ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA
https://doi.org/10.26493/1855-3974.2935.a7b
(Also available at http://amc-journal.eu)

On girth-biregular graphs

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Received 25 July 2022, accepted 12 November 2022

Abstract

Let Γ denote a finite, connected, simple graph. For an edge e of Γ let $n(e)$ denote the number of girth cycles containing e. For a vertex v of Γ let $\{e_1, e_2, \dots, e_k\}$ be the set of edges incident to v ordered such that $n(e_1) \leq n(e_2) \leq \cdots \leq n(e_k)$. Then $(n(e_1), n(e_2), \dots, n(e_k))$ is called the *signature* of v. The graph Γ is said to be *girth-biregular* if it is bipartite, and all of its vertices belonging to the same bipartition have the same signature.

*This research was supported in part by the Hungarian National Research, Development and Innovation Office OTKA grant no. SNN 132625, by the HAS Slovenian-Hungarian bilateral research project "Graph Colourings and Finite Geometries" (NKM-95/2019000206), and by the Slovenian Research Agency (research project J1-9110).

†This research was supported in part by the Slovenian Research Agency (research program P1-0285 and research projects N1-0032, N1-0038, J1-5433, and J1-6720).

‡This research was supported in part by the Hungarian National Research, Development and Innovation Office OTKA grant no. K 124950, by the HAS Slovenian-Hungarian bilateral research project "Graph Colourings and Finite Geometries" (NKM-95/2019000206), and by the Slovenian Research Agency (research project J1-9110). The author would like to thank his coauthors of [1]: Gabriela, Alejandra and Robert. The discussions leading to [1] considerably influenced the present paper. The author author would like to thank the Rényi Institute of Mathematics for the hospitality during this research.

Let Γ be a girth-biregular graph with girth $g = 2d$ and signatures $(a_1, a_2, \dots, a_{k_1})$ and $(b_1, b_2, \dots, b_{k_2})$, and assume without loss of generality that $k_1 \geq k_2$. Our first result is that $\{a_1, a_2, \dots, a_{k_1}\} = \{b_1, b_2, \dots, b_{k_2}\}$. Our next result is the upper bound $a_{k_1} \leq M$, where $M = (k_1 - 1)^{\lfloor g/4 \rfloor}(k_2 - 1)^{\lceil g/4 \rceil}$. We describe the graphs attaining equality. For $d = 3$ or $d \geq 4$ even they are incidence graphs of Steiner systems and generalized polygons, respectively. Finally, we show that when d is even and $a_{k_1} = M - \varepsilon$ for some non-negative integer $\varepsilon < k_2 - 1$, then $\varepsilon = 0$. Similar result is valid for $d = 3$, $\varepsilon \leq 1$ and $k_2 \not| k_1$.

Keywords: Girth cycle, girth-biregular graph, Steiner system, generalized polygons.

Math. Subj. Class. (2020): 05C35, 51E20

1 Introduction

In extremal graph theory one often considers problems of the following type: we fix some graph parameter or some graph property and want to deduce the extremal number of another parameter (in many cases the number of points or edges). Typical questions are Turán type problems, see e.g. the survey of Füredi and Simonovits [7]. The problem considered in our paper is motivated by the cage problem (and the degree/diameter problem), see [4, 12]. The cage problem was extended recently by several authors to bipartite graphs which are biregular in the sense that vertices in the same bipartition set have the same degree, see Jajcay, Ramos-Rivera and their coauthors [1, 6].

The paper by Jajcay, Kiss and Miklavič [8] defined a new type of regularity: a graph is called edge-girth regular if the number of cycles of length g (the girth) containing an edge is independent of the edge. This definition was weakened by Potočnik and Vidali [14] and in [9] it was extended to a stability theorem. One can introduce the signature $(a_1, \dots a_k)$ of a point as the ordered sequence of the number of girth cycles containing the edges emanating from the point (see Definition 2.1). A graph is called girth-regular if all of its points have the same signature. For such graphs with valency $k \geq 3$, it was shown in [14] that $a_k \leq (k-1)^{2d}$, where $d = \lfloor g/2 \rfloor$. In [9], the upper bound was improved for $g = 2d$ in the sense that it is either $(k-1)^{2d}$ or at most $(k-1)^{2d} - (k-1)$. In the former case the graph has to be the incidence graph of a thick generalized d-gon of order $(k-1, k-1)$. In particular, we must have $d = 2, 3, 4, 6$.

The aim of the present paper is to extend some of the results of [9] to the bipartite biregular case. If the valencies in the bipartition classes are $k_1 > k_2 > 2$, then we prove that the maximum number of girth-cycles containg an edge is at most $M = (k_1 - 1)^{\lfloor g/4 \rfloor}(k_2 - 1)^{\lceil g/4 \rceil}$, see Theorem 2.6. For $g = 4$, we show that when the graph is girth regular and the largest element of the signature of a point is equal to $M - \varepsilon$, with $\varepsilon \leq k_2 - 1$, then $\varepsilon = 0$, and the graph is the complete bipartite graph K_{k_1,k_2}. In Section 3, we prove an analogous result for $g = 2d \geq 8$, d even, relating the $\varepsilon = 0$ case to incidence graphs of a finite thick generalized d-gon, see Theorem 3.4(vi). For $q = 2d$, d odd, we have partial results. In particular, similarly to the results of [1, 6], when $g = 6$, we could find a connection of $a_k = M$ and block designs. For particular k_1 and k_2, the connection is with affine planes, see Corollary 6.3.

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2 Definitions and basic properties

In this section we collect basic notation and terminology. First, for the sake of completeness, we recall some definitions from design theory and finite geometries. In the second subsection we define girth-biregular graphs and present some simple, important properties of them.

2.1 Block designs, Steiner systems, generalized polygons

Here we give only the most necessary definitions. A detailed introduction to block designs and Steiner systems we refer the reader to [2] and [3], while the concepts from finite geometries we use can be found for example in [10] and [11].

Definition 2.1. Let $v \geq k \geq t \geq 2$ and $\lambda \geq 1$ be integers. A t-(v, k, λ) *design* is a collection of k-subsets (blocks) of a v-set S (points) such that every t-subset of S is contained in exactly λ of the blocks.

A t-$(v, k, 1)$ design is called a *Steiner system*. In particular, the blocks of a Steiner system with $t = 2$ are often called lines.

A *parallelism* of a design is a partition of its blocks into classes $C_1, C_2, \ldots, C_r$ with the property that any point belongs to a unique block of each class. A design is called *resolvable*, if it has a parallelism.

Let $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a connected, finite point-line incidence geometry. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively, $\mathrm{I} \subset (\mathcal{P}\times\mathcal{L})\cup(\mathcal{L}\times\mathcal{P})$ is a symmetric relation, called *incidence*. A *chain* of length h is a sequence $x_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, \ldots \mathrm{I} x_h$ where $x_i \in \mathcal{P} \cup \mathcal{L}$. The *distance* of the elements u and v, denoted by $d(u, v)$, is the length of the shortest chain joining them.

Definition 2.2. Let $n > 1$ be a positive integer. The incidence geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *thick generalized n-gon* if it satisfies the following axioms.

- $d(x, y) \leq n \;\forall\, x, y \in \mathcal{P} \cup \mathcal{L}$.
- If $d(x, y) = k < n$, then there is a unique chain of length k joining x and y.
- $\forall\, x \in \mathcal{P} \cup \mathcal{L} \;\exists\, y \in \mathcal{P} \cup \mathcal{L}$ such that $d(x, y) = n$.
- $\forall\, x \in \mathcal{P} \cup \mathcal{L}$ there exist at least three elements $y_i \in \mathcal{P} \cup \mathcal{L}$ such that $d(x, y_i) = 1$.

For any finite thick generalized n-gon $\mathcal{G}$ there exist integers $s, t \geq 2$ such that every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines. The pair (s, t) is called the *order of* $\mathcal{G}$.

In particular, for $n = 3$, the generalized 3-gons are the finite projective planes, for $n = 4$, the generalized 4-gons are the finite generalized quadrangles (GQ-s for short). The GQ-s have an alternative definition:

Definition 2.3. Let $s > 1$ and $t > 1$ be positive integers. A *thick generalized quadrangle of order* (s, t) is a point-line incidence structure which satisfies the following axioms:

- every line is incident with exactly $s + 1$ points;
- every point is incident with exactly $t + 1$ lines;

- there exists a non-incident point-line pair;
- for every point P and every line ℓ not incident with P, there is exactly one line through P which intersects ℓ.

2.2 Girth-biregular graphs

Let Γ denote a finite, connected, simple graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. Let d denote the minimal path-length distance function of Γ and let $D = \max\{d(v,w) \mid v,w \in V\}$ denote the *diameter* of Γ. For $v \in V$ and an integer i we let $\Gamma_i(v) = \{w \in V \mid d(v,w) = i\}$. We abbreviate $\Gamma(v) = \Gamma_1(v)$ and observe that $\Gamma_i(v) = \emptyset$ whenever $i < 0$ or $i > D$. For an edge uv of Γ, let $D^i_j(u,v) = \Gamma_i(u) \cap \Gamma_j(v)$.

We say that Γ is *biregular with valencies* k_1, k_2 $(k \in \mathbb{Z})$, whenever Γ is bipartite with bipartition sets A, B, and $|\Gamma(v)| = k_1$ ($|\Gamma(v)| = k_2$, respectively) for every $v \in A$ ($v \in B$, respectively). If Γ is not a tree, then the *girth* g of Γ is the length of a shortest cycle in Γ. If C is a cycle of Γ of girth length g, then we refer to C as a *girth cycle* of Γ.

The *incidence graph* (also known as *Levi graph*) of a point-line incidence geometry is a bipartite graph whose bipartition sets correspond to the set of points and lines, respectively, and there is an edge between two vertices if and only if the corresponding point is incident with the corresponding line.

