

On stratifications for planar tensegrities with a small number of vertices

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Abstract

In this paper we discuss several results about the structure of the configuration space of two-dimensional tensegrities with a small number of points. We briefly describe the technique of surgeries that is used to find geometric conditions for tensegrities. Further we introduce a new surgery for three-dimensional tensegrities. Within this paper we formulate additional open problems related to the stratified space of tensegrities.

Keywords: Tensegrities, equilibrium, surgeries.

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1 Introduction

In this paper we study the stratified spaces of tensegrities with a small number of points. We work mostly with planar tensegrities. In the case of 4 and 5 point configurations we give an explicit description of all the strata and present a visualization of the entire stratified space. Further we give a geometric description of the strata for 6 and 7 points and use the technique of surgeries to find new geometric conditions adding to the list of already known ones. In particular, we introduce a new surgery for tensegrities in \mathbb{R}^3 .

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1.1 Configuration space of tensegrities

The first steps in the study of rigidity and flexibility of tensegrities were made by B. Roth and W. Whiteley in [9] and further developed by R. Connelly and W. Whiteley in [3], see also the survey about rigidity in [13]. N. L. White and W. Whiteley in [12] started the investigation of geometric conditions for a tensegrity with prescribed bars and cables. In the preprint [7] M. de Guzmán describes several other examples of geometric conditions for tensegrities.

Let us recall standard definitions of tensegrities (as in [2], [4], etc.). See also [10] for a collection of open problems and a good bibliography.

Definition 1.1. Fix a positive integer d . Let $G = (V, E)$ be an arbitrary graph without loops and multiple edges. Let it have n vertices v_1, \dots, v_n .

- A *configuration* is a finite collection P of n labeled points (p_1, p_2, \dots, p_n) , where each point p_i (also called a *vertex*) is in a fixed Euclidean space \mathbb{R}^d .
- The embedding of G with straight edges, induced by mapping v_j to p_j is called a *tensegrity framework* and it is denoted as $G(P)$.
- We say that a *load* or *force* F acting on a framework $G(P)$ in \mathbb{R}^d is an assignment of a vector f_i in \mathbb{R}^d to each vertex i of G .
- We say that a *stress* w for a framework $G(P)$ in \mathbb{R}^d is an assignment of a real number $w_{i,j} = w_{j,i}$ (we call it an *edge-stress*) to each edge $p_i p_j$ of G . An edge-stress is regarded as a tension or a compression in the edge $p_i p_j$. For simplicity reasons we put $w_{i,j} = 0$ if there is no edge between the corresponding vertices. We say that w *resolves* a load F if the following vector equation holds for each vertex i of G :

$$f_i + \sum_{\{j|j \neq i\}} w_{i,j}(p_j - p_i) = 0.$$

By $p_j - p_i$ we denote the vector from the point p_i to the point p_j .

- A stress w is called a *self-stress* if, the following equilibrium condition is fulfilled at every vertex p_i :

$$\sum_{\{j|j \neq i\}} w_{i,j}(p_j - p_i) = 0.$$

- A couple $(G(P), w)$ is called a *tensegrity* if w is a self-stress for the framework $G(P)$.
- If $w_{i,j} < 0$ then we call the edge $p_i p_j$ a *cable*, if $w_{i,j} > 0$ we call it a *strut*.

Let $W(n)$ denote the linear space of dimension n^2 of all edge-stresses $w_{i,j}$. Consider a framework $G(P)$ and denote by $W(G, P)$ the subset of $W(n)$ of all possible self-stresses for $G(P)$. By definition the set $W(G, P)$ is a linear subspace of $W(n)$.

Definition 1.2. The *configuration space of tensegrities* corresponding to the graph G is the set

$$\Omega_d(G) := \{(G(P), w) \mid P \in (\mathbb{R}^d)^n, w \in W(G, P)\}.$$

The set $\{G(P) \mid P \in (\mathbb{R}^d)^n\}$ is said to be the *base of the configuration space*, we denote it by $B_d(G)$.

1.2 Stratification of the base of a configuration space of tensegrities

Suppose we have some framework $G(P)$ and we want to find a cable-strut construction on it. Then *which edges can be replaced by cables, and which by struts? What is the geometric position of points for which given edges may be replaced by cables and the others by struts?* These questions lead to the following definition.

Definition 1.3. A set $W(G, P_1)$ is said to be *equivalent* to a set $W(G, P_2)$ if there exists a homeomorphism ξ between $W(G, P_1)$ and $W(G, P_2)$, such that for any self-stress w in $W(G, P_1)$ the self-stress $\xi(w)$ satisfies

$$\operatorname{sgn}(\xi(w)) = \operatorname{sgn}(w).$$

Henceforth we call a set $W(G, P)$ a *linear fiber*. The described equivalence relation on linear fibers gives us a stratification of the base $B_d(G) = (\mathbb{R}^d)^n$. A *stratum* is by definition a maximal connected set of points with equivalent linear fibers. In the paper [4] we prove that all strata are semialgebraic sets (which implies for instance that they are path connected).

The idea of this paper is to make the first steps in the study of particular configuration spaces of tensegrities. We present the techniques to find geometric conditions and open problems for further study that already arise in very simple situations of 9 point configurations.

Let us, first, make the following three *general remarks*.

GR1. The majority of the strata of codimension k can be defined by algebraic equations and inequalities that define the strata of codimension 1. The exceptions here are mostly in high codimension (the simplest one is as follows: for two points connected by an edge there is no codimension 1 stratum, but there is one codimension 2 stratum corresponding to coinciding points; actually it is interesting to find the complete list of such exceptions). So the most important case to study is the codimension 1 case.

GR2. A stratification of a subgraph is a substratification of the original graph (i.e., each stratum for a subgraph is the union of certain strata for the original graph), hence below we skip the description of $B_2(G)$ for graphs with 5 vertices other than K_5 .

GR3. For any stratum there exists a certain subgraph that *locally identifies* the stratum (i.e., for any point x of the stratum there exists a neighborhood $B(x)$ such that any configuration in $B(x)$ has a nonzero self-stress for the subgraph if and only if this point is on the stratum).

