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Hamilton cycles in primitive graphs of order $2rs$*

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Abstract

After long term efforts, it was recently proved by Du, Kutnar and Marušič in 2021 that except for the Petersen graph, every connected vertex-transitive graph of order rs has a Hamilton cycle, where r and s are primes. A natural topic is to solve the hamiltonian problem for connected vertex-transitive graphs of $2rs$. This topic is quite nontrivial, as the problem is still unsolved even for that of $r = 3$ and 5. In this paper, it is shown that except for the Coxeter graph, every connected vertex-transitive graph of order $2rs$ contains a Hamilton cycle, provided the automorphism group acts primitively on vertices.

Keywords: Vertex-transitive graph, Hamilton cycle, primitive group, automorphism group, orbital graph.

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1 Introduction

Throughout this paper graphs are finite, simple and undirected, and groups are finite. Given a graph X, by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ we denote the vertex set, the edge set and the automorphism group of X, respectively. A graph X is *vertex-* or *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on vertices or arcs, respectively.

Given a transitive group G on Ω, a subset B of Ω is called a *block* of G if, for any $g \in G$, we have either $B = B^g$ or $B \cap B^g = \emptyset$. Clearly, G has blocks Ω and $\{\alpha\}$ for any $\alpha \in \Omega$, which are said to be *trivial*. Then G is said to be *primitive* if it has no nontrivial blocks. Moreover, a vertex-transitive graph X is said to be *primitive* if $\mathrm{Aut}(X)$ is primitive on vertices.

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A simple path (resp. cycle) containing all vertices of a graph is called a *Hamilton path* (resp. *cycle*) of this graph. For convenience, a Hamilton-cycle (resp. path) is usually abbreviated by a H-cycle (resp. H-path). A graph containing a Hamilton cycle will be sometimes referred as a *hamiltonian graph.*

In 1970, Lovász asked in [1] that

Does every finite connected vertex-transitive graph have a Hamilton path?

Up to now, this question remains unresolved and no connected vertex-transitive graph without a Hamilton path is known to exist. Moreover, only four (families) of connected vertex-transitive graphs on at least three vertices not having a Hamilton cycle are known, which are Petersen graph, Coxeter graph and triangle-replaced graphs from them. Since all of these graphs are not Cayley graph, we may ask if every connected Cayley graph has a Hamilton cycle.

It has been shown that connected vertex-transitive graphs of orders kp, $k \leq 6$, $10p$ $(p \geq 11)$, p^j $(j \leq 5)$ and $2p^2$, where p is a prime contain a Hamilton path, see [2, 5, 18, 19, 20, 25, 26, 27, 28, 31]. Furthermore, for all of these families, except for the graphs of order $6p$ and $10p$ and that four exceptions, they contain a Hamilton cycle. With the exception of the Petersen graph, Hamilton cycles are also known to exist in connected vertex-transitive graphs whose automorphism groups contain a transitive subgroup with a cyclic commutator subgroup of prime-power order (see [6] and also [9, 17, 24]).

So far we know that Cayley graphs of the following groups contain a Hamilton cycle: nilpotent groups of odd order, with cyclic commutator subgroups (see [6, 11, 12]); dihedral groups of order divisible by 4 (see [3]); and arbitrary p-groups (see [30]). A Hamilton path and in some cases even a Hamilton cycle was proved to exist in cubic Cayley graphs arising from $(2, s, 3)$-generated groups (see [13, 14, 15]).

Recently, Kutnar, Marusic and the first author proved that vertex transitive graphs of order rs have a Hamilton cycle, except for the Petersen graph (see [7, 8]). This work took many years, because of a difficult case, which is a primitive graph with automorphism group $\mathrm{PSL}(2, p)$ and a point-stabilizer $\mathbb{D}_{p-1}$. A natural question is to consider hamiltonian problem for vertex-transitive graphs of order $2rs$. As mentioned above, some special cases have been solved such as that of graphs of order $4p$, $6p$, $10p$ and $2p^2$, where p is a prime (Hamilton path or cycle). To solve the general case, a necessary step is to deal with all primitive graphs of such order. The main result of this paper is the following theorem.

Theorem 1.1. *Except for Coxeter graph, every connected vertex-transitive graph of order* $2rs$ *contains a Hamilton cycle provided the automorphism group acts primitively on its vertices, where* r *and* s *are primes.*

After this introductory section, some notations, basic definitions and useful facts will be given in Section 2 and Theorem 1.1 will be proved in Section 3.

