}$ is not graphic, we have $\frac{d_1^2}{4} + d_1 + 1 > n$, and hence, as d_1 is odd, $\frac{d_1^2}{4} + d_1 + \frac{3}{4} \geq n$. Thus, since $n \geq \left\lfloor \frac{d_1^2}{4} + d_1 \right\rfloor = \frac{d_1^2}{4} + d_1 - \frac{1}{4}$, we have $n = \frac{d_1^2}{4} + d_1 + \frac{3}{4}$ or $n = \frac{d_1^2}{4} + d_1 - \frac{1}{4}$. Thus there are two cases:

(i) $n = x^2 + x - 1$,

(ii) $n = x^2 + x$.

By the Erdős–Gallai Theorem, there exists k with $1 \leq k \leq n$, such that

$$\sum_{i=1}^{k} d_i > k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{2.3}$$

As before, for each i with $1 \leq i \leq k$, replace d_i by d_1 and for each i with $k+1 \leq i \leq n$, replace d_i by d_n, and note that (2.3) again holds. Arguing as in the proof of Theorem 1.2, notice that if $k < d_n$, then (2.3) gives $kd_1 > k(k-1) + (n-k)k = k(n-1)$, and so $d_1 \geq n$. In both cases (i) and (ii) we would have $2x-1 \geq n \geq x^2+x-1$ and hence $x \leq 1$, contrary to our assumption. Thus $k \geq d_n$ and (2.3) reads $kd_1 > k(k-1)+(n-k)d_n$, and consequently, rearranging terms, we obtain in the respective cases:

(i) $d_n x^2 - d_n k + k^2 + d_n x - 2kx - d_n < 0$.

(ii) $d_n x^2 - d_n k + k^2 + d_n x - 2kx < 0$,

In both cases we have $d_n x^2 - d_n k + k^2 + d_n x - 2kx - d_n < 0$. Consider $d_n x^2 - d_n k + k^2 + d_n x - 2kx - d_n$ as a quadratic in k. For this to be negative, its discriminant, $4d_n + d_n^2 + 4x^2 - 4d_n x^2$, must be positive. If $d_n > 1$ we obtain $x^2 < \frac{4d_n + d_n^2}{4(d_n-1)}$. For $d_n = 2$ we have $x^2 < 3$ and so $x = 1$, contrary to our assumption. Similarly, for $d_n = 3$ we have $x^2 < \frac{21}{8}$ and so again $x = 1$. For $d_n \geq 4$, the function $\frac{4d_n+d_n^2}{4(d_n-1)}$ is monotonic increasing in d_n. So, as $d_n \leq d_1$,

$$x^2 < \frac{4d_1 + d_1^2}{4(d_1-1)} = \frac{4x^2+4x-3}{8x-8} < \frac{x^2+x}{2(x-1)},$$

which again gives $x = 1$. We conclude that $d_n = 1$.

So the two cases are:

(i) $x^2 - k + k^2 + x - 2kx - 1 = (k-x)(k-x-1) - 1 < 0$.

(ii) $x^2 - k + k^2 + x - 2kx = (k-x)(k-x-1) < 0$,

In case (ii) we must have $x < k < x+1$, but this is impossible for integer k and x.

In case (i), either $k = x$ or $k = x+1$. Notice that for $1 \leq i \leq k$, if any of the original terms d_i had been less than d_1, we would have obtained $(k-x)(k-x-1) < 0$, which is impossible. Similarly, for $k+1 \leq i \leq n$, all the original terms d_i must have been all equal to one. Thus $\underline{d} = (d_1^k, 1^{n-k})$. Consequently, if $k = x$, we have $\underline{d} = ((2x-1)^x, 1^{x^2-1})$ as $n = x^2+x-1$. In this case, $\underline{d}$ has sum $x(2x-1) + x^2 - 1 = 3x^2 - x - 1$, which is odd, regardless of whether x is even or odd, contradicting the hypothesis. On the other hand, if $k = x+1$, we have $\underline{d} = ((2x-1)^{x+1}, 1^{x^2-2})$. Here, $\underline{d}$ has sum $(2x-1)(x+1) + x^2 - 2 = 3x^2 + x - 3$, which is again odd, regardless of whether x is even or odd, contrary to the hypothesis. □

References

[1] M. D. Barrus, S. G. Hartke, K. F. Jao, and D. B. West, Length thresholds for graphic lists given fixed largest and smallest entries and bounded gaps, *Discrete Math.* **312** (2012), no. 9, 1494–1501.

[2] G. Cairns and S. Mendan, *Symmetric bipartite graphs and graphs with loops*, preprint.

[3] P. L. Hammer, T. Ibaraki, and B. Simeone, Threshold sequences, *SIAM J. Algebraic Discrete Methods* **2** (1981), no. 1, 39–49.

[4] S.Y.R. Li, Graphic sequences with unique realization, *J. Combinatorial Theory Ser. B* **19** (1975), no. 1, 42–68.

[5] A. Tripathi and H. Tyagi, A simple criterion on degree sequences of graphs, *Discrete Appl. Math.* **156** (2008), no. 18, 3513–3517.

[6] A. Tripathi, S. Venugopalan and D. B. West, A short constructive proof of the Erdős-Gallai characterization of graphic lists, *Discrete Math.* **310** (2010), no. 4, 843–844.

[7] J.-H. Yin, Conditions for r-graphic sequences to be potentially $K_{m+1}^{(r)}$-graphic, *Discrete Math.* **309** (2009), no. 21, 6271–6276.

[8] I. È. Zverovich and V. È. Zverovich, Contributions to the theory of graphic sequences, *Discrete Math.* **105** (1992), no. 1-3, 293–303.