According to general remarks GR1 and GR2 the most interesting case is to study the strata of codimension 1 for the complete graph on n vertices (denoted further by K_n). It is possible to find some of the strata of K_n directly. For the other strata one, first, should find an appropriate subgraph that locally identifies the stratum, and then find appropriate surgeries (explained in Section 3) to reduce the complexity of the subgraph to find geometric conditions.

This paper is organized as follows. In Section 2 we study the stratification of configuration spaces of tensegrities in the plane with a small number of vertices. In Subsections 2.1 and 2.2 we briefly describe the trivial cases of two and three point configurations. Further in Subsections 2.3 and 2.4 we study the four and the five point cases. In each of the cases we describe the geometry and the number of strata. In addition we introduce the adjacency

diagram of full dimension and codimension 1 strata. In Subsections 2.5 and 2.6 we describe geometric conditions for the codimension 1 strata of 6, 7, and 8 point tensegrities. In Section 3 we present the technique of surgeries to find geometric descriptions for the strata. In Subsection 3.1 we describe surgeries that do not change graphs, and in Subsection 3.2 we show a couple of surgeries in the two-dimensional case. We introduce a new three-dimensional surgery in Subsection 3.3. In conclusion, we formulate several open questions in Subsection 3.4.

2 Stratification of the space $B_2(K_n)$ for small n

In this section we study the geometry of tensegrity stratifications for graphs with a small number of vertices. The cases of $n = 2, 3, 4, 5$ are studied in full detail. Starting from $n = 6$ there are some gaps in the understanding of tensegrities. Still for $n = 6, 7, 8$ the complete description of the geometric conditions for the strata is known, we briefly describe several results on them here (see [4] for more information).

2.1 Case of two points

Consider, first, the case of two points ($n = 2$). There are only two graphs on two points: a complete one K_2 and a graph without edges (denote it by $G_{0,2}$).

All the fibers of the base $B_2(G_{0,2}) = \mathbb{R}^4$ are of dimension 0, and, therefore, they are equivalent. Hence the stratification is trivial.

The complete graph K_2 here has only one edge. If two points of the graph do not coincide then the stress at this edge should be zero. When two points coincide then the stress at the edge can be arbitrary, and we have a one-dimensional set of solutions (i.e., a fiber). So the base $B_2(K_2) = \mathbb{R}^4$ has a codimension 2 stratum (a 2-dimensional plane). The complement to this stratum is a stratum of codimension 0.

2.2 Three point configurations

There are four different types of graphs here: let $G_{i,3}$ be the graph with i edges for $i = 0, 1, 2, 3$.

In cases $G_{0,3}$ and $G_{1,3}$ the base stratifications are the following direct products:

$$B_2(G_{0,3}) = B_2(G_{0,2}) \times \mathbb{R}^2 \quad \text{and} \quad B_2(G_{1,3}) = B_2(K_2) \times \mathbb{R}^2.$$

So $B_2(G_{0,3})$ is trivial and $B_2(G_{1,3})$ has a 4-dimensional subspace and its complement as strata.

The base $B_2(G_{2,3})$ contains five strata. One of them corresponds to the configuration where three points coincide: the fiber here is 2-dimensional, this stratum is isometric to \mathbb{R}^2 . There are three strata where one of the edges of the graph vanishes: they are isometric to $\mathbb{R}^4 \setminus \mathbb{R}^2$. Finally, the complement to the union of these strata is the only stratum of maximal dimension. There are no nonzero tensegrities for a configuration in this stratum.

For the complete graph on three vertices we have, for the first time, codimension 1 strata. There are three codimension 1 strata, all of them correspond to the following configuration: three points are in one line. Different strata correspond to having a different point between the two others.

Let us briefly describe one of such strata. Let $P_i = (x_i, y_i)$ be the points of the graph ($i = 1, 2, 3$). Then the condition that the three points are in a line is defined by a quadratic

equation:

$$(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) = 0$$

This quadric divides the space into two connected components: corresponding to positively and negatively oriented triangles.

To sum up we present for $B_2(K_3)$ the following table.

Dimension of a stratum	0	1	2	3	4	5	6
Number of such strata	0	0	1	0	3	3	2

2.3 Stratification of $B_2(K_4)$

In this subsection we restrict ourselves to the complete graph K_4 (for its subgraphs we apply the reasoning of GR2 above). A plane configuration of four points in general position admits a unique tensegrity (up to a multiplicative constant), which is called an *atom*. In [8] it was proved that any self-stress for K_n is a sum of self-stressed atoms in K_n (i.e., a sum of certain $K_4 \subset K_n$ with scalars). For K_4 there are exactly 14 strata of general position.

The strata of codimension 1 correspond to three of four points of the graph lying in a line. Actually in this case there is no jump of dimension of the fiber: there is also a unique (up to scalar) solution corresponding to the three points in a line. But the stresses on the edges from the fourth point are all zero, and hence a fiber of this stratum is not equivalent to general fibers. The number of such strata is 24.

In codimension 2 we have two different types of strata corresponding to

- four points in a line: the dimension of a fiber is 2 (twelve strata);
- two points coincide: the dimension of a fiber is 1 (twelve strata).

In codimension 3 there is one type of strata with configurations of four points in a line, two of which coincide. Six of them with the double point in the middle and twelve of them with the double point not in the middle.

In codimension 4, there are two types of strata:

- three points coincide (4 strata);
- two pairs of points coincide (3 strata).

And, finally, there is a codimension 6 stratum when all four points coincide. We remark that for none of the strata the fiber is 3-dimensional.

The cardinalities of strata are shown in the following table.

Dimension of a stratum	0	1	2	3	4	5	6	7	8
Number of strata	0	0	1	0	7	18	24	24	14

2.3.1 The space of formal configurations

Let us draw schematically the adjacency of the strata of maximal dimension via strata of codimension 1. The dimension of the stratified space is 8, let us reduce it to two via factoring by proper affine transformations. We will use the following simple proposition.

Proposition 2.1. *Invertible affine transformations of the plane do not change the equivalence class of a fiber $W(G, P)$. In other words if P is a configuration and T an invertible affine transformation of the plane then*

$$W(G, P) \simeq W(G, T(P)).$$

□

So instead of studying the stratification itself we restrict to the set of formal configurations with respect to proper affine transformations of the plane.