2 Preliminaries

By $\lfloor a \rfloor$ and $\lceil a \rceil$, we denote the largest integer that is smaller than a and smallest integer that is larger than a, respectively. For a prime q, a finite field of order q will be denoted by $\mathbb{F}_q$. Set $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $S = \{t^2 \mid t \in \mathbb{F}_q\}$, $S^* = S \cap \mathbb{F}_q^*$ and $N = \mathbb{F}_q^* \setminus S^*$. Then the elements in S and N are called to be *squares* and *non-squares*, respectively. By $\mathbb{Z}_n$ and $\mathbb{D}_{2n}$ we denote a cycle group of order n and dihedral group of order $2n$, respectively. For a group G and $L \subset G$, by $C_G(L)$ and $N_G(L)$ we denote the centralizer and normalizer of L in G, respectively. A semi-product of K and H is denoted by $K \rtimes H$, where K is normal. Let G be a group with a normal subgroup N, we denote the image of $g \in G$ under the natural homomorphism of G to G/N by $\overline{g}$. For a group G and its subgroup H, $[G : H]$ denotes the set of right cosets of H in G; HgH denotes the orbit containing Hg under the action of H. Recall that the socle of G which is denoted by $\mathrm{soc}(G)$ is defined to be the product of all minimal normal subgroups of G.

Let G act on some set Ω. For some $\alpha \in \Omega$ and $g \in G$, set $\alpha^G = \{\alpha^g \mid g \in G\}$. For $\alpha \in \Omega$, set $H = G_\alpha$. Then the action of G on Ω is equivalent to its right multiplication action on right cosets $[G : H]$ relative to H. For a subset Δ of Ω, by $G_{(\Delta)}$ and $G_{\{\Delta\}}$, we denote the pointwise and setwise stabilizer of Δ in G, respectively.

In a graph X, let $a \in V(X)$ and $B \subset V(X)$, by $d(a, B)$ we denote the number of neighbors of a in B. Given $A, B \subset V(X)$, if $d(a, B) = d(a', B)$ for any $a, a' \in A$, then we denote $d(a, B)$ by $d(A, B)$. Moreover, set $d(B) = d(B, B)$. The neighborhood of any vertex a in the graph X is denoted by $X_1(a)$.

In what follows we recall some definitions related to orbital graphs and semiregular automorphisms.

Let G be a transitive permutation group on Ω. Then G induces a natural action on $\Omega \times \Omega$. We call the orbits of G on $\Omega \times \Omega$ the *orbitals* of G, and in particular the *trivial* orbital is referred to $\{(\alpha, \alpha) \mid \alpha \in \Omega\}$. The *orbital digraph* $X(G, \Gamma)$ relative to an orbital Γ is defined to be the directed graph with vertex set Ω and edge set Γ. Each orbital Γ has an associated *paired orbital* Γ' defined by $\Gamma' = \{(\beta, \alpha) \mid (\alpha, \beta) \in \Gamma\}$, and of course, Γ is said to be *self-paired* if $\Gamma = \Gamma'$ in which case $X(G, \Gamma)$ can be viewed as an undirected graph (*orbital graph*). The G-arc-transitive graphs with vertex-set Ω are precisely the orbital graphs $X(G, \Gamma)$ for the nontrivial self-paired orbitals Γ. In addition, take a point $\alpha \in \Omega$, the orbits of the stabilizer G_α on Ω are called *suborbits* of G relative to α. There is a one-to-one correspondence between the suborbits and the orbitals of G. Each orbital Γ_i corresponds to a suborbit $\Delta_i = \{ \beta \in \Omega \mid (\alpha, \beta) \in \Gamma_i \}$. Conversely, each suborbit Δ_i corresponds to an orbital $\Gamma_i = \{ (\alpha, \beta)^g \mid g \in G, \beta \in \Delta_i \}$. A suborbit of G is said to be *self-paired* if the corresponding orbital is self-paired. Thus we often use $X(G, \Delta_i)$ and $X(G, \Delta_i \cup \Delta_i')$ to denote graphs $X(G, \Gamma)$ and $X(G, \Gamma \cup \Gamma')$ respectively.

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism ρ of a graph X is called (m, n)-*semiregular* (in short, *semiregular*) if as a permutation on $V(X)$ it has a cycle decomposition consisting of m cycles of length n. If $m = 1$ then X is called a *circulant*; it is in fact a Cayley graph of a cyclic group of order n. Let $\mathcal{P}$ be the set of orbits of ρ, that is, the orbits of the cyclic subgroup $\langle \rho \rangle$ generated by ρ. We let the *quotient graph corresponding to* $\mathcal{P}$ be the graph $X_{\mathcal{P}}$ whose vertex set equals $\mathcal{P}$ with $A, B \in \mathcal{P}$ adjacent if there exist vertices $a \in A$ and $b \in B$, such that $a \sim b$ in X.

The following four results will be used later.

