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On girth-biregular graphs

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Abstract

Let Γ denote a finite, connected, simple graph. For an edge e of Γ let $n(e)$ denote the number of girth cycles containing e. For a vertex v of Γ let $\{e_1, e_2, \ldots, e_k\}$ be the set of edges incident to v ordered such that $n(e_1) \leq n(e_2) \leq \cdots \leq n(e_k)$. Then $(n(e_1), n(e_2), \ldots, n(e_k))$ is called the *signature* of v. The graph Γ is said to be *girth-biregular* if it is bipartite, and all of its vertices belonging to the same bipartition have the same signature.

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Let Γ be a girth-biregular graph with girth $g = 2d$ and signatures $(a_1, a_2, \ldots, a_{k_1})$ and $(b_1, b_2, \ldots, b_{k_2})$, and assume without loss of generality that $k_1 \geq k_2$. Our first result is that $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Our next result is the upper bound $a_{k_1} \leq M$, where $M = (k_1 - 1)^{\lfloor g/4 \rfloor}(k_2 - 1)^{\lceil g/4 \rceil}$. We describe the graphs attaining equality. For $d = 3$ or $d \geq 4$ even they are incidence graphs of Steiner systems and generalized polygons, respectively. Finally, we show that when d is even and $a_{k_1} = M - \varepsilon$ for some non-negative integer $\varepsilon < k_2 - 1$, then $\varepsilon = 0$. Similar result is valid for $d = 3$, $\varepsilon \leq 1$ and $k_2 \nmid k_1$.

Keywords: Girth cycle, girth-biregular graph, Steiner system, generalized polygons.

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Ožinsko biregularni grafi

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Povzetek

Naj Γ označuje končen, povezan enostaven graf. Če je e povezava grafa Γ, potem z $n(e)$ označimo število ožinskih ciklov, ki vsebujejo e. Če je v vozlišče grafa Γ, potem naj bo $\{e_1, e_2, \ldots, e_k\}$ množica povezav, incidentnih vozlišču v, urejena tako, da je $n(e_1) \leq n(e_2) \leq \cdots \leq n(e_k)$. Vektor $(n(e_1), n(e_2), \ldots, n(e_k))$ se imenuje *signatura* vozlišča v. Graf Γ je *ožinsko biregularen*, če je dvodelen in če imajo vsa njegova vozlišča, ki pripadajo isti biparticiji, isto signaturo.

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Naj bo Γ ožinsko biregularen graf z ožino $g = 2d$ in signaturami $(a_1, a_2, \ldots, a_{k_1})$ in $(b_1, b_2, \ldots, b_{k_2})$, in privzemimo, brez škode za splošnost, da je $k_1 \geq k_2$. Naš prvi rezultat je, da velja $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Naš naslednji rezultat je zgornja meja $a_{k_1} \leq M$, kjer je $M = (k_1 - 1)^{\lfloor g/4 \rfloor}(k_2 - 1)^{\lceil g/4 \rceil}$. Opišemo grafe, pri katerih je dosežena enakost. Za $d = 3$ so to incidenčni grafi Steinerjevih sistemov, za soda števila $d \geq 4$ pa posplošenih poligonov. Nazadnje pokažemo, da kadar je d sodo število in je $a_{k_1} = M - \varepsilon$ za neko nenegativno celo število $\varepsilon < k_2 - 1$, potem je $\varepsilon = 0$. Podoben rezultat velja za $d = 3$, $\varepsilon \leq 1$ in $k_2 \nmid k_1$.

Ključne besede: Ožinski cikel, ožinsko biregularen graf, Steinerjev sistem, posplošeni poligoni.

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