Definition 2.2. We say that a four point configuration v_1, v_2, v_3, v_4 is *formal* in one of the following cases:

i) nondegenerate case: a configuration $P_{x,y,+}$ with vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (x, y)$, $v_4 = (x, y+1)$ for arbitrary (x, y) .

ii) nondegenerate case: a configuration $P_{x,y,-}$ with $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (x, y)$, $v_4 = (x, y-1)$ for arbitrary (x, y) .

iii) degenerate case: a configuration $P_{\Delta,+}$ with $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (0, 1)$, $v_4 = (\Delta, 1)$ for an arbitrary Δ .

iv) degenerate case: a configuration $P_{\Delta,-}$ with $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (0, -1)$, $v_4 = (\Delta, -1)$ for an arbitrary Δ .

v) closure: we add two formal configurations $P_{\pm\infty}$ with vertices $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (1, 0)$, $v_4 = (1, \pm\infty)$.

We denote the set of all formal configurations by Λ_4 .

In some sense the space Λ_4 is the space of all codimension 0 and codimension 1 configurations factored by the group of proper affine transformations.

Proposition 2.3. For any codimension 0 and codimension 1 configuration there exists a unique formal configuration to which the first configuration can be affinely deformed. □

The space Λ_4 is endowed with a natural topology of a quotient space.

Proposition 2.4. There is a natural topology of a sphere S^2 for the set Λ_4 .

Proof. Let us introduce a topology of the unit sphere S^2 for Λ_4 . Consider the configurations of case i) on the plane $z = 1$: we identify the point $P_{x,y,+}$ with the point $(x, y, 1)$. Consider the projection of this plane to the upper unit hemisphere S^2 from the origin. So we have a one to one correspondence between the configurations of case i) and the upper hemisphere.

Similarly we take the plane $z = -1$ for the case ii) identifying the point $(-x, -y, -1)$ with the configuration $P_{x,y,-}$ and projecting it to the lower hemisphere.

For the equator of the unit sphere we use all the other cases as asymptotic directions. First, we associate the configuration $P_{\Delta,+}$ with the point

$$(\cos(\pi - \operatorname{arccotan} \Delta), \sin(\pi - \operatorname{arccotan} \Delta), 0).$$

Let us explain the topology at one of such points of the equator. Suppose we start with $P_{x,y,+}$. The transformation sending the first three points to $(0, 0)$, $(1, 0)$, and $(0, 1)$ is linear with matrix

$$\begin{pmatrix} 1 & -x/y \\ 0 & 1/y \end{pmatrix}.$$

Then the image of the fourth point of $P_{x,y,+}$ is $(-x/y, 1+1/y)$. While x tends to infinity and x/y tends to Δ the last point tends to $(-\Delta, 1)$, and hence the configuration $P_{x,y,+}$ tends to $P_{-\Delta,+}$, as in the above formula.

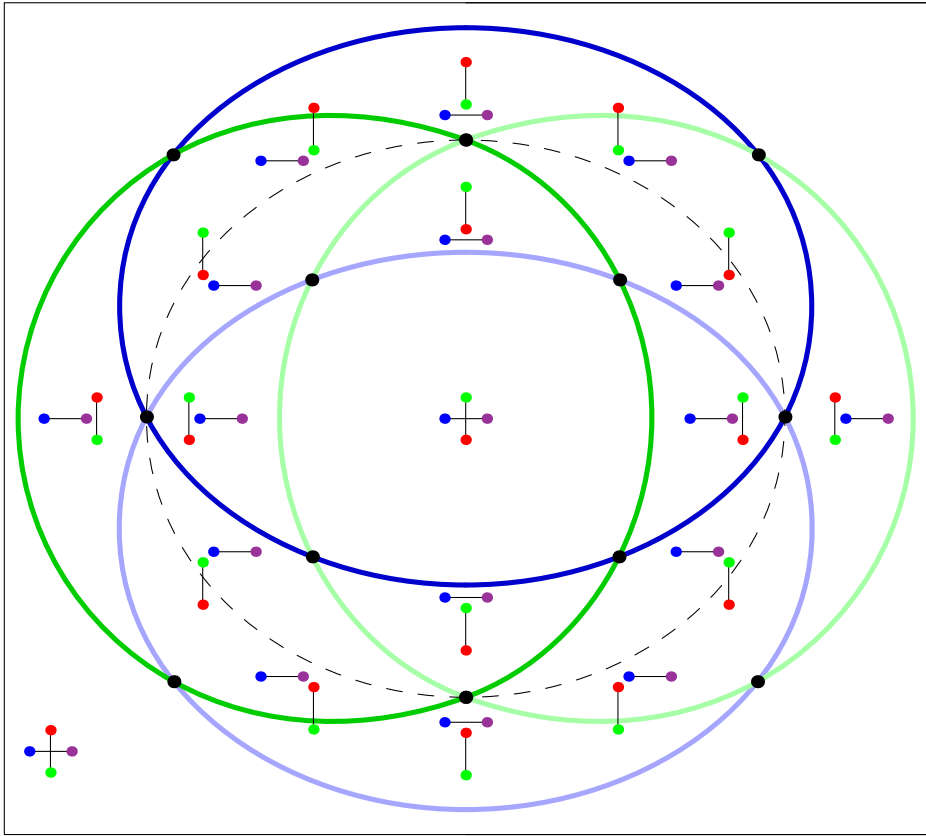


Figure 1: Stratification of $B_2(K_4)$.

Secondly, we associate $P_{\Delta,-}$ with the point

$$(\cos(-\operatorname{arccotan} \Delta), \sin(-\operatorname{arccotan} \Delta), 0)$$

in a similar way.

Finally, we glue $P_{+\infty}$ and $P_{-\infty}$ to the points $(1, 0, 0)$ and $(-1, 0, 0)$ respectively. \square

So, the codimension 0 and 1 stratification of $B_2(K_4)$ can be derived from the stratification of the sphere. We show the stereographic projection of Λ_4 from the point $(0, 0, -1)$ to the plane $z = 1$ on Figure 1. There are four types of strata of codimension 1, they correspond to the fact that certain three points are in a line. They separate the plane into 14 connected components. In each of the connected components we draw a typical type of configuration: (v_1, v_2, v_3, v_4) . Here v_1 is blue, v_2 is purple, v_3 is red and v_4 is green.

Remark 2.5. Different geometric conditions are represented by different colors in the picture, the correspondence is as follows.