Proposition 2.1 ([29, page 167]). *Let F_q be the finite field of odd prime order q. Then*

$$|(S^* + 1) \cap (-S^*)| = \begin{cases} (q-5)/4 & q \equiv 1 \pmod 4, \\ (q+1)/4 & q \equiv 3 \pmod 4. \end{cases}$$

This implies that if $q \equiv 1 \pmod 4$ then

$$|S^* \cap (S^* + 1)| = (q-5)/4, \quad |N \cap (N+1)| = (q-1)/4, \quad |S^* \cap (N \pm 1)| = (q-1)/4.$$

No.	soc(G)	$2rs$	*Action*	*Comment*
1	PSL$(2,\, q)$	$q(q+1)/2$	$G_\alpha \cap \text{soc}(G) = \mathbb{D}_{2(q-1)/d}$	$d = (2,\, q-1)$, $G = \text{PGL}(2,\, 11)$ for $q = 11$
2	PSL$(2,\, q)$	$q(q-1)/2$	$G_\alpha \cap \text{soc}(G) = \mathbb{D}_{2(q+1)/d}$	$d = (2,\, q-1)$
3	PSL$(2, 47)$	$2 \times 47 \times 23$	S_4	
4	PSL$(2, 17)$	$2 \times 17 \times 3$	S_4	
5	PSL$(2, 41)$	$2 \times 41 \times 7$	A_5	

Table 1: Primitive groups of degree $2rs$, where the socle PSL$(2, q)$.

Proposition 2.2 ([16, Theorem 6] (Jackson's Theorem))**.** *Every* 2*-connected regular graph of order* n *and valency at least* $n/3$ *contains a Hamilton cycle.*

Proposition 2.3 ([4, Corollary 3])**.** *If* X *is a connected Cayley graph of an abelian group of order at least* 3*, then every edge of* X *lies in a hamiltonian cycle.*

Lemma 2.4 ([27, Lemma 5])**.** *Let* X *be a graph admitting an* $(m,\, p)$*-semiregular automorphism* ρ*, where* p *is a prime. Let* C *be a cycle of length* m *in the quotient graph* $X_{\mathcal{P}}$*, where* $\mathcal{P}$ *is the set of orbits of* ρ*. Then, the lift of* C *either contains a cycle of length* mp *or it consists of* p *disjoint* m*-cycles. In the latter case we have* $d(S,\, S') = 1$ *for every edge* SS' *of* C*.*

3 Proof of Theorem 1.1

To prove Theorem 1.1, let X be a connected vertex-transitive graph of order $2rs$, where r and s are primes. Set $G = \text{Aut}(X)$. It has been proved that X contains a Hamilton cycle if $2rs = 2p^2$ or $4p$ for a prime p, provided X is not the Coxeter graph which is of order 28. Therefore, in what follows we assume that $r < s$. If G acts 2-transitively on $V(X)$, then X is a complete graph, which contains a H-cycle. Now we need to consider all the primitive groups of degree $2rs$ of rank at least 3 from [10] (or [21]), where r and s are distinct odd primes. Let H be a point stabilizer in soc(G). Checking [10], all the possible groups are listed in Tables 1 and 2.

Table 1 gives the these groups with the socle PSL$(2,\, q)$. The first two cases $H = \mathbb{D}_{q-1}$ and $H = \mathbb{D}_{q+1}$ will be dealt with in Subsections 3.1 and 3.2, respectively. With the help of Magma, we can show that every vertex-transitive graph is hamiltonian, arising from other three groups in Table 1.

Table 2 gives these groups whose socle is a classical simple group which is not PSL$(2,\, q)$, an alternative group or a sporadic simple group. These groups will be dealt with in Subsection 3.3.

3.1 $\text{soc}(G) = \text{PSL}(2,\, q)$ and $H = \mathbb{D}_{q-1}$

Let $G = \text{PSL}(2,\, q)$ and $H = \mathbb{D}_{q-1}$. Consider the action of G on the set $[G : H]$ of cosets of H in G, see row 1 of Table 1. Then the degree is $q(q+1)/2 = 2rs$, thus $q \equiv 3 (\text{mod } 4)$ and in particular $-1 \in N$, the set of non-squares. Set $\mathbb{F}_q^* = \langle \theta \rangle$.

No.	soc(G)	$2rs$	*Action*	*Comment*
1	PSL(4, q)	$\frac{q^3-1}{q-1}(q^2+1)$	2-spaces	$q=3$; or $q=5$; or $q\equiv 11,\ 29(\text{mod } 30)$ and q prime and $q\geq 59$
2	PSL(5, q)	$\frac{q^5-1}{q-1}(q^2+1)$	2-spaces	$q\equiv -1(\text{mod } 10)$, q prime and $q\geq 29$
3	$\mathrm{P}\Omega^-(2m,\ q)$	$\frac{(q^m+1)(q^{m-1}-1)}{q-1}$	on t.s. 1-spaces	m even
4	$\mathrm{P}\Omega^+(2m,\ q)$	$\frac{(q^m-1)(q^{m-1}+1)}{q-1}$	on t.s. 1-spaces	m odd
5	PSL(3, 5)	$2\times 31\times 3$	on (1, 2)-dim. flags	$G=$ PSL(3, 5).2
6	A_c	$\frac{c(c+1)}{2}$	on 2-sets	$c\geq 5$
7	M_{11}	66	S_5	
8	M_{12}	66	$M_{10}:2$	
9	M_{23}	506	A_8	
10	J_1	266	PSL(2, 11)	

Table 2: Primitive groups G of degree $2rs$, where $\mathrm{soc}(G)\neq \mathrm{PSL}(2,q)$.