The next ”folklore” statement gives an important correspondence between generalized polygons and biregular graphs. The proof can be found for example in [11, Lemma 1.3.6], or in [10, Chapter 12].

Theorem 2.4. *A finite thick generalized n-gon $\mathcal{G}$ exists if and only if there exists a connected bipartite biregular graph Γ of diameter n and girth $2n$, such that each vertex has degree at least three. In this case Γ is the incidence graph of $\mathcal{G}$.*

The following definition is a central definition of this paper.

Definition 2.5. Let Γ be a graph and let u, v be adjacent vertices of Γ. For the edge $e = uv$ of Γ let $n(e) = n(uv)$ denote the number of girth cycles containing e. For a vertex w of Γ let $\{e_1, e_2, \ldots, e_{k(w)}\}$ be the set of edges incident to w ordered such that $n(e_1) \le n(e_2) \le \cdots \le n(e_{k(w)})$. Then $(n(e_1), n(e_2), \ldots, n(e_{k_w}))$ is called the *signature* of w. The bipartite graph G is said to be *girth-biregular* if all of its vertices belonging to the same bipartition have the same signature.

Observe that girth-biregular graphs are also biregular. The following straightforward observation will be used through the rest of the paper frequently without explicitly referring to it (see also [14, Subsection 2.2] and Figure 1).

Proposition 2.6. *Let Γ be a biregular graph with valencies k_1, k_2 and girth $2d$, $d \ge 2$. Let uv be an edge of Γ, such that the valency of u is k_1 and valency of v is k_2. Let $D^i_j = D^i_j(u,v)$. Then the following hold.*

(i) *If x, y are vertices of Γ with $d(x,y) \le d-1$, then there is a unique path of length $d(x,y)$ between x and y.*

(ii) *$D^i_i = \emptyset$ for every integer i.*

(iii) *For $1 \le i \le d-1$ and for $z \in D^i_{i+1}$ (resp. $z \in D^{i+1}_i$), we have that $|\Gamma(z) \cap D^{i-1}_i| = 1$ (resp. $|\Gamma(z) \cap D^i_{i-1}| = 1$).*

(iv) *For* $0 \leq i \leq d-2$ *and for* $z \in D^i_{i+1}$, *we have that* $|\Gamma(z) \cap D^{i+1}_{i+2}| = k_1 - 1$ *if* i *is even, and* $|\Gamma(z) \cap D^{i+1}_{i+2}| = k_2 - 1$ *if* i *is odd.*

(v) *For* $0 \leq i \leq d-2$ *and for* $z \in D^{i+1}_i$, *we have that* $|\Gamma(z) \cap D^{i+2}_{i+1}| = k_2 - 1$ *if* i *is even, and* $|\Gamma(z) \cap D^{i+1}_{i+2}| = k_1 - 1$ *if* i *is odd.*

(vi) *For* $0 \leq i \leq d-1$ *we have that*

$$|D^i_{i+1}| = \begin{cases} (k_1-1)^{i/2}(k_2-1)^{i/2} & \text{if } i \text{ is even,} \\ (k_1-1)^{(i+1)/2}(k_2-1)^{(i-1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

$$|D^{i+1}_i| = \begin{cases} (k_1-1)^{i/2}(k_2-1)^{i/2} & \text{if } i \text{ is even,} \\ (k_1-1)^{(i-1)/2}(k_2-1)^{(i+1)/2} & \text{if } i \text{ is odd.} \end{cases}$$

(vii) *There are exactly* $n(uv)$ *edges between* D^{d-1}_d *and* D^d_{d-1}.

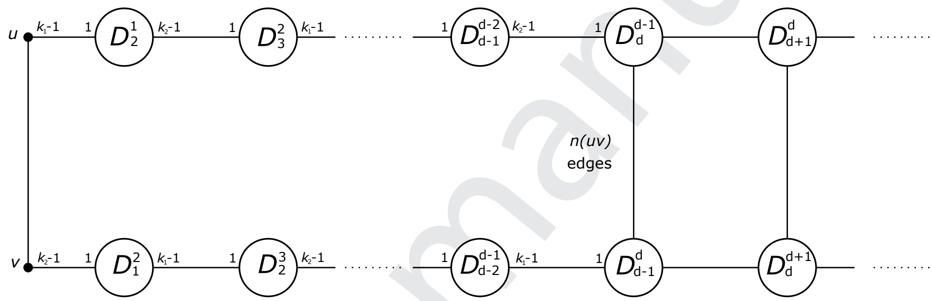

Figure 1: A biregular graph with valencies k_1, k_2 and girth $2d$, d odd. The numbers near the bubble representing the set D^i_j represent the number of neighbours that each vertex of D^i_j has in the neighbouring bubble.

3 Some properties of girth-biregular graphs

In this section we continue to study girth-biregular graphs. We prove several results about these graphs that are interesting on their own, and that will also be useful in the rest of the paper. Keeping in mind Proposition 2.6, one can calculate the number of girth cycles containing two fixed edges.

Lemma 3.1. *Let* Γ *be a girth-biregular graph with valencies* $k_1 \geq k_2$ *and girth* $g = 2d$. *Let* u_1u_2 *and* v_1v_2 *be two edges of* Γ. *Without loss of generality we may assume that* $d(u_1, v_1) = \min\{d(u_i, v_j) : 1 \leq i, j \leq 2\}$. *Let* $m = d(u_1, v_1) + 1$, *and let* c *denote the number of girth cycles containing both* u_1u_2 *and* v_1v_2. *Then* $c = 0$ *if* $m \geq d+1$ *and* $c \leq 1$ *if* $m = d$. *Moreover, if* $m \leq d-1$, *then*

$$c \leq \begin{cases} (k_1-1)^{(d-m)/2}(k_2-1)^{(d-m)/2}, & \text{if } m \text{ and } d \text{ are of the same parity,} \\ (k_1-1)^{(d-1-m)/2}(k_2-1)^{(d+1-m)/2}, & \text{if } m \text{ is even and } d \text{ is odd,} \\ (k_1-1)^{(d-1-m)/2}(k_2-1)^{(d+1-m)/2}, & \text{if } m \text{ is odd, } d \text{ is even and valency of } v_2 \text{ is } k_2, \\ (k_1-1)^{(d+1-m)/2}(k_2-1)^{(d-1-m)/2}, & \text{if } m \text{ is odd, } d \text{ is even and valency of } v_2 \text{ is } k_1. \end{cases}$$

Proof. The statement is obvious if $m \geq d+1$. If $m = d$, then $d-1 = d(u_1, v_1) \leq d(u_2, v_2)$, so there exists a girth cycle containing both u_1u_2 and v_1v_2 if and only if $d(u_2, v_2) = d-1$, hence $c \leq 1$.

Suppose that $m \leq d-1$. Let $D^i_j = D^i_j(u_1, u_2)$ and observe that $v_1 \in D^{m-1}_m$, $v_2 \in D^m_{m+1}$. Note that there is a unique path of length $m-1$ between v_1 and u_1. Let $F = D^{d-m-1}_{d-m}(v_2, v_1)$ and note that by Proposition 2.6(iii) we have that $F \subseteq D^{d-1}_d$. Let us denote the valency of v_2 by k and let k' be the other valency of Γ. Then

$$|F| = (k-1)^{\lceil (d-m-1)/2 \rceil}(k'-1)^{\lfloor (d-m-1)/2 \rfloor},$$

and there is a unique path of length $d-m-1$ between v_2 and any element of F because the girth of Γ is $2d$. Now the number of girth cycles containing both u_1u_2 and v_1v_2 equals to the number of edges between F and D^d_{d-1}. Observe that this number is the same as the number of $(d-m)$-arcs $(v_2, x_1, \ldots, f, r)$ where $f \in F$ and $r \in D^{d-1}_d$. Observe also that the valency of f is k if $d-m-1$ is even and it is k' if $d-m-1$ is odd. Therefore, we have that $c \leq |F|(k-1)$ if $d-m-1$ is even, and $c \leq |F|(k'-1)$ if $d-m-1$ is odd. Now we distinguish four cases. If d and m are of the same parity, then $d-m-1$ is odd, and so

$$c \leq |F|(k'-1) = (k-1)^{(d-m)/2}(k'-1)^{(d-m)/2} = (k_1-1)^{(d-m)/2}(k_2-1)^{(d-m)/2}.$$

If d is odd and m is even, then $\deg(u_2) \neq \deg(v_2)$, so we may assume $\deg(v_2) = k_2$ (otherwise we interchange the roles of edges u_1u_2 and v_1v_2). Hence

$$c \leq |F|(k-1) = |F|(k_2-1) = (k_1-1)^{(d-m-1)/2}(k_2-1)^{(d-m+1)/2}.$$

Finally, if d is even and m is odd, then

$$c \leq |F|(k-1) = (k-1)^{(d-m+1)/2}(k'-1)^{(d-m-1)/2},$$

and this gives the third and fourth estimates of the statement according as $k = k_1$ or $k = k_2$. □

Proposition 3.2. *Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 \geq k_2$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and the signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Then $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$.*

Proof. As Γ is bipartite, each edge e of Γ is incident with one vertex from A and with one vertex from B. It thus follows that $n(e) \in \{a_1, a_2, \ldots, a_{k_1}\}$ if and only if $n(e) \in \{b_1, b_2, \ldots, b_{k_2}\}$. This shows that $\{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. □

Proposition 3.3. *Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 \geq k_2$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and the signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Pick $a \in \{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Let a_A (a_B, respectively) denote the number of appearences of a in the signature $(a_1, a_2, \ldots, a_{k_1})$ ($(b_1, b_2, \ldots, b_{k_2})$, respectively). Then $k_2 a_A = k_1 a_B$.*

Proof. Let us count the number of edges of Γ that are contained in exactly a girth cycles. On the one hand, this number is equal to $|A|a_A$, and on the other hand it is equal to $|B|a_B$. Recall also that $|A|k_1 = |B|k_2$. The claim follows. □

Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 \geq k_2$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Let us comment on the case $k_1 = k_2$. It follows from Proposition 3.3 that in this case we have $a_A = a_B$ for every $a \in \{a_1, a_2, \ldots, a_{k_1}\} = \{b_1, b_2, \ldots, b_{k_2}\}$. Therefore, Γ is in fact girth-regular graph. As girth regular graphs were studied in details in [9] and [14], we will assume $k_1 > k_2$ for the rest of this paper.