- Light blue strata (6 strata forming a circle) correspond to configurations with $v_1, v_2,$ and v_3 in a line.

- Dark blue strata (6 strata) contain configurations with $v_1, v_2,$ and v_4 in a line.
- Light green strata (6 strata) contain configurations with $v_1, v_3,$ and v_4 in a line.
- Dark green strata (6 strata) correspond to configurations with $v_2, v_3,$ and v_4 in a line. We have 24 strata of codimension 1 in total.
- The dashed black line is the projection of the equator. It corresponds to the degenerate case of parallel segments. The dashed line is not a stratum, it has the same fiber as all the points in its neighborhood. While one passes the dashed line the red-green segment "rotates" around the blue-purple segment.

Remark 2.6. The 14 connected components of the plane are in one-to-one correspondence with the 14 faces of a cuboctahedron (accordingly, the 12 points on these circles correspond to its vertices). Thus, the four circles are those circumscribed around the equatorial regular hexagons of the cuboctahedron. The vertices of this polytope lie on a sphere, hence, through stereographic projection the four circumcircles in question project in fact to circles in the image plane.

2.4 Stratification of $B_2(K_5)$

2.4.1 General description of the strata

We have 264 strata of general position.

As in the two previous cases the strata of codimension 1 correspond to three points of the graph lying in a line. The number of such strata is 600.

The following strata are of codimension 2:

- twice three points in a line: 270 strata;
- four points in a line: 120 strata;
- two points coincide: 420 strata.

In codimension 3 we have the following cases:

- three points in a line and one double point: 60 strata;
- four points in a line two of which coincide: 180 strata;
- five points in a line: 60 strata.

For codimension 4 we have the following list:

- one triple point: 20 strata;
- five points in a line two of which coincide: 120 strata;
- two double points: 30 strata.

In codimension 5 we get:

- five points in a line three of which coincide: 30 strata;
- five points in a line with two pairs of points coinciding: 45 strata.

In codimension 6 there are the following strata:

- a triple point and a double point: 10 strata;
- one point and one point of multiplicity four: 5 strata.

And, finally, there is a codimension 8 stratum when all five points coincide.

The cardinalities of the strata are shown in the following table.

Dimension of a stratum	0	1	2	3	4	5	6	7	8	9	10
Number of strata	0	0	1	0	15	75	170	300	810	600	264

2.4.2 Visualization of $B_2(K_5)$

Let us now describe the structure of the stratification $B_2(K_5)$. Like in case of $B_2(K_4)$ we introduce a set Λ_5 which represents the adjacency of strata of full dimension and of codimension 1. By definition we put

$$\Lambda_5 = \Lambda_4 \times \mathbb{R}^2,$$

i.e., we consider all the four point configurations of Λ_4 , and to each configuration we add the fifth point. We take the product topology for Λ_5 .

So at each point of Λ_4 we attach an \mathbb{R}^2 -fiber. It will soon become clear that for any full dimension stratum of Λ_4 the corresponding fibration is trivial, but the adjacency is not.

On Figures 2 and 3 we show Λ_5 in the following way. We draw the stratification of Λ_4 and inside each connected component we show the typical fiber of the component. The first four points are represented by purple, blue, green, and red points. The lines passing through any pair of them divide the fiber into 18 connected components, that correspond to strata of full dimension. At each such component we write a letter of the Latin alphabet (we consider capital and small letters as distinct).

- Two regions denoted by the same letter and lying in neighboring connected components of Λ_4 separated by light red, dark red, and black strata are in the same stratum.
- Two regions denoted by the same letter and lying in neighboring connected components of Λ_4 separated by light blue, dark blue, light green, and dark green strata are in distinct strata which are adjacent to the same codimension 1 stratum.
- Two regions denoted by a distinct letter and lying in neighboring connected components of Λ_4 are not in one stratum and are not adjacent to the same codimension 1 stratum.

The light blue, dark blue, light green, and dark green strata represent the same geometric conditions as in Remark 2.5 above. For the remaining strata we have:

- The dark red stratum symbolizes that the line through the red and blue points is parallel to the line through the green and purple points.
- The light red stratum symbolizes that the line through the red and purple points is parallel to the line through the green and blue points.
- The black stratum symbolizes that the line through the red and green points is parallel to the line through the purple and blue points.

Remark 2.7. The configuration space $B_2(K_5)$ has several obvious symmetries. First, there is the group of permutations S_5 that acts on the points of $B_2(K_5)$; these symmetries are hardly seen from Figures 2 and 3 since the representation is not S_5 -symmetric. Secondly, there is a symmetry about the origin that sends configurations from $B_2(K_5)$ to themselves, on Figures 2 and 3 we used capital and small letters to indicate this symmetry (for instance, the strata of "a" contain centrally symmetric configurations to the configurations of the strata "A").

As in the case of 4 point configurations we skip the subgraphs of K_5 , see the second general remark above (GR2).