For any $g\in \mathrm{SL}(2,\ q)$, set $\overline{g}=gZ(\mathrm{SL}(2,\ q))$. In SL(2, q), set

$$u=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ u'=\begin{pmatrix}1&0\\1&1\end{pmatrix},\ l=\begin{pmatrix}\theta&0\\0&\theta^{-1}\end{pmatrix},\ t=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Since PSL(2, q) has only one conjugacy class of subgroups isomorphic to $\mathbb{D}_{q-1}$, we may set $H=\langle \overline{l},\ \overline{t}\rangle$. Let V be the row vector space so that the action of $g\in \mathrm{GL}(2,\ q)$ on a vector $(x,\ y)$ is just defined as $(x,\ y)\cdot g$. Set $\frac{y}{x}=\langle (x,\ y)\rangle$. Then all the projective points are $\{\infty,\ 0,\ 1,\ 2,\ \cdots,\ q-1\}$. The action of G on $[G:H]$ is equivalent to its action on the set of unordered pairs of distinct projective points, where $H=G_{\{0,\ \infty\}}$. Thus we have

$$\begin{aligned}&\overline{u}':\{\infty,\ 0\}\to\{1,\ 0\},\quad \overline{l^i}:\{j,\ j+1\}\to\{j\theta^{-2i},\ (j+1)\theta^{-2i}\},\\ &\overline{l^it}:\{j,\ j+1\}\to\{-j^{-1}\theta^{2i},\ -(j+1)^{-1}\theta^{2i}\}.\end{aligned}$$

Then the all $\langle\overline{u}\rangle$-orbits are

$$B_\infty=\{\{\infty,\ i\}|i\in\mathbb{F}_q\},\quad B_j=\{\{i,\ i+j\}|i\in\mathbb{F}_q\},\ j\in\{1,\ 2,\ 3,\ \cdots,\ \frac{q-1}{2}\}.$$

Set $\mathbf{B}=\{B_j \mid j\in\{1,\ 2,\ 3,\ \cdots,\ \frac{q-1}{2}\}\}$. Considering the action of $N_G(\langle\overline{u}\rangle)=\langle\overline{u}\rangle\rtimes\langle\overline{l}\rangle$ on the vertices, we know that $N_G(\langle\overline{u}\rangle)$ fixes the block B_∞ setwise and acts transitively on other vertices. In particular, $\langle\overline{l}\rangle$ fixes B_∞ and acts regularly on $\frac{q-1}{2}$ remaining blocks B_j in $\mathbf{B}$.

The suborbits of G have been determined in [22] and an alternative description is given below.

Lemma 3.1. *Suppose $q\equiv 3(\text{mod } 4)$. Then every nontrivial suborbit of G relative to H can be written as $\{j,\ j+1\}^H$, where $j\in\mathbb{F}_q$, with length $\frac{q-1}{2}$ and $q-1$ if and only if $j^2+j\in N$ and $j^2+j\in S$, respectively. Moreover, $\{j,\ j+1\}^H$ is self-paired if and only if either $j+1\in N$ or $j\in S$, and if it is non self-paired, then its paired suborbit is $\{-j,\ -j-1\}^H$.*

Proof. For $i \in \mathbb{F}_q^*$, direct computations show that $\{\infty, i\}$ belongs to $\{0, 1\}^H$ or $\{0, -1\}^H$ depending on whether $i \in S^*$ or $i \in N$, respectively. Since $\langle \bar{l} \rangle \leq H$ acts regularly on $\mathbf{B}$, any other suborbits can also be written as $\{j, j+1\}^H$. The length of $\{j, j+1\}^H$ is $\frac{q-1}{2}$ and $q-1$ if and only if the order of the stabilizer for $\{j, j+1\}$ in H is 2 and 1, respectively. But the former holds if and only if there exists some $k \in \mathbb{Z}_q$ such that $\overline{l^k t}$ fixes $\{j, j+1\}$, i.e., $j+1 = -j^{-1}\theta^{2k}$. Therefore we deduce that the length of the suborbit is $\frac{q-1}{2}$ or $q-1$ depending on $j^2+j \in N$ or $j^2+j \in S$, respectively.

Let $\Delta = \{j, j+1\}^H$. If $j+1 = 0$, then $\Delta^* = \{0, 1\}^H$. If $j+1 \neq 0$, then $\Delta^* = \{\frac{-j}{j+1}, -1\}^H$. Now, Δ is self-paired if and only if there exists some element of H mapping $\{j, j+1\}$ to $\{\frac{-j}{j+1}, -1\}$. From

$$\{\overline{l^k}(j), \overline{l^k}(j+1)\} = \{j\theta^{-2k}, (j+1)\theta^{-2k}\} = \{\frac{-j}{j+1}, -1\} \quad \text{and}$$
$$\{\overline{l^k t}(j), \overline{l^k t}(j+1)\} = \{-j^{-1}\theta^{2k}, -(j+1)^{-1}\theta^{2k}\} = \{\frac{-j}{j+1}, -1\},$$

we know that such element of H exists if and only if $j+1 \in N$ or $j \in S$, as desired.