Observe also that connected biregular graphs with valencies $k_1, k_2 = 1$ are just the star graphs, which contain no cycles at all (and are therefore girth-biregular with signatures $(0, 0, \ldots, 0)$ and (0)).

Let Γ be a girth-biregular graph with bipartition A, B and valencies $k_1 > k_2 = 2$. Then for any vertex $w \in B$ there are two edges, say u_1w and u_2w through w, hence a cycle contains u_1w if and only if it contains u_2w. In particular, $n(u_1w) = n(u_2w)$ which implies $b_1 = b_2$. Now, define the graph Γ' in the following way: $V(\Gamma') = A$ and there is an edge between vertices u and v if and only if $d(u, v) = 2$ in Γ. Then Γ' is an edge-girth-regular graph with valency k_1. These graphs were studied in [8]. Therefore, in the rest of this paper we also assume $k_1 > k_2 > 2$.

The following theorem is a generalization of the result of *Potočnik* and *Vidali* [14, Theorem 1.3].

Theorem 3.4. *Let Γ be a girth-biregular graph with bipartition A, B, valencies $k_1 > k_2 > 2$ and girth $2d$. Let us denote the signature of vertices from A by $(a_1, a_2, \ldots, a_{k_1})$ and the signature of vertices from B by $(b_1, b_2, \ldots, b_{k_2})$. Let $M = (k_1 - 1)^{g/4}(k_2 - 1)^{g/4}$ if d is even, and $M = (k_1 - 1)^{(g-2)/4}(k_2 - 1)^{(g+2)/4}$ if d is odd. Then $a_{k_1} = b_{k_2} \leq M$.*

When the upper bound is attained, $a_{k_1} = b_{k_2} = M$, the following (i)-(vii) hold.

(i) *For every edge uv of Γ with $u \in A$ and $n(uv) = M$ we have $D_i^{i+1}(u, v) = \emptyset$ for $i \geq d$.*

(ii) *The signature of each vertex of Γ is $(M, M, \ldots, M)$, hence $n(e) = M$ for all $e \in E(\Gamma)$.*

(iii) *Every path on $d + 2$ vertices of Γ, starting in a vertex that is contained in A, is contained in a unique girth cycle;*

(iv) *If d is even and uv is an edge of Γ with $u \in A$, then $D_{i+1}^i(u, v) = \emptyset$ for $i \geq d$.*

(v) *if d is odd and uv is an edge of Γ with $u \in A$, then $D_{d+1}^d(u, v) \neq \emptyset$ and $D_{i+1}^i = \emptyset$ for $i \geq d + 1$.*

(vi) *if d is even, then Γ is the incidence graph of a generalized d-gon of order $(k_1 - 1, k_2 - 1)$;*

(vii) *if $d = 3$, then Γ is the incidence graph of a $2 - (k_1k_2 - k_1 + 1, k_2, 1)$-design.*

Proof. Pick adjacent vertices $u \in A, v \in B$ such that $n(uv) = a_{k_1} = b_{k_2}$.

We prove the upper bound on a_{k_1} in the case when d is odd. The proof for the case when d is even is similar. By Proposition 2.6(vi) we have that $|D_{d-1}^d(u, v)| = (k_1 - 1)^{(d-1)/2}(k_2 - 1)^{(d-1)/2}$. As $D_{d-1}^d(u, v) \subseteq B$ and as every vertex from $D_{d-1}^d(u, v)$ has exactly one neighbour in $D_{d-2}^{d-1}(u, v)$, it follows that every vertex from $D_{d-1}^d(u, v)$ has at most $k_2 - 1$ neighbours in $D_d^{d-1}(u, v)$. Therefore, there are at most

$(k_1-1)^{(d-1)/2}(k_2-1)^{(d+1)/2}$ edges between $D^d_{d-1}(u,v)$ and $D^{d-1}_d(u,v)$. The result now follows from Proposition 2.6(vii).

Now, suppose that $a_{k_1}=M$.

(i): By Proposition 2.6(vii), there are M edges between $D^d_{d-1}(u,v)$ and $D^{d-1}_d(u,v)$. Recall that by Proposition 2.6(iii), every vertex from $D^d_{d-1}(u,v)$ has exactly one neighbour in $D^{d-1}_{d-2}(u,v)$. It follows that every vertex from $D^d_{d-1}(u,v)$ has all other neighbours in D^{d-1}_d, and so $D^{d+1}_d(u,v)=\emptyset$. Consequently, $D^{i+1}_i(u,v)=\emptyset$ for every $i\geq d$.

(ii): Let $w\in D^2_1(u,v)$ be any vertex. Then we have that

$$D^{d-1}_d(v,w)=D^{d-2}_{d-1}(u,v)\cup\left(D^d_{d-1}(u,v)\setminus D^{d-2}_{d-1}(w,v)\right),$$

and, as $D^{d+1}_d(u,v)=\emptyset$ by (i) above, also

$$D^d_{d-1}(v,w)=D^{d-1}_d(u,v).$$

By Proposition 2.6(iv), the number of edges between $D^{d-2}_{d-1}(u,v)$ and $D^{d-1}_d(u,v)$ is equal to $|D^{d-2}_{d-1}(u,v)|(k_2-1)$ if d is odd, and to $|D^{d-2}_{d-1}(u,v)|(k_1-1)$ if d is even. As every vertex from $D^d_{d-1}(u,v)$ has exactly one neighbour in $D^{d-1}_{d-2}(u,v)$ and as $D^{d+1}_d(u,v)=\emptyset$, the number of edges between $\left(D^d_{d-1}(u,v)\setminus D^{d-2}_{d-1}(w,v)\right)$ and $D^{d-1}_d(u,v)$ is equal to

$$\left(|D^d_{d-1}(u,v)|-|D^{d-2}_{d-1}(w,v)|\right)(k_2-1)\text{ if }d\text{ is odd,}$$

and to

$$\left(|D^d_{d-1}(u,v)|-|D^{d-2}_{d-1}(w,v)|\right)(k_1-1)\text{ if }d\text{ is even.}$$

Observe that by Proposition 2.6(vi) we have that $|D^{d-2}_{d-1}(u,v)|=|D^{d-2}_{d-1}(w,v)|$, and so Proposition 2.6(vii) and the above comments imply that $n(vw)=(k_2-1)|D^d_{d-1}(u,v)|$ if d is odd and $n(vw)=(k_1-1)|D^d_{d-1}(u,v)|$ if d is even. Finally, Proposition 2.6(vi) implies that $n(vw)=M$. Hence the signature of v is $(M,M,\dots,M)$, so the girth-biregularity of Γ implies that $n(e)=M$ for all $e\in E(\Gamma)$.

(iii): Pick any path $x_0x_1\dots x_{d+1}$ with $x_0\in A$ and consider the sets $D^i_j(x_0,x_1)$. It follows from Proposition 2.6 that $x_i\in D^i_{i-1}$ for $1\leq i\leq d$. Recall that $n(x_0x_1)=M$ by (ii) above, and so $D^{d+1}_d(x_0,x_1)=\emptyset$ by (i) above. It follows that $x_{d+1}\in D^{d-1}_d$. The result now follows from Proposition 2.6(iii).

(iv): Recall that by (ii) above we have $n(uv)=M$, and so there are exactly M edges between $D^{d-1}_d(u,v)$ and $D^d_{d-1}(u,v)$. Recall also that by Proposition 2.6(iii), every vertex from $D^{d-1}_d(u,v)$ has exactly one neighbour in $D^{d-2}_{d-1}(u,v)$. It follows that every vertex from $D^{d-1}_d(u,v)$ has all other neighbours in D^d_{d-1}, and so $D^d_{d+1}(u,v)=\emptyset$. Consequently, $D^i_{i+1}(u,v)=\emptyset$ for every $i\geq d$.

(v): By Proposition 2.6(vi) we have $|D^{d-1}_d(u,v)|=(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$. As vertices of $D^{d-1}_d(u,v)$ have valency k_1, there are therefore $k_1(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$ edges going out of $D^{d-1}_d(u,v)$. As $n(uv)=M$ by (ii) above, $M=(k_1-1)^{(d-1)/2}(k_2-1)^{(d+1)/2}$ of these edges are between $D^{d-1}_d(u,v)$ and $D^d_{d-1}(u,v)$. By Proposition 2.6(iii), $(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$ of these edges are between $D^{d-1}_d(u,v)$ and $D^{d-2}_{d-1}(u,v)$. It follows that there are exactly $(k_1-k_2)(k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}$ edges between $D^{d-1}_d(u,v)$ and $D^d_{d+1}(u,v)$. As $k_1>k_2\geq 3$, this number is nonzero, implying that $D^d_{d+1}(u,v)\neq\emptyset$.