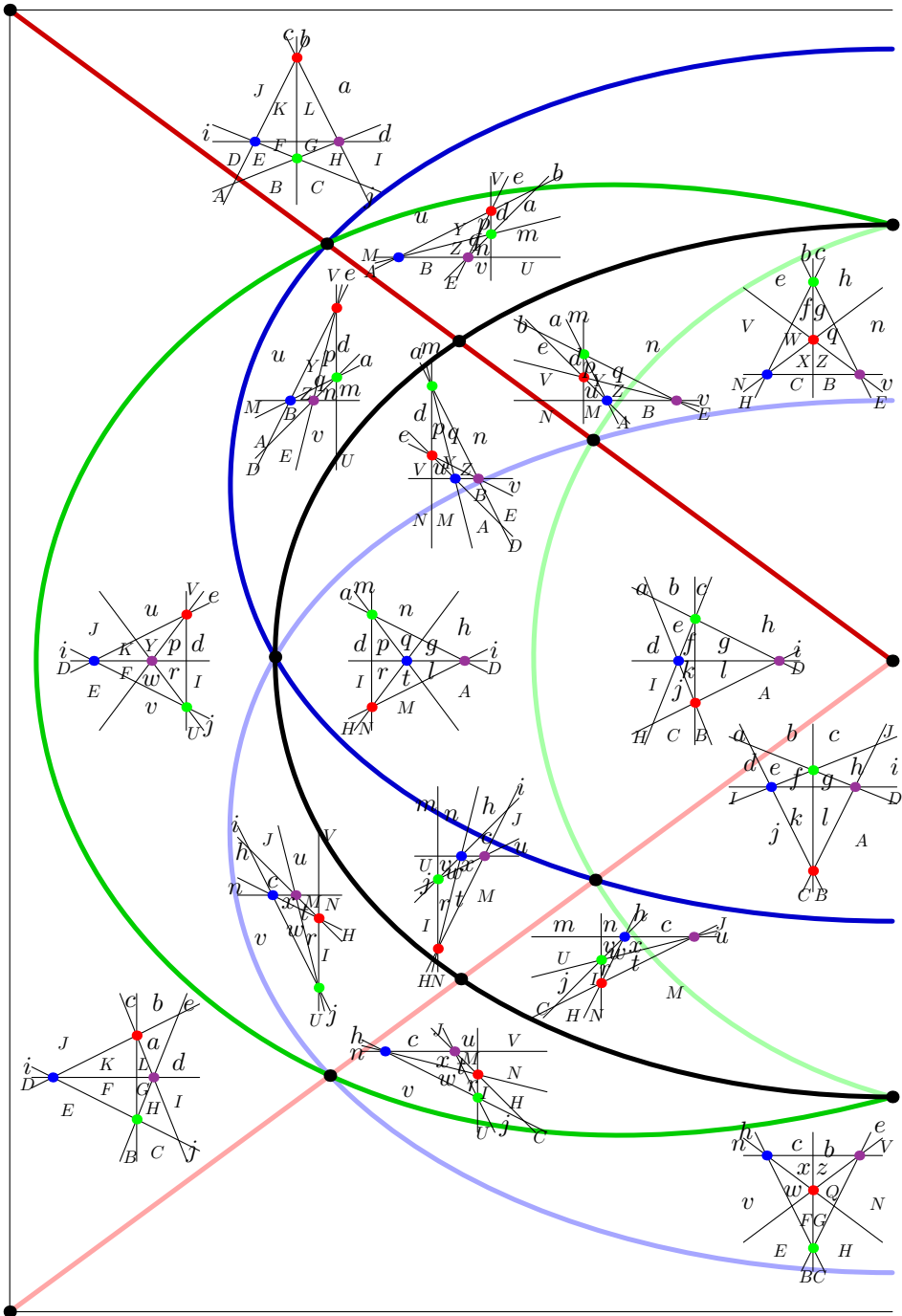


Figure 2: Stratification of $B_2(K_5)$ (Left part).

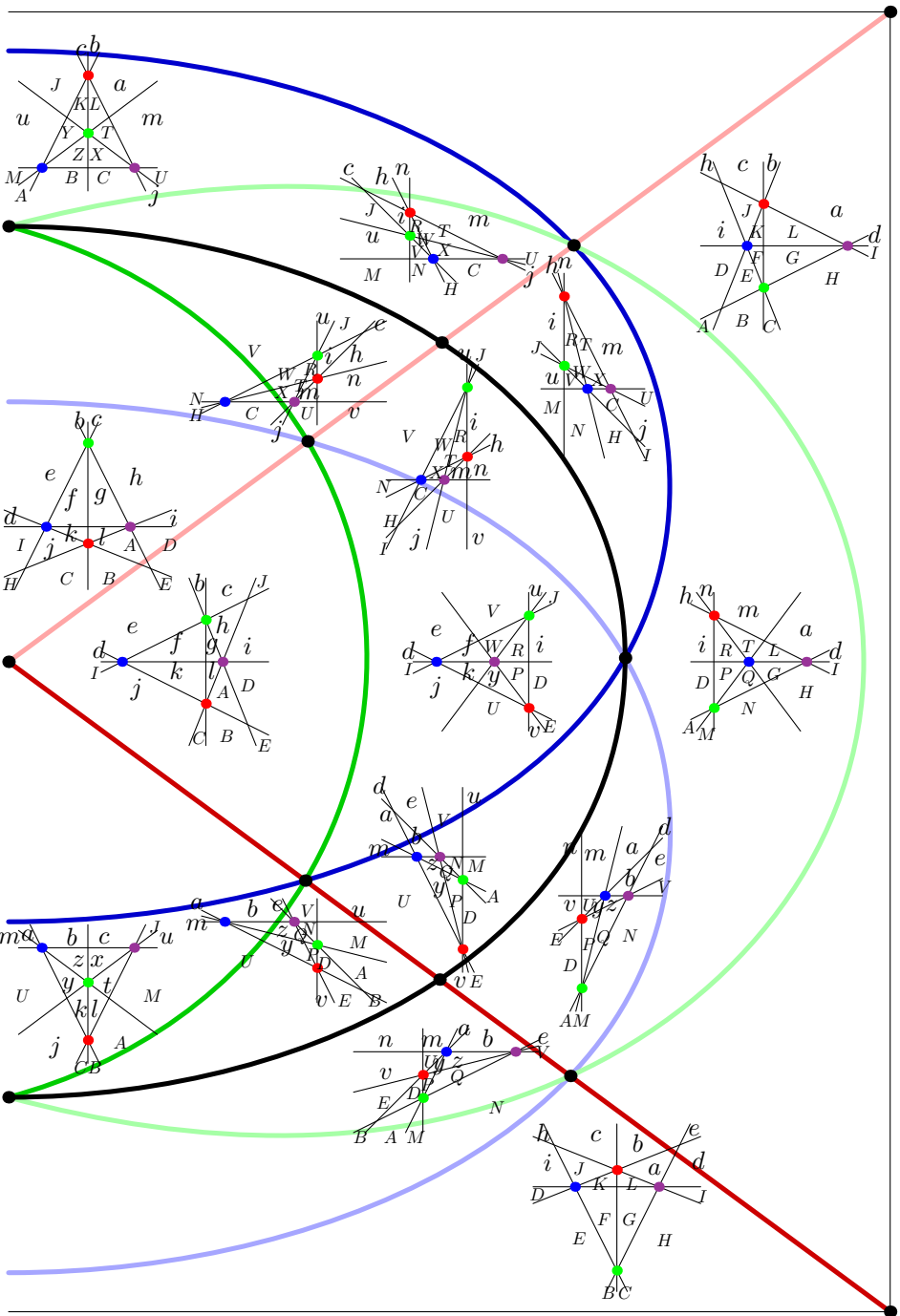


Figure 3: Stratification of $B_2(K_5)$ (Right part).