Suppose that Δ is not self-paired and $j+1 \neq 0$. Then $j+1 = \theta^{-2k} \in S$ and $\overline{l^k}$ maps $\{\frac{-j}{j+1}, -1\}$ to $\{-j, -j-1\}$, that is $\Delta^* = \{\frac{-j}{j+1}, -1\}^H = \{-j, -j-1\}^H$. □

Remark 3.2. By Lemma 3.1, it is easy to determine the number of nontrivial suborbits of length $\frac{q-1}{2}$ or $q-1$, and the number of nontrivial paired suborbits. But we do not need these numbers in here.

Before going to prove the main result, we first give a technical lemma on number theory.

Lemma 3.3. *Suppose that q is an odd prime. If $a, b \in \mathbb{F}_q^*$ and $a \neq b$. Then*

$$\begin{aligned}
|(S^*+a) \cap (S^*+b) \cap N| &\leq \lceil \tfrac{1}{8}(q+11+2\sqrt{q}) \rceil, \\
|(S^*+a) \cap (N+b) \cap N| &\leq \lceil \tfrac{1}{8}(q+11+2\sqrt{q}) \rceil, \\
|(S^*+a) \cap (N+b) \cap S^*| &\geq \lfloor \tfrac{1}{8}(q-11-2\sqrt{q}) \rfloor, \\
|(N+a) \cap (N+b) \cap S^*| &\geq \lfloor \tfrac{1}{8}(q-11-2\sqrt{q}) \rfloor.
\end{aligned}$$

Proof. Set $\eta \colon \mathbb{F}_q^* \to \{\pm 1\}$ by assigning the elements of S^* to 1 and that of N to -1 and moreover, set $\eta(0) = 0$. This η is exactly that in [23, Example 5.10]. Also we need to quote the following three results from [23, Theorems 5.4, 5.48, 5.41]:

(i) $\sum_{x \in \mathbb{F}_q} \eta(x) = 0$;

(ii) $\sum_{x \in \mathbb{F}_q} \eta(x^2 + Ax + B) = q-1$ for $A^2 - 4B = 0$ or -1 for otherwise, where $A, B \in \mathbb{F}_q$;

(iii) $|m| \leq 2\sqrt{q}$, where $m := \sum_{x \in \mathbb{F}_q} \eta(x(x-1)(x-t))$ and $t \in \mathbb{F}_q$.

For four inequalities of the lemma, we have the same arguments and here we just prove the first one. Set $W = (S^*+a) \cap (S^*+b) \cap N$, that is

$$W = \{x \in \mathbb{F}_q \mid \eta(x-a) = \eta(x-b) = 1, \eta(x) = -1\}.$$

Now let $a,\, b \in S^*$. Then by the above three formulas (i) – (iii), we have

$$|W| = \frac{1}{8}\sum_{x\in\mathbb{F}_q\setminus\{0,a,b\}}(1+\eta(x-a))(1+\eta(x-b))(1-\eta(x))$$
$$= \frac{1}{8}\sum_{x\in\mathbb{F}_q\setminus\{0,a,b\}}(1-\eta(x)+\eta(x-a)+\eta(x-b)-\eta(x(x-a))-\eta(x(x-b))$$
$$+\,\eta((x-a)(x-b))-\eta((x-a)(x-b)x)$$
$$= \frac{1}{8}[(q-3)-(-\eta(b)-\eta(a))-(\eta(-a)+\eta(b-a))-(\eta(-b)+\eta(a-b))$$
$$-\,(-1-\eta b(b-a))-(-1-\eta a(a-b))+(-1-\eta(ab))+m]$$
$$\le \lceil\frac{1}{8}(q+11+2\sqrt{q})\rceil.$$

□

According to Lemma 3.1, we shall deal with the orbital graphs $X = X(G,\,\Delta)$ or $X = X(G,\,\Delta\cup\Delta^*)$, according to that Δ is self-paired and of length $\frac{q-1}{2}$, non self-paired and of length $\frac{q-1}{2}$, self-paired and of length $q-1$, and non self-paired and of length $q-1$, respectively, in the following four lemmas.

Lemma 3.4. *Suppose that Δ is a self-paired suborbit of length $\frac{q-1}{2}$. Then $X(G,\,\Delta)$ is hamiltonian.*

Proof. Let $X = X(G,\,\Delta)$, where Δ is self-paired and of length $\frac{q-1}{2}$. Let Y be the quotient graph induced by $\langle\overline{u}\rangle$, with vertices $\mathbf{B}\cup\{B_\infty\}$. Then by Lemma 3.1, we may set $\Delta = \{j,\, j+1\}^H$, where $j(j+1)\in N$, $j+1\in N$ and $j\in\mathbb{F}_q$. Then the neighborhood of $\{0,\infty\}$ is:

$$X_1(\{0,\,\infty\}) = \Delta = \{\{j\theta^{-2k},\,(j+1)\theta^{-2k}\} \mid k\in\mathbb{F}_q\}.$$

Since $|\Delta| = \frac{q-1}{2}$ and $\langle\overline{l}\rangle$ acts regularly on $\mathbf{B}$, $d(B_\infty,\,B_i) = 1$ for any $i = 1,\,2,\,3,\,\cdots,\,\frac{q-1}{2}$.

