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On the connectivity of Cartesian product of graphs*

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Abstract

We give a new alternative proof of Liouville's formula which states that for any graphs G and H on at least two vertices, $\kappa(G \square H) = \min\{\kappa(G)|H|,\ |G|\kappa(H),\ \delta(G) + \delta(H)\}$, where κ and δ denote the connectivity number and minimum degree of a given graph, respectively. The main idea of our proof is based on construction of a vertex-fan which connects a vertex from $V(G \square H)$ to a subgraph of $G \square H$. We also discuss the edge version of this problem as well as formula for products with more than two factors.

Keywords: Connectivity, Cartesian product.

Math. Subj. Class.: 05C40, 05C76

1 Introduction

The Cartesian product has been studied extensively since the 1950's. Despite of its simple definition, answering many underlying questions is far from being trivial. One such question refers to a connectivity of a Cartesian product and how it depends on invariants of its factors.

The first result of this type appeared in an article written by Sabidussi [5] in 1957. He proved that for arbitrary graphs G and H, $\kappa(G \square H) \geq \kappa(G) + \kappa(H)$. In 1978, Liouville [4] conjectured that for graphs G and H on at least two vertices, $\kappa(G \square H) = \min\{\kappa(G)|H|,\ |G|\kappa(H),\ \delta(G) + \delta(H)\}$, where κ and δ denote the connectivity number

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and minimum degree of a given graph, respectively. The result of Subidussi was improved by Xu and Yang [7]. They showed that $\kappa(G\square H) \geq \min\{\kappa(G)+\delta(H), \kappa(H)+\delta(G)\}$, where G and H are connected undirected graphs. Finally, the Liouville's formula has been recently proved by Špacapan in [6]. For more information on the topic, see also [1, 2, 3].

We use the following terminology. The *Cartesian product* of two graphs G and H, denoted by $G\square H$, is a graph with vertex set $V(G)\times V(H)$, where two vertices (g,h) and (g',h') are adjacent if $gg' \in E(G)$ and $h = h'$ or $g = g'$ and $hh' \in E(H)$. The graphs G and H are called the *factors* of $G\square H$. For any $h \in V(H)$, we denote by G^h the subgraph of $G\square H$ induced by $V(G)\times\{h\}$ and name it *G-fiber*. Similarly, we can define *H-fiber*.

For a graph G and $v \in V(G)$, the *degree* of a vertex v is denoted by $d_G(v)$, or simply $d(v)$ if the graph G is known from the context. Furthermore, we denote by $\delta(G)$ the minimum degree of a graph G. The minimum degree is additive under Cartesian products, i.e. $\delta(G\square H) = \delta(G)+\delta(H)$. Recall that the symbol $N_G(v)$ denotes the set of neighbours of a vertex v in a graph G.

For a connected graph G a subset $S \subseteq V(G)$ is a *separating set* if $G-S$ has more than one component. The *connectivity* $\kappa(G)$ of G is the minimum size of $S \subseteq V(G)$ such that $G-S$ is disconnected or a single vertex. For any $k \leq \kappa(G)$, we say that G is *k-connected*. A subset of edges $S' \subseteq E(G)$ is *disconnecting set* if $G-S'$ has more than one component. The *edge-connectivity* $\lambda(G)$ of G is the maximum k for which G is k-edge-connected, i.e. every disconnecting set consists of at least k edges.

A set of (u,W)-paths, where $W \subset V(G)$ and $u \in V(G)\setminus W$, is called a *vertex-fan* (resp. an *edge-fan*) if any two of the considered paths have only vertex u in common (resp. edge-disjoint paths). We say that a vertex-fan *avoids a vertex* v if its paths do not contain v.

In this paper, we provide an alternative poof of Liouville's formula. Our approach to proving this result includes construction of a vertex-fan which connects a vertex from $V(G\square H)$ to a subgraph of $G\square H$. Namely, for every vertex $(a,b) \in V(G\square H)$ there exists a vertex-fan of minimum size $d(a)+d(b)$ which connects a chosen vertex and a connected subgraph of $G\square H$ comprised of a fiber of G and a fiber of H. We also discuss the edge version of this problem as well as formula for products with more than two factors.

2 Connectivity of Cartesian product

Here, we construct the fan.

Proposition 2.1. Let G and H be connected graphs, $a, c \in V(G)$ and $b, d \in V(H)$ distinct vertices. Then, there exists a vertex-fan F from (a,b) to $G^d \cup {}^cH$ of size $d(a)+d(b)$ that avoids the vertex (c,d). Moreover $d(a)$ paths of F are ended in G^d and $d(b)$ paths of F are ended in ${}^cH$.