2.5 Essentially new strata in $B_2(K_6)$

The stratification of $B_2(K_6)$ is much more complicated, at this moment we do not even know how many strata of distinct dimension are present in the stratification.

According to GR1 the first step in studying the stratification of $B_2(K_6)$ is to study all possible distinct types of strata of codimension 1. In the examples of K_n for $n < 6$ we only have strata corresponding to the following geometric condition: three points are in a line. For the case of 6 points we get two additional types of strata: six points on a conic, and three lines passing through three pairs of points have a unique point of intersection.

So the following are three codimension 1 strata (appeared in [12] by N. L. White and W. Whiteley):

- three points in a line;
- the lines v_1v_2 , v_3v_4 , and v_5v_6 meet in one point (or all parallel);
- all the six points are on a conic.

We conclude this subsection with the following problems.

Problem 2.8. Find a description of $B_2(K_6)$, $B_3(K_4)$ and $B_3(K_5)$ similar to the ones for $B_2(K_4)$ and $B_2(K_5)$ shown in the previous subsections.

2.6 A few words about the case $n > 6$

In [4] we have studied strata of the 7 and 8 point configurations. There are 4 distinct types of codimension 1 strata for 7 points and 17 types for 8 points.

The 4 types of codimension 1 strata for 7 points are defined by the following geometric conditions:

- three points in a line;
- the lines v_1v_2 , v_3v_4 , and v_5v_6 meet in one point (or all parallel);
- the lines v_1v_2 , v_3v_4 , and v_5p (where p is the intersection of the lines v_2v_6 and v_3v_7) have a common nonempty intersection;
- the six points v_1 , v_2 , v_3 , v_4 , v_5 , and p (where p is the intersection of the lines v_1v_6 and v_3v_7) are on a conic.

For the list of strata of 8 point configurations we refer to [4].

It turns out that the geometric conditions of any codimension 1 stratum can be obtained by the following procedure. Consider the points of configuration P ; for each two pairs of points (v_i, v_j) and (v_k, v_l) of this configuration consider the point of intersection of the lines v_iv_j and v_kv_l . This leads to a bigger configuration of points including P and the above intersections, we denote it by $U(P)$. This operation can be iteratively applied infinitely many times, which results in a *universal set*

$$U^\infty(P) = \bigcup_{m=0}^{\infty} U^m(P).$$

Any condition for a codimension 1 stratum is always as follows: *three certain points of $U^\infty(P)$ are in a line* (for the details, see for instance [9] and [4]).

Example 2.9. The condition *the lines v_1v_2 , v_3v_4 , and v_5v_6 meet in one point* in terms of points of $U^1(P) = U(P)$ is as follows. *The points v_1 , v_2 , and $p = v_3v_4 \cap v_5v_6$ are in a line.*

Remark 2.10. For simplicity reasons we omit discussions of cases where certain lines $v_i v_j$ and $v_k v_l$ are parallel, due to the fact that this situation is never generic for codimension 1 strata. In general one may think that if the lines $v_i v_j$ and $v_k v_l$ are parallel, then their intersection point is in the line at infinity in the projectivization of \mathbb{R}^2 .

Remark 2.11. At first glance, the condition *six points are on a conic* is of different nature. Nevertheless, it is a relation on the points of the configuration in $U^1(P)$ described by Pascal’s theorem: *The intersections of the extended opposite sides of a hexagon inscribed in a conic lie on the Pascal line.* See also Example 2.15 below.

Problem 2.12. Describe all the possible different types of strata for 9 points.

Problem 2.13. How to calculate the number of different types of strata for n points with arbitrary n ?

It is also interesting to have an answer for the following question: *how many iterations does one need to perform (i.e., find the minimal m for $U^m(P)$) to describe all conditions for the codimension 1 strata of n -point configurations P ?*

Problem 2.14. Which configurations of $U^m(P)$ define the same geometric condition?

This problem is a kind of question of finding generators and relations for the set of all conditions. Let us show one type of such "relations" in the following example.

Example 2.15. Consider the condition: six points v_1, v_2, \dots, v_6 are on a conic. This condition is described by configurations contained in $U^1(P)$ via Pascal’s theorem:

$$\text{The points } p, q, r \text{ are in a line for } \begin{cases} p = v_{\sigma(1)}v_{\sigma(2)} \cap v_{\sigma(4)}v_{\sigma(5)} \\ q = v_{\sigma(2)}v_{\sigma(3)} \cap v_{\sigma(5)}v_{\sigma(6)} \\ r = v_{\sigma(3)}v_{\sigma(4)} \cap v_{\sigma(6)}v_{\sigma(1)} \end{cases} ,$$

where σ is an arbitrary permutation of the set of six elements. So, there are 60 different configurations of $U^1(P)$ defining the same geometric condition.

3 Further study of strata: surgeries

We now look into subgraphs contained in a particular stratum and ask the basic question on the dimension of the fiber.

Even graphs of very low connectivity admit non-zero tensegrities, for disconnected or one-connected graphs we may simply examine the connected or 2-connected components. Also 2-connected graphs may be decomposed via the 2-sum, see [11]: Consider graphs G_1 and G_2 , their configurations P_1 and P_2 admitting tensegrities with $p_1 q_1$ a cable in $G_1(P_1)$ and $p_2 q_2$ a strut in $G_2(P_2)$. We form the 2-sum $G_1 \oplus G_2$ by identifying p_1 with p_2 and q_1 with q_2 and removing the identified edge. We can inherit a configuration P from P_1 and P_2 by fixing P_1 and properly dilating, rotating and translating P_2 . It is clear that

$$\dim W(G_1 \oplus G_2, P) = \dim W(G_1, P_1) + \dim W(G_2, P_2) - 1.$$

Since 2-sum decomposition is canonical, we can describe geometric conditions for 2-connected graphs by geometric conditions on their 3-blocks. For example the geometric condition for G in Figure 4 is that the lines $v_1 v_2, v_3 v_4$, and $v_5 v_6$ meet in one point.