The lemma will be proved by the following three steps:

Step 1: Show $d(B_m,\,B_i)\le 2$ for any $i,\, m = 1,\,2,\,3,\,\cdots,\,\frac{q-1}{2}$.

Since $\langle\overline{l}\rangle$ is regular on $\mathbf{B}$ and $\{0,\,1\}\in B_1$, we may just consider $d(B_1,\,B_i) = d(\{0,\,1\},\,B_i)$ for any $i = 1,\,2,\,3,\,\cdots,\,\frac{q-1}{2}$. Since $\overline{u}'$ maps $\{\infty,\,0\}$ to $\{0,\,1\}$, we know that

$$X_1(\{0,\,1\}) = \Delta^{\overline{u}'} = \{\{j\theta^{-2k},\,(j+1)\theta^{-2k}\} \mid k\in\mathbb{F}_q\}^{\overline{u}'}$$
$$= \{\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}},\,\frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\} \mid k\in\mathbb{F}_q\}.$$

So a vertex in $X_1(\{0,\,1\})$ is contained in B_i if and only if

$$\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}},\,\frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\} = \{t,\,t+i\}\text{ for some }t,$$

if and only if one of the following two systems of equations has solutions:

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t,\qquad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t+i; \tag{3.1}$$

and

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t+i, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t. \tag{3.2}$$

Solving Equation (3.1), we get

$$ij(j+1)u^2 + (2ij+i-1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_1 := (2ij+i-1)^2 - 4i^2j(j+1) = i^2 - (2+4j)i + 1 \in S^*.$$

Suppose that the above equation has solutions, say u_1 and u_2. Since $u_1u_2 = (j(j+1))^{-1}$, a non-square, we know that $u_1, u_2 \neq 0$, one of them is a non-square and the other one is a square. Therefore, there exists exactly one solution for $\theta^{-2k} = u$ if and only if $\delta_1 \in S^*$, noting that every θ^{-2k} gives a unique t, equivalently, a unique vertex in the block B_i.

Solving Equation (3.2), we get

$$ij(j+1)u^2 + (2ij+i+1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_2 := (2ij+i+1)^2 - 4i^2j(j+1) = i^2 + (2+4j)i + 1 \in S^*.$$

Similarly, there exists exactly one solution for θ^{-2k} if and only if $\delta_2 \in S^*$.

Summarizing Equation (3.1) and Equation (3.2), we get $d(\{0,\, 1\},\, B_i) \leq 2$.

Step 2: Show that for a given j, there exists some i such that $d(B_j,\, B_i) = 2$.

It suffices to show $d(\{0,\, 1\},\, B_i) = 2$ for some $i \neq 0$, equivalently, to show that the number of B_i $(i \neq 1)$ such that $d(B_1,\, B_i) = 1$ is less than $\frac{q-1}{2} - 1 - 2 = \frac{q-7}{2}$.

Now, $d(B_1,\, B_i) = 1$ if and only if

$$\delta_1\delta_2 = (i^2 - (2+4j)i + 1)(i^2 + (2+4j)i + 1) = y \in N,$$

that is

$$u^2 + (2 - (2+4j)^2)u + 1 - y = 0, \tag{3.3}$$

where $u = i^2$. Note that for a given $u \in S^*$, i and $-i$ give the same block B_i. Thus a solution of u can provide at most one block B_i satisfying our conditions.

In what follows, we analyse the number of solutions for u.

Equation (3.3) has some solutions for u if and only if

$$\delta := (2 - (2+4j)^2)^2 - 4(1-y) \in S,$$

that is

$$y \in S + t, \text{ where } t = -4j(j+1) \in S.$$

Now $y \in (S+t) \cap N$. First suppose that $1 - y \in N$. Then $y \in (S+t) \cap N \cap (1+S)$. By Lemma 3.3, we have at most $\lceil \frac{1}{8}(q + 11 + 2\sqrt{q}) \rceil + 1$ choices for y, and then for u as well.

Secondly, suppose that $1-y \in S$. Then $y \in (S+t)\cap N \cap (1+N)$. By Lemma 3.3, we have at most $\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 1$ choices for y. Since every y may give two solutions for u, we have at most $2\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 2$ solutions for u.

In summary, we have at most

$$\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 2\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 3$$

blocks B_i such that $d(B_0, B_i) = 1$. Now

$$\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 2\lceil \frac{1}{8}(q+11+2\sqrt{q})\rceil + 3 \leq \frac{q-7}{2},$$

provided $q > 169$. In other words, if $q > 169$ there exists some i such that $d(B_0, B_i) = 2$. For $7 \leq q \leq 169$, only the primes $19, 43, 67$ and 163 satisfy $\frac{q(q+1)}{2} = 2rs$. For these primes, we can get a Hamilton cycle by Magma.