Assume now that $D^{d+1}_{d+2}(u,v) \neq \emptyset$. Pick $w \in D^{d+1}_{d+2}(u,v)$ and let $ux_1x_2\dots x_dw$ be arbitrary path between u and w such that $x_i \in D^i_{i+1}$ for $1 \leq i \leq d$. Note that this path is not contained in a girth cycle of Γ, contradicting (iii) above. Therefore $D^{d+1}_{d+2}(u,v) = \emptyset$ and consequently $D^i_{i+1}(u,v) = \emptyset$ for every $i \geq d+1$.
(vi): Observe that (i), (ii) and (iv) above implies that the diameter of Γ is d. As $k_1 > k_2 \geq 3$, Theorem 2.4 implies that Γ is the incidence graph of a generalized d-gon.
(vii): Finally, suppose that $d = 3$. We call the vertices in A *points* and the the vertices in B *lines* and we use the geometric terminology. We claim that there is a unique line through any pair of distinct points. As the girth of Γ is 6, there is at most one line through any pair of points. Pick now distinct points $x, y \in A$. Pick an arbitrary line z through x. It follows from (i) and (v) above, that either $y \in D^2_1(x,z)$ or $y \in D^2_3(x,z)$. If $y \in D^2_1(x,z)$, then z is the unique line through x and y. If however $y \in D^2_3(x,z)$, then, by Proposition 2.6(iii), there is a unique line $w \in D^1_2(x,z)$ which is adjacent to both x and y in Γ. Therefore, in this case w is the unique line through x and y. □

In the rest of this paper we use the following notation.

Notation 3.5. Let Γ be a girth-biregular graph with bipartition A, B, valencies $k_1 > k_2 \geq 3$, girth $g = 2d$, signatures $(a_1, a_2, \dots, a_{k_1})$ and $(b_1, b_2, \dots, b_{k_2})$. Let $M = (k_1-1)^{g/4}(k_2-1)^{g/4}$ if d is even, and $M = (k_1-1)^{(g-2)/4}(k_2-1)^{(g+2)/4}$ if d is odd and suppose that $a_{k_1} = M - \varepsilon$ for some $\varepsilon < k_2 - 1$. Let uv be an edge with $u \in A, v \in B$ and $n(uv) = a_{k_1}$, and let $D^i_j = D^i_j(u,v)$. Note that $D^i_i = \emptyset$ for every i and that there are no edges between D^{i-1}_i and D^i_{i-1} for $1 \leq i \leq d-1$.

For every $r \in D^d_{d-1}$ ($s \in D^{d-1}_d$, respectively) we let $h(r) = |\Gamma(r) \cap D^{d+1}_d|$ ($h(s) = |\Gamma(s) \cap D^d_{d+1}|$, respectively). Let $\{r_1, r_2, \dots, r_m\} \subseteq D^d_{d-1}$ be the set of vertices of D^d_{d-1}, for which the value of the function h is positive, that is, the set of vertices of D^d_{d-1}, that have a neighbour in D^{d+1}_d. Choose the indices in such a way that $h(r_i) \leq h(r_j)$ for $i < j$. Similarly, let $\{s_1, s_2, \dots, s_n\} \subseteq D^{d-1}_d$ be the set of vertices of D^{d-1}_d, for which the value of the function h is positive. Again, choose the indices in such a way that $h(s_i) \leq h(s_j)$ for $i < j$. We also set $\gamma = h(r_m)$, $\sigma = h(s_n)$, $\mu = h(r_1)$ and $\nu = h(s_1)$.

Proposition 3.6. *Suppose that $g = 2d$ with d even. With reference to* Notation 3.5, *we have*

$$\sum_{r \in D^d_{d-1}} h(r) = \sum_{i=1}^{m} h(r_i) = \sum_{s \in D^{d-1}_d} h(s) = \sum_{i=1}^{n} h(s_i) = \varepsilon. \tag{3.1}$$

Proof. The first and the third of the above equalities are clear. We now prove that $\sum_{i=1}^n h(s_i) = \varepsilon$. The proof that $\sum_{i=1}^m h(r_i) = \varepsilon$ is similar. Let $\mathcal{E}$ denote the set of edges, that have one endpoint in D^{d-1}_d, and the other endpoint in D^d_{d+1}. Note that $\mathcal{E} = \sum_{i=1}^n h(s_i)$, and so it is enough to prove $|\mathcal{E}| = \varepsilon$. As d is even, it follows from Proposition 2.6(vi) that $|D^{d-1}_d| = (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}$. As $D^{d-1}_d \subseteq B$, there are total $(k_1-1)^{d/2}(k_2-1)^{(d-2)/2}k_2$ edges, having one endpoint in D^{d-1}_d. By Proposition 2.6(iii), $(k_1-1)^{d/2}(k_2-1)^{(d-2)/2}$ of these edges have the other endpoint in D^{d-2}_{d-1}. Since $a_k = M - \varepsilon$, it follows from Proposition 2.6(vii) that there are $(k_1-1)^{d/2}(k_2-1)^{d/2} - \varepsilon$ edges between D^{d-1}_d and D^d_{d-1}. Combining these observations, we get the desired result. □

Lemma 3.7. *Suppose that $g = 2d$ with d even. With reference to* Notation 3.5, *we have $m \geq \sigma$ and $n \geq \gamma$.*

Proof. Set $\Gamma(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_{k_1-1}\}$ and $\Gamma(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_{k_2-1}\}$. Moreover, for $1 \le i \le k_1 - 1$ ($1 \le i \le k_2 - 1$, respectively) set $U_i = \Gamma_{d-2}(u_i) \cap D_d^{d-1}$ ($V_i = \Gamma_{d-2}(v_i) \cap D_{d-1}^d$, respectively). Note that as girth of Γ is $2d$, the sets U_i (V_i, respectively) are pairwise disjoint, and $|U_i| = |V_i| = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$. Moreover, each $r \in D_{d-1}^d$ ($s \in D_d^{d-1}$, respectively) could have at most one neighbour in U_i (V_i, respectively) for each i. It is now clear that if $s \in D_d^{d-1}$ has no neighbours in V_i for some $1 \le i \le k_2 - 1$, then there is at least one vertex $r \in V_i$ with $h(r) \ge 1$. It follows $m \ge \sigma$. Similarly we show that $n \ge \gamma$. □

Equation (3.1) and Lemma 3.7 obviously imply the following inequalities:

$$\mu\sigma \le \mu m \le \varepsilon, \quad \nu\gamma \le \nu n \le \varepsilon. \tag{3.2}$$

If $\gamma \le \sigma$, then observe also that it follows from the above comments that

$$\mu^2 \le \mu\gamma \le \mu\sigma \le \mu m \le \varepsilon,$$

while if $\sigma \le \gamma$ then

$$\nu^2 \le \nu\sigma \le \nu\gamma \le \nu n \le \varepsilon.$$

This shows that if $\gamma \le \sigma$ then $\mu \le \sqrt{\varepsilon}$, while if $\sigma \le \gamma$ then $\nu \le \sqrt{\varepsilon}$.

First, we give a lower bound on a_1 using the vertex u.

Lemma 3.8. *With reference to* Notation 3.5 *we have that*

$$a_1 \ge (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\max\{(k_2-1-\sigma)(k_1-1), (k_1-1-\gamma)(k_2-1)\} - \varepsilon. \tag{3.3}$$

Proof. We prove that $a_1 \ge (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}(k_2-1-\sigma) - \varepsilon$. The proof of $a_1 \ge (k_1-1)^{(d-2)/2}(k_2-1)^{d/2}(k_1-1-\gamma) - \varepsilon$ is similar.

Recall that $n(uv) = a_k$ and that $D_j^i = D_j^i(u,v)$. Let $s \ne v$ be a neighbour of u such that $n(us) = a_1$. Abbreviate $K = D_d^{d-1} \cap \Gamma_{d-2}(s)$. For $s' \in K$ abbreviate $L(s') = D_{d-1}^d \cap \Gamma(s')$. Note that as girth of Γ is $2d$, we have that sets $L(s')$ are pairwise disjoint, and so by (3.1) we have that

$$\sum_{s' \in K} \sum_{r' \in L(s')} h(r') \le \varepsilon.$$

Pick $r' \in L(s')$ and observe that for each $\tilde{r} \in (\Gamma(r') \cap (D_d^{d-1} \cup D_{d-2}^{d-1})) \setminus \{s'\}$, there is a unique girth cycle containing the arc us and the 2-arc $s'r'\tilde{r}$. Note that

$|K| = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$, and so, by (3.1), we have

$$\begin{aligned} a_1 = n(us) &\geq \sum_{s'\in K}\sum_{r'\in L(s')}(k_1-1-h(r')) \\ &= \sum_{s'\in K}\sum_{r'\in L(s')}(k_1-1) - \sum_{s'\in K}\sum_{r'\in L(s')} h(r') \\ &\geq (k_1-1)\sum_{s'\in K}(k_2-1-h(s')) - \varepsilon \\ &\geq (k_1-1)\sum_{s'\in K}(k_2-1-\sigma) - \varepsilon \\ &= (k_1-1)(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(k_2-1-\sigma) - \varepsilon \\ &= (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}(k_2-1-\sigma) - \varepsilon. \end{aligned}$$ □

4 The case $g = 4$

In this section we consider the case $g = 4$. Throughout this section we will use Notation 3.5. Recall that m (n, respectively) denotes the number of vertices of D^d_{d-1} (D^{d-1}_d, respectively), for which the value of the function h is positive.