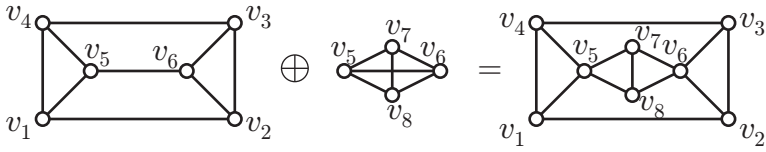


Figure 4: The 2-sum of a triangular prism with K_4

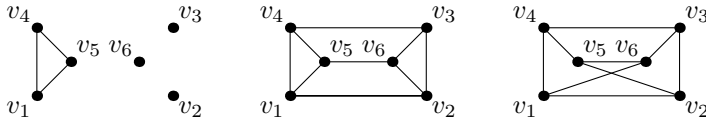


Figure 5: Examples of subgraphs of K_6 admitting tensegrities at codimension 1 strata of $B_2(K_6)$.

3.1 Subgraphs related to codimension 1 strata

As we have already mentioned in GR3, for any codimension 1 stratum there exists at least one subgraph of K_n that generically does not admit tensegrities but at this stratum admits a one-dimensional family of tensegrities. Let us show such subgraphs for the codimension one strata of $B_2(K_6)$ and $B_2(K_7)$.

Example 3.1. In the case of K_6 we have three strata of different geometrical nature. The first triangular subgraph (Figure 5, left) is related to the strata with three points in a line. The second (Figure 5, middle) corresponds to the strata whose three pairs of points generate lines passing through one point. The last one (Figure 5, right) corresponds to the configurations of six points on a conic.

Example 3.2. In the case of K_7 there are the following new examples of subgraphs, corresponding to the main 4 different types of strata.

From the left to the right we have the following geometric conditions

- $v_1, v_2,$ and v_3 are in a line;
- the lines $v_1v_2, v_3v_4,$ and v_5v_6 meet in one point;
- the lines $v_1v_2, v_3v_4,$ and v_5p (where $p = v_2v_6 \cap v_3v_7$) have a common point;
- the six points $v_1, v_2, v_3, v_4, v_5,$ and p (where $p = v_1v_6 \cap v_3v_7$) are on a conic.

Note that the example for three points in a line is actually the 2-sum of a triangle with two atoms, so the only way for a non-zero self-stress on the edges is to have $v_1, v_2,$ and $v_3,$

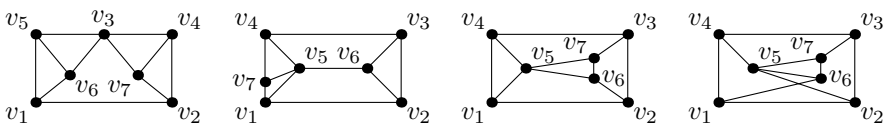


Figure 6: Examples of subgraphs of K_7 admitting tensegrities at codimension 1 strata of $B_2(K_7)$.

the vertices of the triangle, in a line.

Remark 3.3. Geometric conditions for the graphs with 8 and fewer vertices are given in [4]. Several of those geometric conditions were described before in terms of bracket polynomials in [12] by N. L. White and W. Whiteley. We also refer to the paper [1] by E. D. Bolker and H. Crapo for the relation of bipartite graphs with rectangular bar constructions.

3.2 Surgeries on subgraphs that change geometric conditions in a predictable way

In this subsection we present several surgeries that allow to guess the geometric conditions for new strata (characterized by certain subgraphs) via other strata (characterized by these graphs modified in a certain way). We call such modifications of graphs *surgeries*.

3.2.1 Surgeries that do not change geometric conditions

Let G be a graph, denote by G_e the graph with an edge e removed.

Proposition 3.4. (Edge exchange) Consider a graph G and a subgraph H , and let e_1 and e_2 be two edges of H . Let P be a configuration for which $\dim W(H, P) = 1$. Suppose also that the self-stresses of H do not vanish at the edges e_1 and e_2 . Then we have

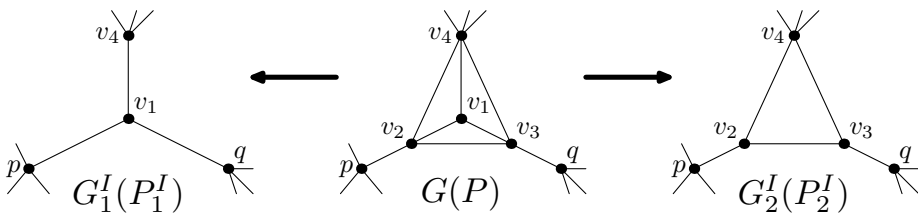
$$\dim W(G_{e_1}, P) = \dim W(G_{e_2}, P). \quad \square$$

In the situation of Proposition 3.4 the strata of $G_{e_1}(P)$ and $G_{e_2}(P)$ are defined by the same geometrical conditions.

3.2.2 Two two-dimensional surgeries that change geometric conditions

The first surgery is described in the following proposition.

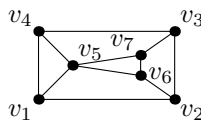
Proposition 3.5. Consider the frameworks $G(P)$, $G_1^I(P_1^I)$, and $G_2^I(P_2^I)$ as on the figure:



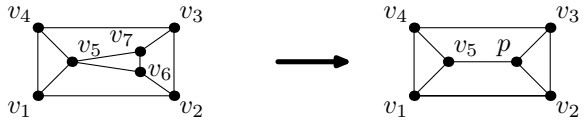
If none of the triples of points (p, v_2, v_3) , (q, v_2, v_3) , (p, v_2, v_4) , (q, v_3, v_4) and (v_2, v_3, v_4) are on a line then we have

$$\dim W(G_1^I, P_1^I) = \dim W(G_2^I, P_2^I).$$

Example 3.6. Let us consider a simple example of how to get a geometric condition for the graph



to admit a tensegrity knowing all geometric conditions for 6-point graphs. Let us apply Surgery I to the points v_5, v_6, v_7 . We have:



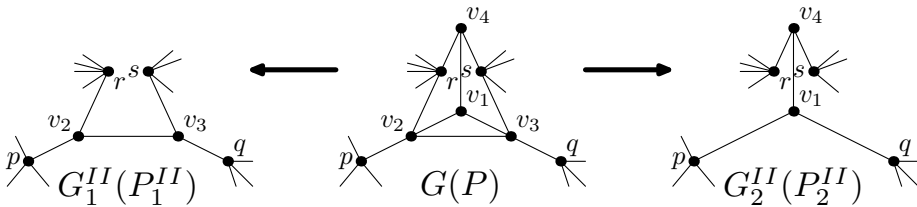
The geometric condition to admit a tensegrity for the graph on the right is:
the lines v_1v_2, v_3v_4 and v_5p intersect in a point.