Step 3: Show the existence of a H-cycle.

Let us come back to the proof of the lemma. Let $Y_1 = Y[\mathbf{B}]$, the subgraph of Y induced by $\mathbf{B}$. Then Y_1 is a Cayley graph on $\mathbb{Z}_{\frac{q-1}{2}}$. Since the valency of X is $\frac{q-1}{2}$, $d(B_1, B_\infty) = 1$, and $d(B_1, B_i) \leq 2$, it follows from

$$\frac{1}{2}(\frac{q-1}{2} - 1 - 2) \geq \frac{1}{3}\cdot\frac{q-1}{2}$$

that Y_1 has at most two connected components. Since $\frac{q-1}{2}$ is odd, Y_1 must be connected. Now there are double edges between B_1 and B_i for some i. By Proposition 2.3, Y_1 contains a cycle passing the edge B_1B_i, say $\cdots B_jB_1B_i\cdots$. In Y, replacing the edge B_jB_1 by the path $B_jB_\infty B_1$, we get a H-cycle, say C for Y. By Proposition 2.4, C can be lifted to a H-cycle of X. □

Lemma 3.5. *Suppose that Δ is a non self-paired suborbit of length $\frac{q-1}{2}$. Then $X(G, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G, \Delta\cup\Delta^*)$, where Δ is non self-paired and of length $\frac{q-1}{2}$. Let Y be the quotient graph induced by $\mathbf{B}\cup\{B_\infty\}$. Then by Lemma 3.1, we may set $\Delta = \{j, j+1\}^H$ and $\Delta^* = \{-j, -j-1\}^H$ where $j(j+1) \in N$, $j+1 \in S$, $j \in N$ and $j \in \mathbb{F}_q$. Then the neighborhood of $\{0, \infty\}$ is:

$$X_1(\{0, \infty\}) = \Delta\cup\Delta^* = \{\{j\theta^{-2k}, (j+1)\theta^{-2k}\}, \{(-j)\theta^{-2k}, (-j-1)\theta^{-2k}\} \mid k \in \mathbb{F}_q\}.$$

Since $|\Delta\cup\Delta^*| = q-1$ and $\langle \bar{l}\rangle$ acts regularly on $\mathbf{B}$, $d(B_\infty, B_i) = 2$ for any $i = 1, 2, \cdots, \frac{q-1}{2}$.

The lemma will be proved by the following two steps:

Step 1: $d(B_k, B_i) \in \{0, 2, 4\}$ for any $i, k = 1, 2, \cdots, \frac{q-1}{2}$.

Since $\langle \bar{l}\rangle$ is regular on $\mathbf{B}$ and $\{0, 1\} \in B_1$, we may just consider $d(B_1, B_i) = d(\{0,1\}, B_i)$ for any $i = 1, 2, 3, \cdots, \frac{q-1}{2}$. Since $\overline{u}'$ maps $\{\infty, 0\}$ to $\{0, 1\}$, we know that

$$X_1(\{0,\,1\}) = \{\Delta,\,\Delta^*\}^{\overline{u}'}$$
$$= \{\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}},\,\frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\},\,\{\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}},\,\frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}}\}\mid k\in\mathbb{F}_q\}.$$

A vertex in $X_1(\{0,\,1\})$ is contained in B_i if and only if some of the following four systems of equations has solutions:

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t,\quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t+i; \tag{3.4}$$

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t+i,\quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t; \tag{3.5}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t,\quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t+i; \tag{3.6}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t+i,\quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t. \tag{3.7}$$

Solving Equation (3.4) and Equation (3.6), we get the respective equation

$$ij(j+1)u^2 \pm (2ij+i-1)u + i = 0,$$

where $u = \theta^{-2k}$. For each of these two equations, it has solutions for u if and only if

$$\delta_1 := (2ij+i-1)^2 - 4i^2j(j+1) = i^2 - (2+4j)i + 1 \in S^*.$$

Since the product of two solutions u_1 and u_2 is $(j(j+1))^{-1}$, a non-square, we know that either $u_1 \in S^*$ or $u_2 \in S^*$ if the above equation has solutions. Therefore, there exists exactly one solution for $\theta^{-2k} = u$ if and only if $\delta_1 \in S^*$, noting that every θ^{-2k} gives a unique t, equivalently, a unique vertex in the block B_i. Totally, two systems of equations give two vertices in the B_i.

Solving Equation (3.5) and Equation (3.7), we get respective equation

$$ij(j+1)u^2 \pm (2ij+i+1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_2 := (2ij+i+1)^2 - 4i^2j(j+1) = i^2 + (2+4j)i + 1 \in S^*.$$

Similarly, there exists exactly one solution for θ^{-2k} if and only if $\delta_2 \in S^*$. Totally, two systems of equations give two vertices in the B_i.