Lemma 4.1. *Assume that $g = 4$ and $\varepsilon \geq 1$. Pick $1 \leq i \leq n$, $1 \leq j \leq m$, $w \in \Gamma(s_i) \cap D^2_3$ and $\tilde{w} \in \Gamma(r_j) \cap D^3_2$. Then the following (i) – (iv) holds.*

(i) *There are at most $(h(s_i)-1)(k_1-1)$ girth cycles of the form (w, s_i, x, y, w) such that $x \in \Gamma(s_i) \cap D^2_3$.*

(ii) *There are at most ε girth cycles of the form (w, s_i, x, y, w) such that $x \in D^2_1$ and $y \notin D^1_2$.*

(iii) *There are at most $(h(r_j)-1)(k_2-1)$ girth cycles of the form $(\tilde{w}, r_j, x, y, \tilde{w})$ such that $x \in \Gamma(r_j) \cap D^3_2$.*

(iv) *There are at most ε girth cycles of the form $(\tilde{w}, r_j, x, y, \tilde{w})$ such that $x \in D^1_2$ and $y \notin D^2_1$.*

Proof. (i): Note that there are $h(s_i)-1$ choices for x, and for each such choice there are at most k_1-1 choices for y. The result follows.
(ii): As $x \in D^2_1$ and $y \notin D^1_2$, it follows that $y \in D^3_2$. It follows from Proposition 3.6 that there are at most ε choices for the edge xy. For each such edge xy there is clearly at most one girth cycle containing also the edge ws_i. The result follows.
(iii), (iv): Similar to the proofs of (i) and (ii) above. □

Lemma 4.2. *Assume that $g = 4$ and $\varepsilon \geq 1$. Then $m \geq 2$ and $n \geq 2$.*

Proof. We prove that $n \geq 2$. The proof that $m \geq 2$ is similar. Suppose on the contrary that $n = 1$. Note that in this case $\sigma = \nu = \varepsilon$, $\gamma = 1$, $m = \varepsilon$ and $h(r_i) = 1$ for $1 \leq i \leq m$. Let w be the unique neighbour of r_1 in D^3_2. Let $t = |\Gamma(w) \cap D^2_1|$ and note that $t \leq m = \varepsilon$. Note that the girth cycles containing the edge r_1w are exactly the cycles of form (w, r_1, x, y, w), where $x \in \{v\} \cup (D^1_2 \setminus \{s_1\})$ and $y \in (\Gamma(w) \cap D^2_1) \setminus \{r_1\}$.

Therefore, $n(r_1w) \leq (k_1-1)(t-1) \leq (k-1)(\varepsilon-1)$. Since $\gamma = 1$ and $\sigma = \varepsilon$, we have by Lemma 3.8 that

$$a_1 \geq \max\{(k_2-1-\varepsilon)(k_1-1), (k_1-2)(k_2-1)\} - \varepsilon \geq (k_1-2)(k_2-1) - \varepsilon,$$

and so

$$(k_1-2)(k_2-1) - \varepsilon \leq a_1 \leq n(r_1w) \leq (k_1-1)(\varepsilon-1).$$

It follows that $k_1k_2 - 2k_2 + 1 \leq k_1\varepsilon$, and so

$$k_2 - 2 + \frac{1}{k_1} \leq k_2 - \frac{2k_2}{k_1} + \frac{1}{k_1} \leq \varepsilon < k_2 - 1,$$

contradicting the fact that ε is an integer. □

We now give an upper bound for a_1.

Lemma 4.3. *Assume that $g = 4$ and $\varepsilon \geq 1$. Let $\alpha = h(s_{n-1})$ and $\beta = h(r_{m-1})$. Then*

$$a_1 \leq (\alpha-1)(k_1-1) + \varepsilon + (k_2-\alpha)(\varepsilon-\alpha-\sigma+1). \tag{4.1}$$

and

$$a_1 \leq (\beta-1)(k_2-1) + \varepsilon + (k_1-\beta)(\varepsilon-\beta-\gamma+1). \tag{4.2}$$

Proof. We prove inequality (4.1). The proof of inequality (4.2) is similar. Let $\{w_1, \ldots, w_\alpha\} = \Gamma(s_{n-1}) \cap D_3^2$. We estimate $n(s_{n-1}w_1)$. To do this we split the girth cycles $(w_1, s_{n-1}, x, y, w_1)$ into two types depending on the vertex x. We say that the girth cycle is of type 1 if $x \in \{w_2, \ldots, w_\alpha\}$, and of type 2 if $x \in \{u\} \cup D_1^2$. By Lemma 4.1(i) there are at most $(\alpha-1)(k_1-1)$ girth cycles of type 1. To estimate the number of girth cycles of type 2, we further split these girth cycles into two subfamilies depending on the vertex y. Let us say that the girth cycle $(w_1, s_{n-1}, x, y, w_1)$ with $x \in \{u\} \cup D_1^2$ is of type 2a if $y \in D_2^1$, and of type 2b if $y \in D_2^3$.

If the girth cycle is of type 2b, then $x \in D_1^2$, and so by Lemma 4.1(ii) there are at most ε such girth cycles. To estimate the number of girth cycles of type 2a, observe that s_{n-1} has $k_2 - \alpha$ neighbours in $\{u\} \cup D_1^2$, and that w_1 has at most $\varepsilon - \alpha - \sigma + 1$ neighbours in $D_2^1 \setminus \{s_{n-1}\}$. This shows that the number of girth cycles of type 2a is at most $(k_2-\alpha)(\varepsilon - \alpha - \sigma + 1)$. As $a_1 \leq n(s_{n-1}w_1)$, the result follows. □

Lemma 4.4. *Assume that $g = 4$ and $\varepsilon \geq 1$. Then $\varepsilon = k_2 - 2$ and $k_2 - 1 \geq 2k_1/3$.*

Proof. As in Lemma 4.3, let $\alpha = h(s_{n-1})$. Then, by Lemmas 3.8 and 4.3, we get that

$$(k_1-1)(k_2-1) - \sigma(k_1-1) - \varepsilon \leq (\alpha-1)(k_1-1) + \varepsilon + (k_2-\alpha)(\varepsilon-\alpha-\sigma+1).$$

Rearranging the above inequality we find this is equivalent to

$$(k_1-1)(k_2-1) \leq (k_1-k_2-1+\alpha)(\alpha+\sigma-1) + \varepsilon(k_2-\alpha+2). \tag{4.3}$$

Taking into account that $\alpha + \sigma \leq \varepsilon$ and that $\alpha \geq 1$, inequality (4.3) implies that

$$(k_1-1)(k_2-1) \leq (k_1+1)\varepsilon - k_1 + 1 + k_2 - \alpha \leq (k_1+1)\varepsilon - k_1 + k_2,$$

and so

$$\varepsilon \geq \frac{(k_1-1)(k_2-1)+k_1-k_2}{k_1+1} = k_2 - \frac{3k_2-1}{k_1+1}. \tag{4.4}$$

As $k_1 \geq k_2$, the above inequality yields

$$\varepsilon \geq k_2 - \frac{3k_1-1}{k_1+1} = k_2 - 3 + \frac{4}{k_1+1} > k_2 - 3.$$

Recall that $\varepsilon < k_2 - 1$ by assumption, and so $\varepsilon = k_2 - 2$ as claimed. Plugging $\varepsilon = k_2 - 2$ into (4.4) we easily get that $k_2 - 1 \geq 2k_1/3$. □

Theorem 4.5. *Assume that* $g = 4$. *Then* $\varepsilon = 0$ *and* Γ *is the complete bipartite graph* K_{k_1,k_2}.

Proof. Suppose on the contrary that $\varepsilon \geq 1$. Recall that $\varepsilon = k_2 - 2$. As in Lemma 4.3, let $\beta = h(r_{m-1})$. Then, by Lemmas 3.8 and 4.3, we get that

$$(k_1-1)(k_2-1) - \gamma(k_2-1) - \varepsilon \leq (\beta-1)(k_2-1) + \varepsilon + (k_1-\beta)(\varepsilon-\beta-\gamma+1).$$

Rearranging the terms of the above inequality we get

$$(k_1-1)(k_2-1) \leq \varepsilon(k_1-\beta+2) + (\beta+\gamma-1)(k_2-k_1+\beta-1). \tag{4.5}$$

If $\beta = 1$, then inequality (4.5) together with $\varepsilon = k_2 - 2$ yields $k_1 - 1 \leq \gamma(k_2 - k_1)$. But this is a contradiction as $k_1 > k_2 > 0$.

If $k_2 - k_1 + \beta - 1 \leq 0$, then inequality (4.5) together with $\varepsilon = k_2 - 2$ and $\beta \geq 2$ yields

$$(k_1-1)(k_2-1) \leq (k_2-2)(k_1-\beta+2) \leq (k_2-2)k_1,$$

implying $k_1 \leq k_2 - 1$, a contradiction.

Therefore, we have that $k_2 - k_1 + \beta - 1 > 0$ and $\beta \geq 2$. Recall that $\beta + \gamma \leq \varepsilon = k_2 - 2$, and so inequality (4.5) gives us

$$(k_1-1)(k_2-1) \leq \varepsilon(k_1-\beta+2)+(\varepsilon-1)(k_2-k_1+\beta-1) = (k_2-2)(k_2+1)-k_2+k_1-\beta+1.$$

It follows that $2 \leq \beta \leq k_2^2 - k_2 - 2 + 2k_1 - k_1k_2$, or

$$k_1(k_2-2) \leq k_2^2 - k_2 - 4.$$

As $k_1 \geq k_2 + 1$ this yields $-2 \leq -4$, a contradiction. This shows that $\varepsilon = 0$ as claimed. It is now easy to see that Γ is isomorphic to the complete bipartite graph K_{k_1,k_2}. □

5 The case $g = 2d \geq 8$, where d is even

In this section we study girth-biregular graphs with girth $g = 2d \geq 8$, d even. Throughout this section we will use Notation 3.5. Assume that $g = 2d \geq 8$. For every $z \in D_1^2$ we define

$$\beta(z) = \sum_{r \in D_{d-1}^d \cap \Gamma_{d-2}(z)} h(r).$$

Note that for $z \in D_1^2$ we have $|D_{d-1}^d \cap \Gamma_{d-2}(z)| = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$ and that for $z, z' \in D_1^2$ $(z \neq z')$, the sets $D_{d-1}^d \cap \Gamma_{d-2}(z)$ and $D_{d-1}^d \cap \Gamma_{d-2}(z')$ are disjoint as the girth of Γ is $2d$. Therefore,

$$\sum_{z \in D_1^2} \beta(z) = \sum_{r \in D_{d-1}^d} h(r) = \varepsilon. \tag{5.1}$$

In particular, $\beta(z) \leq \varepsilon$. Recall also that for an edge e of Γ we denoted by $n(e)$ the number of girth cycles passing through e.