Hence the geometric condition for the original graph is:

the lines v_1v_2, v_3v_4 and v_5p intersect in a point, where $p = v_2v_6 \cap v_3v_7$.

Now let us show the second surgery.

Proposition 3.7. Consider the frameworks $G(P), G_1^{II}(P_1^{II}),$ and $G_2^{II}(P_2^{II})$ as on the following figure:



If none of the triples of points $(p, q, v_1), (p, v_1, v_4), (r, v_1, v_4), (q, v_1, v_4), (s, v_1, v_4),$ or (r, s, v_4) lie on a line then we have

$$\dim W(G_1^{II}, P_1^{II}) = \dim W(G_2^{II}, P_2^{II}). \quad \square$$

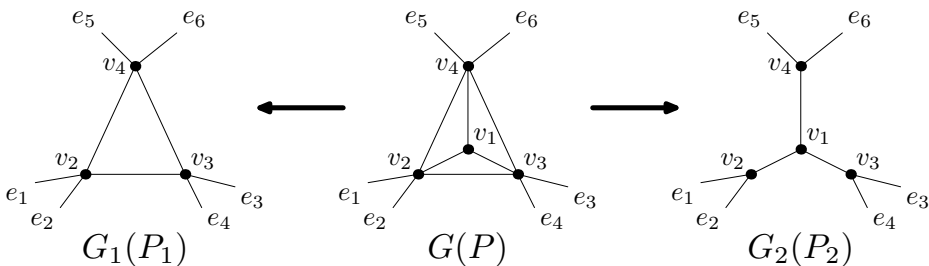
Remark 3.8. Both surgeries were shown in [4]. There is a certain analogy of the first surgery to ΔY exchange in matroid theory (see for instance [13] and [5] for the connections between matroids and rigidity theory), but it is not exactly the same.

Remark 3.9. Actually these surgeries are valid in the multidimensional case as well under the condition that certain points are in one plane.

3.3 A new tensegrity surgery in \mathbb{R}^3

We conclude this paper with a single surgery for tensegrities in \mathbb{R}^3 .

Proposition 3.10. Consider a graph G and frameworks $G(P), G_1(P_1),$ and $G_2(P_2)$ as follows:



Denote the plane $v_2v_3v_4$ by π_1 . Suppose that the couples of edges e_1 and e_2 , e_3 and e_4 , e_5 and e_6 define planes π_2 , π_3 , and π_4 , different from π_1 . Assume that $\pi_2 \cap \pi_3 \cap \pi_4$ is a one point intersection.

If $G_1(P_1)$ and $G_2(P_2)$ have nonzero stress on the edges connecting v_1 , v_2 , v_3 , and v_4 then

$$\pi_1 \cap \pi_2 \cap \pi_3 \cap \pi_4 = v_1.$$

In this case we additionally have

$$\dim W(G_1, P_1) = \dim W(G_2, P_2).$$

Proof. The first statement follows since v_1 only has valency 3 in $G_2(P_2)$, so v_1 , v_2 , v_3 , and v_4 need to be coplanar to have a nonzero edge-stress. Now we explain how to map $W(G_1, P_1)$ to $W(G_2, P_2)$. The inverse map is simply given by the reverse construction. By the conditions v_1 is the intersection point of the planes π_1 , π_2 , and π_3 . We add the uniquely defined plane atom on v_1, v_2, v_3, v_4 to $G_1(P_1)$ that cancels the edge-stress on v_2v_3 . Since the plane π_1 does not coincide with the plane π_2 spanned by the forces on e_1 and e_2 , the edge-stress on v_2v_4 is also canceled. By the same reasons the edge-stress on v_3v_4 is canceled as well. This uniquely defines a self-stress on $G_2(P_2)$. \square

3.4 Some related open problems

The next goal in this approach is to continue to study the geometry of the strata. Ideally one would like to find techniques that will give geometric conditions for a graph via its combinatorics. This question seems to be a very hard open problem. The study of surgeries is the first step to solve it at least in codimension 1.

For a start we propose the following open question.

Problem 3.11. Find all geometric conditions for the strata of 9 point tensegrities.

The surgeries introduced in this section were extremely useful for the study of 8 point configurations (see in [4]). We think that it is not enough to know only these surgeries to find all the geometric conditions. This gives rise to another question.

Problem 3.12. Find other surgeries on graphs that predictably change the geometric conditions.

As far as we know there is no systematic study of strata for tensegrities in \mathbb{R}^3 or higher dimensions: these cases are much more complicated than the planar case. At least the stratification of $B_3(K_5)$ should have a description similar to that of $B_2(K_4)$, since 5 points in general position in \mathbb{R}^3 admit a unique non-zero self-stress.

Additionally one should examine the rigidity properties of subgraphs in a stratum. For K_4 we have 14 strata of full dimension. For 8 of them the convex hull is a triangle, in 5 of the strata the points are in convex position. A tensegrity for the convex position has 4 struts (cables) and two cables (struts), while in the non-convex case there are three cables and three struts. All of these tensegrities are (infinitesimally) rigid and struts and cables may be exchanged without destroying rigidity. However, when viewed as graphs embedded in \mathbb{R}^3 only half of them are rigid. For the convex case, there must be cables on the convex hull and two struts. In the non-convex case there must be a triangle of struts on the convex hull and three cables in the interior, termed a spider web by R. Connelly. None of these

are proper forms in the sense of B. Grünbaum. They are minimally rigid, but in the convex case they have members intersecting in a vertex other than a vertex of the graph, in the non-convex case there is a vertex without a strut. B. Grünbaum in his lectures on lost mathematics [6] asks about the number of proper forms given n struts. On 3 struts, there is only one tensegrity which is minimally rigid with edges only intersecting at vertices and such that every vertex is endpoint of at least one strut. For 4 struts there are at least 20 forms, but it is not known how many there are. The number of forms on n struts is bounded by the number of strata on $B_3(K_n)$. For the hierarchies of the various kinds of rigidity see [3].

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