Unramified Brauer groups and isoclinism

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Abstract

We show that the Bogomolov multipliers of isoclinic groups are isomorphic.

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1 Introduction

Let \( G \) be a finite group and \( V \) a faithful representation of \( G \) over an algebraically closed field \( k \) of characteristic zero. Suppose that the action of \( G \) upon \( V \) is generically free. A relaxed version of Noether’s problem [11] asks as to whether the fixed field \( k(V)^G \) is purely transcendental over \( k \), i.e., whether the quotient space \( V/G \) is rational. A question related to the above mentioned is whether \( V/G \) is stably rational, that is, whether there exist independent variables \( x_1, \ldots, x_r \) such that \( k(V)^G(x_1, \ldots, x_r) \) becomes a pure transcendental extension of \( k \). This problem has close connection with Lüroth’s problem [12] and the inverse Galois problem [14, 13]. By the so-called no-name lemma, stable rationality of \( V/G \) does not depend upon the choice of \( V \), but only on the group \( G \), cf. [4, Theorem 3.3 and Corollary 3.4]. Saltman [13] found examples of groups \( G \) of order \( p^9 \) such that \( V/G \) is not stably rational over \( k \). His main method was application of the unramified cohomology group \( H^2_{\text{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z}) \) as an obstruction. A version of this invariant had been used before by Artin and Mumford [1] who constructed unirational varieties over \( k \) that were not rational. Bogomolov [2] proved that \( H^2_{\text{nr}}(k(V)^G, \mathbb{Q}/\mathbb{Z}) \) is canonically isomorphic to

\[
B_0(G) = \bigcap_{A \leq G, \text{ A abelian}} \ker \text{res}^G_A,
\]

where \( \text{res}^G_A : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z}) \) is the usual cohomological restriction map. Following Kunyavskiĭ [7], we say that \( B_0(G) \) is the Bogomolov multiplier of \( G \).
We recently proved \cite{9} that $B_0(G)$ is naturally isomorphic to $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, where $\tilde{B}_0(G)$ is the kernel of the commutator map $G \times G \to [G, G]$, and $G \times G$ is a quotient of the non-abelian exterior square of $G$ (see Section 2 for further details). This description of $B_0(G)$ is purely combinatorial, and allows for efficient computations of $B_0(G)$, and a Hopf formula for $\tilde{B}_0(G)$. We also note here that the group $\tilde{B}_0(G)$ can be defined for any (possibly infinite) group $G$.

Recently, Hoshi, Kang, and Kunyavski\text{"}ï [6] classified all groups of order $p^5$ with non-trivial Bogomolov multiplier; the question was dealt with independently in \cite{10}. It turns out that the only examples of such groups appear within the same isoclinism family, where isoclinism is the notion defined by P. Hall in his seminal paper \cite{5}. The following question was posed in [6]:

**Question 1.1** ([6]). Let $G_1$ and $G_2$ be isoclinic $p$-groups. Is it true that the fields $k(V)^{G_1}$ and $k(V)^{G_2}$ are stably isomorphic, or at least, that $B_0(G_1)$ is isomorphic to $B_0(G_2)$?

The purpose of this note is to answer the second part of the above question in the affirmative:

**Theorem 1.2.** Let $G_1$ and $G_2$ be isoclinic groups. Then $\tilde{B}_0(G_1) \cong \tilde{B}_0(G_2)$. In particular, if $G_1$ and $G_2$ are finite, then $B_0(G_1)$ is isomorphic to $B_0(G_2)$.

The proof relies on the theory developed in \cite{9}. We note here that we have recently become aware of a paper by Bogomolov and Böhnling [3] who fully answer the above question using different techniques. We point out that our approach here is purely combinatorial and does not require cohomological machinery.

## 2 Proof of Theorem 1.2

We first recall the definition of $G \times G$ from \cite{9}. For $x, y \in G$ we write $x y = x y x^{-1}$ and $[x, y] = x y x^{-1} y^{-1}$. Let $G$ be any group. We form the group $G \times G$, generated by the symbols $g \times h$, where $g, h \in G$, subject to the following relations:

$$
g g' \times h = (g g' \times h)(g \times h),
g \times h h' = (g \times h)(h g \times h'),
x \times y = 1,
$$

for all $g, g', h, h' \in G$, and all $x, y \in G$ with $[x, y] = 1$. The group $G \times G$ is a quotient of the non-abelian exterior square $G \times G$ of $G$ defined by Miller [8]. There is a surjective homomorphism $\kappa : G \times G \to [G, G]$ defined by $\kappa(x \times y) = [x, y]$ for all $x, y \in G$. Denote $\tilde{B}_0(G) = \ker \kappa$. By [9] we have the following:

**Theorem 2.1** ([9]). Let $G$ be a finite group. Then $B_0(G)$ is naturally isomorphic to $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, and thus $B_0(G) \cong \tilde{B}_0(G)$.