Proof. Let $N_G(a) = \{a_1, a_2, \ldots, a_{d(a)}\}$ and $N_H(b) = \{b_1, b_2, \ldots, b_{d(b)}\}$. Let $P_G = ax_1x_2x_3\cdots x_k(=c)$ (resp. $P_H = by_1y_2y_3\cdots y_l(=d)$) be a shortest path that connects a and c in G (resp. b and d in H). Thus, x_1 (resp. y_1) is the only neighbour of a (resp. b) that is contained in P_G (resp. P_H). So, without loss of generality, we can assume that $x_1 = a_1$ and $y_1 = b_1$.

Now, we construct a vertex-fan F from (a,b) to $G^d \cup {}^cH$. First, we define $d(a)$ vertex-disjoint paths that use copies of the path P_H to reach the fiber G^d:

$$\begin{array}{lll} (a,b)P_{^aH} & \text{where} & P_{^aH} = (a,y_1)(a,y_2)(a,y_3)\cdots(a,y_{l-1})(a,d), \\ (a,b)(a_i,b)P_{^{a_i}H} & \text{where} & P_{^{a_i}H} = (a_i,y_1)(a_i,y_2)(a_i,y_3)\cdots(a_i,y_{l-1})(a_i,d), \end{array}$$

$i = 2, 3, \ldots, d(a)$. Furthermore, we construct $d(b)$ vertex-disjoint paths that use copies of the path P_G to reach the fiber ${}^cH$. For every $j = 2, 3, \ldots, d(b)$ define:

$$\begin{array}{lll} (a,b)P_{G^b} & \text{where} & P_{G^b} = (x_1,b)(x_2,b)(x_3,b)\cdots(x_{k-1},b)(c,b), \\ (a,b)(a,b_j)P_{G^{b_j}} & \text{where} & P_{G^{b_j}} = (x_1,b_j)(x_2,b_j)(x_3,b_j)\cdots(x_{k-1},b_j)(c,b_j). \end{array}$$

As can be easily seen from the construction, defined paths are vertex disjoint and none contains the vertex (c,d).

□

Now, we prove the formula.

Theorem 2.2. Let G and H be graphs on at least two vertices. Then,

$$\kappa(G\Box H) = \min\{\kappa(G)|H|,\ |G|\kappa(H),\ \delta(G)+\delta(H)\}.$$

Proof. First we show that $\kappa(G\Box H)$ is at most the claimed minimum. Let S be a separating set of the graph G. Then $S \times V(H)$ is a separating set of $G\Box H$. Consequently, $\kappa(G\Box H) \leq \kappa(G)|H|$ and analogously, $\kappa(G\Box H) \leq \kappa(H)|G|$. Since $\kappa(G\Box H) \leq \delta(G\Box H) = \delta(G) + \delta(H)$, it follows that $\kappa(G\Box H) \leq \min\{\kappa(G)|H|,\ |G|\kappa(H),\ \delta(G)+\delta(H)\}$ and we have shown desired inequality.

Now, we show that $\kappa(G\Box H)$ is at least the claimed minimum. Suppose it is false and $G\Box H$ has a separating set S with $|S| < \min\{\kappa(G)|H|,\ |G|\kappa(H), \delta(G)+\delta(H)\}$. Then, there exist vertices $c \in V(G)$ and $d \in V(H)$ such that $|V(G^d) \cap S| < \kappa(G)$ and $|V({}^cH) \cap S| < \kappa(H)$. In particular, $G^d - S$ and ${}^cH - S$ are connected.

Notice that Proposition 2.1 implies that each vertex of $G\Box H - S$ is connected to $G^d \cup {}^cH$ by a path avoiding S. Hence, if $(c,d) \notin S$, then $G^d \cup {}^cH - S$ is connected, and so is $G\Box H - S$.

So assume $(c,d) \in S$. Then $G\Box H - S$ has at most two components, one containing $G^d - S$ and other containing ${}^cH - S$. Now we will find a vertex adjacent to both of these subgraphs, and this will imply connectedness of $G\Box H - S$.

If we can choose $c_1 \in N_G(c)$ and $d_1 \in N_H(d)$ such that none of vertices (c,d_1), (c_1,d_1), (c_1,d) belongs to S, then (c_1,d_1) connects $G^d - S$ and ${}^cH - S$ and hence connectedness of $G\Box H - S$ follows. Suppose that we cannot make such a choice. Let x (resp. y) be the number of neighbours of (c,d) in $G^d - S$ (resp. ${}^cH - S$). Then $x\,y$ vertices from $N_G(c) \times N_H(d)$ must be in S, because of the assumption. So there are at least $d_G(c) - x + d_H(d) - y + x\,y + 1$ vertices in S.