Lemma 5.1. *Assume that $g = 2d \geq 8$ and $\varepsilon \geq 1$. Then*

$$a_1 \geq (k_1-1)^{d/2}(k_2-1)^{d/2} - k_2\varepsilon.$$

Proof. Abbreviate $\ell = (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$. Pick $z \in D_1^2$ with $n(vz) = a_1$ and let $w_1, w_2, \ldots, w_\ell$ be the vertices of $D_{d-1}^d \cap \Gamma_{d-2}(z)$. For $1 \leq j \leq \ell$ consider the $2d$-cycles of the form $(v, z, \ldots, w_j, b, r, r', \ldots)$ with $b \in D_d^{d-1}$, where $(v, z, \ldots, w_j)$ is the unique path from v to w_j of length $d-1$. Observe that for fixed w_j and r, there is only one such cycle (recall that as $g \geq 8$, w_j and r have a unique common neighbour), and that for fixed w_j and b, we could choose r in $k_2 - 1 - h(b)$ different ways. Therefore,

$$\begin{aligned} a_1 = n(vz) \geq & \sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} (k_2 - 1 - h(b)) \\ = & \sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} (k_2 - 1) - \sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} h(b). \end{aligned} \tag{5.2}$$

Furthermore, observe that for a fixed w_j we could choose b in $(k_1 - 1 - h(w_j))$ different ways, and so

$$\sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} (k_2-1) = (k_2-1) \sum_{j=1}^{\ell} (k_1-1-h(w_j)) = \ell(k_1-1)(k_2-1) - (k_2-1)\beta(z).$$

Finally, the sets $\Gamma(w_j) \cap D_d^{d-1}$ and $\Gamma(w_\ell) \cap D_d^{d-1}$ are disjoint if $j \neq \ell$ (otherwise we would get a cycle of length $2d-2$), and so

$$\sum_{j=1}^{\ell} \sum_{b \in \Gamma(w_j) \cap D_d^{d-1}} h(b) \leq \sum_{b \in D_d^{d-1}} h(b) = \varepsilon.$$

This, together with $\beta(z) \leq \varepsilon$, shows that

$$a_1 = n(vz) \geq \ell(k_1-1)(k_2-1) - (k_2-1)\beta(z) - \varepsilon \geq (k_1-1)^{d/2}(k_2-1)^{d/2} - k_2\varepsilon. \quad \square$$

Lemma 5.2. *Assume that $g = 2d \geq 8$ and $\varepsilon \geq 1$. Then*

$$a_1 < (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big(k_1\varepsilon - k_1 + 2\big).$$

Proof. Let

$$D = \bigcup_{i=0}^{d-1} \left(D^i_{i+1} \cup D^{i+1}_i\right).$$

For vertices $x, y \in D$, let $d_D(x, y)$ denote the distance between x and y in the subgraph $\Gamma[D]$, that is, in the subgraph of Γ, that is induced by D. Observe that $d_D(x, y) \leq 2d - 1$ for all $x, y \in D$.

Pick a vertex $r \in D^d_{d-1}$ with $h(r) \geq 1$ and abbreviate $\alpha = h(r)$. Pick $w \in \Gamma(r) \cap D^{d+1}_d$ and consider the set C of $2d$-cycles $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w)$ through wr. Note that, as $w \notin D$ at most $2d - 2$ edges of such a cycle have both endpoints in D. For $1 \leq i \leq 2d - 1$ let C_i denote the subset of C defined as follows. A cycle $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w)$ is an element of C_i if and only if $\{x_1, \ldots, x_i\} \subseteq D$ and $x_{i+1} \notin A$, where the addition in subscripts is computed modulo $2d$. For example, cycles in C_1 are those $2d$-cycles $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w)$, for which $x_2 \notin D$, while cycles in C_{2d-1} are those for which $\{x_1, x_2, \ldots, x_{2d-1}\} \subseteq D$. Note that the sets C_i are pairwise disjoint, and so

$$a_1 \leq n(wr) \leq |C_1| + |C_2| + \cdots + |C_{2d-1}|.$$

Let us now estimate the above sum. To do this we introduce the following notation. For $i \in \{1, 3, \ldots, 2d - 1\}$ we define

$$\varepsilon_i = \sum_{\substack{b \in D^{d-1}_d \\ d_D(r,b) = i}} h(b).$$

Note that as $\Gamma[D]$ is bipartite with diameter at most $2d - 1$, we have that

$$\varepsilon_1 + \varepsilon_3 + \cdots + \varepsilon_{2d-1} = \varepsilon.$$

We also define

$$\kappa = |\Gamma(w) \cap (D^d_{d-1} \setminus \{r\})| = |\Gamma(w) \cap D^d_{d-1}| - 1.$$

Note that $\alpha + \kappa \leq \varepsilon$.

Consider a $2d$-cycle $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w) \in C_1$. Observe that there are $\alpha - 1$ choices for x_2. For each such choice of x_2, there are, by Lemma 3.1, at most $(k_1 - 1)^{(d-2)/2}(k_2 - 1)^{d/2}$ girth cycles containing both edges wr and rx_2. Therefore,

$$|C_1| \leq (\alpha - 1)(k_1 - 1)^{(d-2)/2}(k_2 - 1)^{d/2}.$$

Consider a $2d$-cycle $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w) \in C_2 \cup C_4 \cup \cdots \cup C_{2d-2}$. Assume that this cycle is an element of C_{2j} $(1 \leq j \leq d - 1)$. Observe that in this case we have that $x_{2j} \in D^{d-1}_d$ and that $d_D(r, x_{2j}) = 2j - 1$ (otherwise there would be a cycle of length less than $2d$). Therefore, we could choose an edge $x_{2j}x_{2j+1}$ in ε_{2j-1} different ways. For each such choice of an edge $x_{2j}x_{2j+1}$, there are, by Lemma 3.1, at most $(k_1 - 1)^{(d-2)/2}(k_2 - 1)^{(d-2)/2}$ girth cycles containing edges wr and $x_{2j}x_{2j+1}$, and so

$$\begin{aligned} |C_2| + |C_4| + \cdots + |C_{2d-2}| &\leq (\varepsilon_1 + \varepsilon_3 + \cdots + \varepsilon_{2d-3})(k_1 - 1)^{(d-2)/2}(k_2 - 1)^{(d-2)/2} \\ &= \varepsilon(k_1 - 1)^{(d-2)/2}(k_2 - 1)^{(d-2)/2}. \end{aligned}$$

Consider a $2d$-cycle $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w) \in C_3 \cup C_5 \cup \cdots \cup C_{2d-3}$. If this cycle is an element of C_{2j+1} $(1 \le j \le d-2)$, then it is easy to see that $x_{2j+1} \in D^d_{d-1}$, and so $x_{2j+2} \in D^{d+1}_d \setminus \{w\}$. Therefore, there are at most $\varepsilon - \kappa - \alpha$ choices for an edge $x_{2j+1}x_{2j+2}$. For each such choice there are, by Lemma 3.1, at most $(k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}$ girth cycles containing edges wr and $x_{2j+1}x_{2j+2}$, and so

$$|C_3| + |C_5| + \cdots + |C_{2d-3}| \le (\varepsilon - \kappa - \alpha)(k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}.$$

Finally, consider a $2d$-cycle $(x_0 = w, x_1 = r, x_2, \ldots, x_{2d-1}, w) \in C_{2d-1}$. Note that we have at most $k_1 - \alpha$ choices for a vertex x_2. For each choice of vertices $x_2, x_3, \ldots, x_{i-1}$, where $i \le d$, we have at most $k_1 - 1$ choices for vertex x_i if i is even, and $k_2 - 1$ choices for x_i if i is odd. Therefore, there are at most $(k_1-\alpha)(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}$ choices for vertices $x_2, x_3, \ldots, x_d$. On the other hand, there are at most κ choices for a vertex x_{2d-1}. For each such choice of vertices $x_2, x_3, \ldots, x_d$ and x_{2d-1}, there is at most one girth cycle containing the edges wr, rx_2, $x_2x_3, \ldots x_{d-1}x_d$ and $x_{2d-1}w$. Therefore,

$$|C_{2d-1}| \le \kappa(k_1-\alpha)(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}.$$

To further estimate the sum $|C_1| + |C_2| + \cdots + |C_{2d-1}|$, we first note that

$$\begin{aligned}
|C_1| + |C_{2d-1}| &\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2} \\
&\qquad \Big((\alpha-1)(k_1-1)(k_2-1) + \kappa(k_1-\alpha)(k_1-1)\Big) \\
&< (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2} \\
&\qquad \Big((\alpha-1)(k_1-1)^2 + \kappa(k_1-\alpha)(k_1-1)\Big) \\
&= (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2} \\
&\qquad \Big((\alpha-1+\kappa)(k_1-1)^2 - \kappa(\alpha-1)(k_1-1)\Big) \\
&\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}(\alpha-1+\kappa)(k_1-1)^2 \\
&\le (k_1-1)^{d/2}(k_2-1)^{(d-2)/2}(\varepsilon-1),
\end{aligned}$$

while

$$\begin{aligned}
|C_2| + |C_3| + \cdots + |C_{2d-2}| &\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}(\varepsilon(k_1-1) + (\varepsilon-\kappa-\alpha)) \\
&\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1-1) + \varepsilon - 1\big) \\
&= (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}(k_1\varepsilon - 1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_1 \le n(wr) &\le |C_1| + |C_2| + \cdots + |C_{2d-1}| \\
&\le (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big((\varepsilon-1)(k_1-1)^2 + k_1\varepsilon - 1\big) \\
&= (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1^2-k_1) + \varepsilon - (k_1-1)^2 - 1\big) \\
&< (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1^2-k_1) + (k_2-1) - (k_1-1)^2\big) \\
&< (k_1-1)^{(d-4)/2}(k_2-1)^{(d-2)/2}\big(\varepsilon(k_1^2-k_1) + (k_1-1) - (k_1-1)^2\big) \\
&= (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big(k_1\varepsilon - k_1 + 2\big).
\end{aligned}$$