Let $L$ be a group. A function $\phi : G \times G \to L$ is called a $\tilde{B}_0$-pairing if for all $g, g', h, h' \in G$, and for all $x, y \in G$ with $[x, y] = 1$, $\phi(g g', h) = \phi(g g', h) \phi(g, h)$, $\phi(g, h h') = \phi(g, h) \phi(h g, h')$, $\phi(x, y) = 1$. 


Clearly a $\tilde{B}_0$-pairing $\phi$ determines a unique homomorphism of groups $\phi^* : G \times G \to L$ such that $\phi^*(g \cdot h) = \phi(g, h)$ for all $g, h \in G$.

We now turn to the proof of Theorem 1.2. Let $G_1$ and $G_2$ be isoclinic groups, and denote $Z_1 = Z(G_1)$, $Z_2 = Z(G_2)$. By definition [5], there exist isomorphisms $\alpha : G_1/Z_1 \to G_2/Z_2$ and $\beta : [G_1, G_1] \to [G_2, G_2]$ such that whenever $\alpha(a_1 Z_1) = a_2 Z_2$ and $\alpha(b_1 Z_1) = b_2 Z_2$, then $\beta([a_1, b_1]) = [a_2, b_2]$ for $a_1, b_1 \in G_1$. Define a map $\phi : G_1 \times G_1 \to G_2 \times G_2$ by $\phi(a_1, b_1) = a_2 \cdot b_2$, where $a_i, b_i$ are as above. To see that this is well defined, suppose that $\alpha(a_1 Z_1) = a_2 Z_2 = \tilde{a}_2 Z_2$ and $\alpha(b_1 Z_1) = b_2 Z_2 = \tilde{b}_2 Z_2$. Then we can write $\tilde{a}_2 = a_2 z$ and $\tilde{b}_2 = b_2 w$ for some $w, z \in Z_2$. By the definition of $G_2 \times G_2$ we have that $\tilde{a}_2 \cdot \tilde{b}_2 = a_2 z \cdot b_2 w = a_2 \cdot b_2$, hence $\phi$ is well defined.

Suppose that $a_1, b_1 \in G_1$ commute, and let $a_2, b_2 \in G_2$ be as above. By definition, $[a_2, b_2] = \beta([a_1, b_1]) = 1$, hence $a_2 \cdot b_2 = 1$. This, and the relations of $G_2 \times G_2$, ensure that $\phi$ is a $\tilde{B}_0$-pairing. Thus $\phi$ induces a homomorphism $\gamma : G_1 \times G_1 \to G_2 \times G_2$ such that $\gamma(a_1 \cdot b_1) = a_2 \cdot b_2$ for all $a_1, b_1 \in G_1$. By symmetry there exists a homomorphism $\delta : G_2 \times G_2 \to G_1 \times G_1$ defined via $\alpha^{-1}$. It is straightforward to see that $\delta$ is the inverse of $\gamma$, hence $\gamma$ is an isomorphism.

Let $\kappa_1 : G_1 \times G_1 \to [G_1, G_1]$ and $\kappa_2 : G_2 \times G_2 \to [G_2, G_2]$ be the commutator maps. Since $\beta \kappa_1(a_1 \cdot b_1) = \beta([a_1, b_1]) = [a_2, b_2] = \kappa_2 \gamma(a_1 \cdot b_1)$, we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & \tilde{B}_0(G_1) & \rightarrow & G_1 \times G_1 & \overset{\kappa_1}{\rightarrow} [G_1, G_1] & \rightarrow & 0 \\
\downarrow{\tilde{\gamma}} & & \downarrow{\gamma} & & \downarrow{\beta} & & \\
0 & \rightarrow & \tilde{B}_0(G_2) & \rightarrow & G_2 \times G_2 & \overset{\kappa_2}{\rightarrow} [G_2, G_2] & \rightarrow & 0
\end{array}
\]

Here $\tilde{\gamma}$ is the restriction of $\gamma$ to $\tilde{B}_0(G_1)$. Since $\beta$ and $\gamma$ are isomorphisms, so is $\tilde{\gamma}$. This concludes the proof.

References