By the assumption on S, $d_G(c) - x + d_H(d) - y + x\,y + 1 < \delta(G) + \delta(H) < d_G(c) + d_H(d)$ and after simple transformation one can obtain $x\,y + 1 < x + y$. But the latter inequality holds if and only if $x = 0$ or $y = 0$. Since $\kappa(G) \leq \delta(G)$, this contradicts the assumptions that both fibers G^d and ${}^cH$ contain less than $\kappa(G)$ resp. $\kappa(H)$ vertices of S.

Hence, we showed that $G^d \cup {}^cH - S$ is connected, and so is $G\Box H - S$. Therefore, $|S| \geq \delta(G) + \delta(H)$. □

We discuss the edge version of this problem. A similar result for the edge-connectivity of the Cartesian product was proved by Xu and Yang [7] in 2006 using the edge version of Menger's theorem:

Theorem 2.3. Let G and H be graphs on at least two vertices. Then,

$$\lambda(G\Box H) = \min\{\lambda(G)|H|,\ |G|\lambda(H),\ \delta(G)+\delta(H)\}.$$

In 2008 new short version of the proof, avoiding Menger's theorem, appeared in [3].

We can use the fan from Proposition 2.1 and a very simplified argument of Theorem 2.2 (mainly the first two paragraphs) to prove Theorem 2.3. Moreover, instead of the fan of Proposition 2.1, we can use the following simpler one (using notation from the proof of Proposition 2.1): for every $i = 1, 2, \dots, d(a)$ define:

$$(a,b)(a_i,b)P_{^{a_i}H}, \quad \text{where} \quad P_{^{a_i}H} = (a_i,y_1)(a_i,y_2)(a_i,y_3)\cdots(a_i,y_{l-1})(a_i,d),$$

and for every $j = 1, 2, \dots, d(b)$ define

$$(a,b)(a,b_j)P_{G^{b_j}}, \quad \text{where} \quad P_{G^{b_j}} = (x_1,b_j)(x_2,b_j)(x_3,b_j)\cdots(x_{k-1},b_j)(c,b_j).$$

It is easy to see that this is an edge-fan of size $d(a)+d(b)$ from (a,b) to $G^d \cup {}^cH$.

Finally, we would like to stress that above approach enables us to generalize Liouville's formula for Cartesian products with more than two factors. We state this result in the following theorem.

Theorem 2.4. Let G_i, $i = 1, 2, \dots, n$ be graphs on at least two vertices and let $G = G_1\Box G_2\Box\cdots\Box G_n$. Then,

$$\kappa(G) = \kappa(G_1\Box\cdots\Box G_n) = \min\left\{\frac{\kappa(G_1)}{|G_1|}|G|,\ \dots,\ \frac{\kappa(G_n)}{|G_n|}|G|, \delta(G_1)+\cdots+\delta(G_n)\right\}.$$

Proof. We prove given equality using induction on the number of factors n. The base case for $n = 2$ is proved in Theorem 2.2. In order to prove that equality holds for n, we consider $\kappa(G) = \kappa(G_1\Box\cdots\Box G_n)$ as the Cartesian product of two graphs $G' = G_1\Box\cdots\Box G_{n-1}$ and G_n.

Then, by the induction hypothesis,

$$\begin{aligned}\kappa(G'\Box G_n) &= \min\{\kappa(G')|G_n|,\ \kappa(G_n)|G'|,\ \delta(G')+\delta(G_n)\}\\ &= \min\left\{\kappa(G')|G_n|,\ \tfrac{\kappa(G_n)}{|G_n|}|G|,\ \delta(G_1)+\cdots+\delta(G_{n-1})+\delta(G_n)\right\}\\ &= \min\left\{\tfrac{\kappa(G_1)}{|G_1|}|G|,\ \dots,\ \tfrac{\kappa(G_{n-1})}{|G_{n-1}|}|G|,\ \tfrac{\kappa(G_n)}{|G_n|}|G|,\ \delta(G_1)+\cdots+\delta(G_n)\right\}\end{aligned}$$

where the last equality holds by the induction hypothesis applied on $\kappa(G') = \kappa(G_1\Box\cdots\Box G_{n-1})$ and an obvious inequality $\delta(G_1)+\cdots+\delta(G_n) < (\delta(G_1)+\cdots+\delta(G_{n-1}))|G_n|$. □

Regarding the above formula, if all G_i's are isomorphic to a graph H, then we obtain the formula

$$\kappa(H^n) = \min\left\{\kappa(H)|H|^{n-1}, n\delta(H)\right\} = n\delta(H),$$

which was observed in [3].

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