The result follows. □

Theorem 5.3. *Assume that $g = 2d \geq 8$ and d is even. Then $\varepsilon = 0$ and Γ is the incidence graph of a finite thick generalized d-gon, hence either $d = 4$ or $d = 8$.*

Proof. Suppose first that ε is positive. By Lemma 5.1 and 5.2 we have

$$(k_1-1)^{d/2}(k_2-1)^{d/2} - k_2\varepsilon \leq a_1 < (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big(k_1\varepsilon - k_1 + 2\big).$$

This implies

$$\begin{aligned} k_2 - 1 > \varepsilon &> \frac{(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(k_1k_2-k_2-1)}{k_2 + k_1(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}} \\ &> \frac{(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(k_1k_2-k_2-1)}{k_1\big(1+(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big)} \\ &= k_2 - 2 + \frac{(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(2k_1-k_2-1) - k_1(k_2-2)}{k_1\big(1+(k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}\big)} \end{aligned}$$

As $k_1(k_2-2) < (k_1-1)(k_2-1) < (k_1-1)^{(d-2)/2}(k_2-1)^{(d-2)/2}(2k_1-k_2-1)$, the above inequality implies

$$k_2 - 1 > \varepsilon > k_2 - 2,$$

contradicting the fact that ε is an integer. Therefore, $\varepsilon = 0$. □

6 The case $g = 2d$, where d is odd

In this section we consider the case $g = 2d$ with d odd, in particular the case $g = 6$ when we provide a characterization of affine planes. Unfortunately, the method we applied in the proof of Lemma 5.2 for giving an upper estimate on b_1 does not work for odd d, but we can calculate the exact value of b_1 if $\varepsilon = 1$. Throughout this section we will use Notation 3.5.

Theorem 6.1. *Assume that d is odd and suppose that $a_{k_1} = b_{k_2} = M - 1$. Then $b_1 = M - k_2 + 1$ and $b_2 = \cdots = b_{k_2} = M - 1$.*

Proof. Pick adjacent vertices $u \in A, v \in B$ such that $n(uv) = a_{k_1} = b_{k_2} = M - 1$. Let D_j^i denote $D_j^i(u,v)$ and

$$D = \bigcup_{i=0}^{d-1} \big(D_{i+1}^i \cup D_i^{i+1}\big).$$

For vertices $x, y \in D$, let $d_D(x,y)$ denote the distance between x and y in the subgraph $\Gamma[D]$, that is, in the subgraph of Γ, that is induced by D. Observe that $d_D(x,y) \leq 2d-1$ for all $x, y \in D$.

By Proposition 2.6(vi) and (vi) we have that

$$|D_d^{d-1}| = |D_{d-1}^d| = (k_1-1)^{(g-2)/4}(k_2-1)^{(g-2)/4} = \frac{M}{k_2-1},$$

and there are $M-1$ edges between D_d^{d-1} and D_{d-1}^d. Hence all but one vertices in D_{d-1}^d have $k_2 - 1$ neighbours in D_d^{d-1}. Let $p \in D_{d-1}^d$ denote the unique vertex which has only $k_2 - 2$ neighbours in D_d^{d-1}.

We claim that all but one vertices in D_d^{d-1} have $k_2 - 1$ neighbours in D_{d-1}^d, too. Let x be any vertex in D_d^{d-1}. Then for each vertex $y \in D_1^2$ there is at most one vertex $z \in D_{d-1}^d \cap \Gamma(x)$ so that $d(y,z) = d-2$, because otherwise a cycle of length $2(d-1)$ would appear. Thus

$$|\Gamma(x) \cap D_{d-1}^d| \leq |D_1^2| = k_2 - 1.$$

This implies, by the pigeonhole principle, that there is a unique vertex $r \in D_d^{d-1}$ which has only $k_2 - 2$ neighbours in D_{d-1}^d. Then r has one neighbour in D_{d-1}^{d-2} and it has $k_1 - k_2 + 1$ neighbours outside D.

Now, let w be an arbitrary vertex in D_1^2 and let $S = D_{d-1}^d \setminus D_{d-1}^{d-2}(w,v)$. Then

$$D_{d-1}^d(w,v) = D_{d-1}^{d-2} \cup S.$$

We now describe the set $D_d^{d-1}(w,v)$. Observe that

$$D_d^{d-1}(w,v) \subseteq D_d^{d-1} \cup \{p_1\}, \tag{6.1}$$

where p_1 is the unique neighbour of p outside D. There are two possibilities we have to consider, namely either w is the unique vertex of D_1^2 for which $d_D(p,w) = d-2$, or $d_D(p,w) = d$. Let us first consider the case $d_D(p,w) = d-2$. Note that in this case $p_1 \in D_d^{d-1}(w,v)$, so there is a unique vertex $w_1 \in D_d^{d-1}$ which is not contained in $D_d^{d-1}(w,v)$. Observe that every vertex from D_d^{d-1}, which has $k_2 - 1$ neighbours in D_{d-1}^d, is at distance $d-1$ from w, and so $w_1 = r$. Therefore (6.1) implies

$$D_d^{d-1}(w,v) = \big(D_d^{d-1} \setminus \{r\}\big) \cup \{p_1\}.$$

We now count the number of neighbours between $D_d^{d-1}(w,v)$ and $D_{d-1}^d(w,v)$. Recall that each vertex from D_d^{d-1} has a unique neighbour in D_{d-1}^{d-2} and that each vertex from $D_d^{d-1} \setminus \{r\}$ has $k_2 - 1$ neighbours in D_{d-1}^d. Pick $x \in D_d^{d-1} \setminus \{r\}$. As $x \in D_d^{d-1}(w,v)$, x has at least one neighbour in $D_{d-1}^d \setminus S$. On the other hand, if x has more than one neighbour in $D_{d-1}^d \setminus S$, then this would imply a cycle of length $2(d-1)$, a contradiction. Using the above observations we now have

$$\begin{aligned} n(vw) &= \big(|D_d^{d-1}| - 1\big)(k_2 - 1) \\ &= \big((k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2} - 1\big)(k_2 - 1) \\ &= M - k_2 + 1. \end{aligned}$$

In the case when $d_D(p,w) = d-2$ we have that $d(w,p_1) = d+1$ (note that p is the only neighbour of p_1 in D), and so by (6.1) we have $D_d^{d-1}(w,v) = D_d^{d-1}$. Observe also that $|S| = |D_{d-1}^d| - |D_{d-1}^{d-2}(w,v)| = (k_1-1)^{(d-1)/2}(k_2-1)^{(d-3)/2}(k_2-2)$. Similar arguments as in the previous case now show that

$$\begin{aligned} n(vw) &= |D_d^{d-1}| + |S|(k_2 - 1) - 1 \\ &= (k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2} + (k_1-1)^{(d-1)/2}(k_2-1)^{(d-1)/2}(k_2-2) - 1 \\ &= M - 1. \end{aligned}$$

This proves the statement. □

Theorem 6.2. *Assume that d is odd and k_2 does not divide k_1. If $a_{k+1} = b_k = M - \varepsilon$ for a non-negative integer $\varepsilon \le 1$, then $\varepsilon = 0$ and $a_1 = \cdots = a_{k+1} = b_1 = \cdots = b_k = M$.*

Proof. We first assume $\varepsilon = 1$ and derive a contradiction. If $\varepsilon = 1$, then it follows from Theorem 6.1 that the signature of any vertex from B is $(M - k_2 + 1, M - 1, \dots, M - 1)$. Now Proposition 3.2 yields that the signature of any vertex from A is $(M-k_2+1, \dots, M-k_2+1, M-1, \dots, M-1)$. Let $a = M - k_2 + 1$ and let a_A and a_B be as in Proposition 3.3. Observe that $a_B = 1$ and so we have $k_2 a_A = k_1$ by Proposition 3.3. Hence k_1 is divisible by k_2, a contradiction. Therefore $\varepsilon = 0$ and the result now follows from Theorem 3.4. □

In particular, we consider the case $k_1 - 1 = k_2 = k$ and $d = 3$. Then $k_1 k_2 - k_1 + 1 = k^2$ and it is well-known that a $2-(k^2, k, 1)$ design is a finite affine plane of order k. Combining Theorems 3.4(vii) and 6.2 we get the following characterization.

Corollary 6.3. *Assume that $k_1 - 1 = k_2 = k$ and that $d = 3$. If $a_{k+1} = b_k = M - \varepsilon$ for a non-negative integer $\varepsilon \le 1$, then $\varepsilon = 0$ and Γ is the incidence graph of a finite affine plane of order k.*

7 Examples

In this section we provide some examples where a_{k_1} is close to the upper bound given in Theorem 3.4. In all cases, the signatures of the points are constants, hence each edge is contained in the same number of girth cycles. So our examples are edge-girth-regular graphs, too. Let us start with the $g = 4$ case.

Example 7.1. Let $f_1 > f_2 \ge 1$ and $h > 2$ be integers and consider the complete bipartite graph $\Gamma' = K_{f_1 h, f_2 h}$ with bipartition A and B. Label the vertices so that

$$A = \bigcup_{i=1}^{f_1} \{u_{1,i}, u_{2,i}, \dots, u_{h,i}\}, \quad B = \bigcup_{j=1}^{f_2} \{v_{1,j}, v_{2,j}, \dots, v_{h,j}\}.$$

Let Γ denote a graph that is obtained from Γ' by deleting all edges of the form $u_{\ell,i} v_{\ell,j}$, where $\ell \in \{1, 2, \dots, h\}$, $i \in \{1, 2, \dots, f_1\}$ and $j \in \{1, 2, \dots, f_2\}$. Then Γ is a bipartite biregular graph with $g = 4$, $k_1 = f_2(h-1)$ and $k_2 = f_1(h-1)$.

Take any edge $e = u_{\ell_1,i} v_{\ell_2,j}$ in Γ. Then $\ell_1 \ne \ell_2$, and there are $((f_2(h-1)-1)(f_1(h-1)-1)$ 3-arcs of Γ which contain e. Let us now count how many of these 3-arcs are not contained in a 4-cycle. Let $\mathcal{A} = v_{\ell',j'} u_{\ell_1,i} v_{\ell_2,j} u_{\ell'',i''}$ be any 3-arc containing edge e. Note that $\ell' \ne \ell_1$ and $\ell'' \ne \ell_2$. Then $\mathcal{A}$ is not contained in a 4-cycle if and only if vertices $v_{\ell',j'}$ and $u_{\ell'',i''}$ are not adjacent in Γ, which happens if and only if $\ell' = \ell''$. As $\ell' \ne \ell_1$ and $\ell'' \ne \ell_2$, there are $h-2$ choices for $\ell' = \ell''$, hence there are $f_1 f_2 (h-2)$ 3-arcs containing e, that are not contained in a 4-cycle. So the number of girth cycles through e in Γ is $((f_2(h-1)-1)(f_1(h-1)-1) - f_1 f_2 (h-2)$. It follows that Γ is girth-biregular with

$$a_1 = \cdots = a_{k_1} = b_1 = \cdots = b_{k_2} = (k_1 - 1)(k_2 - 1) - f_1 f_2 (h-2) = M - f_1 f_2 (h-2).$$

For $g = 6$ we follow the examples of the paper [1].

Example 7.2. Take an affine plane of order q and remove i parallel classes. Consider the incidence graph of this structure. The lines still have size q and the points have degree $q+1-i$, so it is a bipartite biregular graph with valencies q and $q+1-i$. To count the girth cycles containing the edge corresponding to an incident point-line pair (e_0, P_0), we have to choose a point $P_0 \neq P_1 \in e_0$, and a line $e_0 \neq e_1$ through P_0 and complete it to a girth cycle (of length 6) by choosing a point $P_0 \neq P_2 \in e_1$ and a line e_2 joining P_1 and P_2. There are $q-1$ ways to choose P_1 and $q-i$ ways of choosing e_1. For e_2 we have to choose a line different from e_1, not parallel to e_0, so we have $(q-1-i)$ possibilities, since the point P_2 will just be the unique point of $e_0 \cap e_2$. So, in total there are $M' = (q-1)(q-i)(q-1-i)$ girth cycles through the edge (e_0, P_0).

In particular, when we have an affine plane of order q, its incidence graph is a bipartite biregular graph with valencies $q+1$ and q, and we have $M = (q-1)^2 q$ girth cycles through an edge. If there is an affine plane of order $q+1$ as well, then removing $i = 2$ parallel classes will also give us a bipartite biregular graph with valencies $q+1$ and q and this graph will have $M' = q(q-1)(q-2) = M - q(q-1)$ girth cycles through every edge.

Another construction from the paper [1] is the following.

Example 7.3. Let us consider a Steiner system on v points and line size k. Delete a point P^* and all the lines through the deleted point. The incidence graph of the resulting structure will be a bipartite biregular graph with valencies k and $r-1$, again with $r = (v-1)/(k-1)$. One can more or less copy the argument in the previous example: using the same notation, the point P_1 can be chosen in $(k-1)$ ways. Now consider the line e^* in the original Steiner system that joins P_1 and P^*. If the line e_1 intersects e^*, then we have $(k-2)$ choices for P_2 and e_2, and there are $(k-2)$ such lines in the original Steiner system. So, this case gives $(k-1)(k-2)^2$ girth cycles. There remain $(r-2)-(k-2) = r-k$ lines through P_0, not intersecting e^*. If e_1 is one of them, then there are $(k-1)$ ways to extend it to a girth cycle. This is $(k-1)^2(r-k)$ possibility, so in total we have $(k-1)((k-2)^2 + (r-k)(k-1))$ girth cycles containing the edge (e_0, P_0).

It is easy to extend Example 7.2 to resolvable Steiner systems.

Example 7.4. Consider a resolvable Steiner system and denote by v the number of points, by r the degrees of points, where $r = (v-1)/(k-1)$. In this case k divides v, and the original design will have $(k-1)^2(r-1)$ girth cycles through any edge. If we remove i parallel classes of lines, then the incidence graph of the resulting structure will have degrees k and $r-i$). For determining the number of girth cycles containing an edge start from an incident point-line pair (P_0, e_0) as before. Take a point P_1 on e_0 and let U be the set of points which are on the lines through P_1 that belong to the deleted parallel classes. This implies that $|U| = i(k-1)$. Let r_j, $j = 0, \ldots, k-1$, be the number of lines through P_0 which intersect U in exactly j points. Clearly, we have $\sum_j r_j = r-1$, and $\sum_j jr_j = |U| = i(k-1)$. On a line ℓ through having j points in U, we can choose the point P_2 of the girth cycle in $(k-1-j)$ ways. This way we get in total

$$\sum_{j=0}^{k-1}(k-1-j)r_j = (k-1)(r-1) - i(k-1)$$

girth cycles for a given choice of P_1, so the total number of girth cycles will be $(k-1)^2((r-1)-i)$. For small i this is close to our upper bound.

In particular, we mention two examples arising from higher dimensional finite spaces.

1. Let $n = 2m + 1$. Remove the $q^m + 1$ elements of a line spread from $\mathrm{PG}(n, q)$ and denote the correponding point-line incidence graph by Γ. Then Γ is a girth-biregular bipartite graph with $g = 6$, $k_1 = q^{2m} + \cdots + q$ and $k_2 = q + 1$ and its signature is

$$a_1 = \cdots = a_{k_1} = b_1 = \cdots = b_{k_2} = q^2(q^{2m} + \cdots + q - 2) = M - q^2.$$

2. Let us remove the q^{n-1} elements of a class of parallel lines from $\mathrm{AG}(n, q)$ and denote the corresponding point-line incident graph by Γ. Then Γ is a girth-biregular bipartite graph with $g = 6$, $k_1 = q^{n-1} + \cdots + q$ and $k_2 = q$ and its signature is

$$a_1 = \cdots = a_{k_1} = b_1 = \cdots = b_{k_2} = (q-1)^2(q^{n-1} + \cdots + q - 2) = M - (q-1)^2.$$

In both cases the magnitude of ε is only $k_1^{2/(n-1)}$.

In the case $g = 8$ our examples come from incidence graphs of generalized quadrangles. For a detailed descriptions of generalized quadrangles, their ovoids and spreads, we refer the reader to the book of Payne and Thas [13].

Example 7.5. Let $\mathcal{G}' = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized quadrangle of order (s, t) and Γ' be the Levi graph of $\mathcal{G}'$.

Suppose that $\mathcal{G}'$ admits a spread $\mathcal{S}$ (a set of $st + 1$ lines, no two of which intersect). Delete the lines of $\mathcal{S}$. Then the Levi graph Γ of $\mathcal{G} = (\mathcal{P}, \mathcal{L} \setminus \mathcal{S}, \mathrm{I})$ is a bipartite graph with bipartition $|A| = (s + 1)(st + 1)$ and $|B| = t(st + 1)$, valencies $s + 1$ and t and $g = 8$. We claim that it is also girth-biregular with

$$a_1 = \cdots = a_{s+1} = b_1 = \cdots = b_t = s^2(t^2 - 3t + 2) = M - s^2(t-1).$$

Dually, if $\mathcal{G}'$ admits an ovoid $\mathcal{O}$ (a set of $st + 1$ points, no two of which are collinear), then the Levi graph Γ of $\mathcal{G} = (\mathcal{P} \setminus \mathcal{O}, \mathcal{L}, \mathrm{I})$ is a girth-biregular graph with valencies s and $t + 1$, and

$$a_1 = \cdots = a_{s+1} = b_1 = \cdots = b_t = t^2(s^2 - 3s + 2) = M - t^2(s-1).$$

In $\mathcal{G}$ for any incident point-line pair (P, ℓ) there are $(t - 1)s$ points in $\mathcal{P}$ which are collinear with P but are not incident with ℓ, and there are $s(t-1)$ lines in which meet ℓ but are not incident with P. Let R be one of these points and e be one of these lines. Then there is a unique point-line pair (T, f) in $\mathcal{G}'$ so that $R \,\mathrm{I}\, f \,\mathrm{I}\, T \,\mathrm{I}\, e$. Thus in Γ' there are $s^2(t-1)^2$ girth cycles through the edge which corresponds to the pair (P, ℓ). For a fixed R there is a unique element $f \in \mathcal{S}$ through R. All the s other points on f determines a unique 8-cycle which contains (P, ℓ). No two elements of $\mathcal{S}$ intersect, hence there are $(t-1)s \cdot s$ deleted 8-cycles. Thus in Γ' the total number of girth cycles through the edge corresponding to (P, ℓ) is

$$s^2(t-1)^2 - s(t-1)s = s^2(t^2 - 3t + 2) = s^2(t-1)(t-2).$$

Among the known generalized quadrangles only a few admit a spread or an ovoid. In particular, the classical generalized quadrangle $\mathcal{Q}(5, q)$ admits a spread. In this case Γ has valencies $q+1$ and q^2, and the number of girth cycles through every edge is $q^2(q^2-1)(q^2-2) = M - q^2(q^2-1)$. So the magnitude of ε is $M^{2/3}$.

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