In summary, $d(B_1,\,B_i) = 2$ if and only if $\delta_1\delta_2 \in N$; and $d(B_1,\,B_i) = 0$ or 4 provided $\delta_1\delta_2 \in S$.

Step 2: Show the existence of a H-cycle.

Let $Y_1 = Y[\mathbf{B}]$, the subgraph of Y induced by $\mathbf{B}$. Then Y_1 is a Cayley graph on $\mathbb{Z}_{\frac{q-1}{2}}$. Since the valency of X is $q-1$, $d(B_1, B_\infty) = 2$, and $d(B_1, B_i) \leq 4$, it follows from

$$\frac{1}{4}(q-1-4-2) \geq \frac{1}{3} \cdot \frac{q-1}{2}$$

that Y_1 has at most two connected components. Then, using the same arguments in Step 3 of Lemma 3.4, one may get a H-cycle of X. □

Lemma 3.6. *Suppose that Δ is a self-paired suborbit of length $q-1$. Then $X(G, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G, \Delta)$, where Δ is self-paired and of length $q-1$. Let Y be the quotient graph induced by $\mathbf{B} \cup \{B_\infty\}$. Then by Lemma 3.1, we may set $\Delta = \{j, j+1\}^H$ where $j(j+1) \in S^*$ and either $j+1 \in N$ or $j \in S^*$. Then the neighborhood of $\{0, \infty\}$ is:

$$X_1(\{0, \infty\}) = \Delta = \{\{j\theta^{-2k}, (j+1)\theta^{-2k}\}, \{(-j)\theta^{-2k}, (-j-1)\theta^{-2k}\} \mid k \in \mathbb{F}_q\}.$$

Since $|\Delta| = q-1$ and $\langle \bar{l} \rangle$ acts regularly on $\mathbf{B}$, $d(B_\infty, B_i) = 2$ for any $i = 1, 2, \cdots, \frac{q-1}{2}$.

The lemma will be proved by the following two steps:

Step 1: $d(B_m, B_i) \leq 4$ for any $i, m \in \mathbb{F}_q^*$.

Since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$ and $\{0, 1\} \in B_1$, we may just consider $d(B_1, B_i) = d(\{0, 1\}, B_i)$ for any $i \in \mathbb{F}_q^*$. Now,

$$X_1(\{0, 1\}) = \{\Delta\}^{\overline{u}'} = \{\{\frac{j\theta^{-2k}}{1+j\theta^{-2k}}, \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}}\},$$
$$\{\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}}, \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}}\} \mid k \in \mathbb{F}_q\}.$$

A vertex in $X_1(\{0, 1\})$ is contained in B_i if and only if one of the following four systems of equations has solutions:

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t+i; \tag{3.8}$$

$$\frac{j\theta^{-2k}}{1+j\theta^{-2k}} = t+i, \quad \frac{(j+1)\theta^{-2k}}{1+(j+1)\theta^{-2k}} = t; \tag{3.9}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t, \quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t+i; \tag{3.10}$$

$$\frac{(-j)\theta^{-2k}}{1+(-j)\theta^{-2k}} = t+i, \quad \frac{(-j-1)\theta^{-2k}}{1+(-j-1)\theta^{-2k}} = t. \tag{3.11}$$

Solving Equation (3.8) and Equation (3.10) we get the respective equation

$$ij(j+1)u^2 \pm (2ij+i-1)u + i = 0,$$

where $u = \theta^{-2k}$. Each of these two equations has solutions for u only if

$$\delta_1 := (2ij+i-1)^2 - 4i^2 j(j+1) = i^2 - (2+4j)i + 1 \in S.$$

(1) $\delta_1 \in S^*$: Since the product of two solutions u_1 and u_2 is $(j(j+1))^{-1}$, a square, we know that either $u_1,\, u_2 \in S^*$ or $u_1,\, u_2 \in N^*$. Therefore, there exist two solutions for $\theta^{-2k} = u$ only if $\delta_1 \in S^*$. Noting that every θ^{-2k} gives a unique t, equivalently, one vertex in the block B_i. Thus two systems of equations give two vertices in B_i.

(2) $\delta_1 = 0$: For these two equations, there is just one solution for u and it gives a unique t. Thus two systems of equations give one vertice in B_i.

Solving Equation (3.9) and Equation (3.11), we get respective equation

$$ij(j+1)u^2 \pm (2ij+i+1)u + i = 0,$$

where $u = \theta^{-2k}$. This equation has solutions for u if and only if

$$\delta_2 := (2ij+i+1)^2 - 4i^2 j(j+1) = i^2 + (2+4j)i + 1 \in S.$$

Similarly, if $\delta_2 \in S^*$, there exist exactly two solutions for θ^{-2k}. Thus two equations give two vertices in B_i. If $\delta_2 = 0$, there exists one solution for θ^{-2k}. Thus we only get one vertex in B_i.

In summary, $d(B_1,\, B_i) = 2$ if and only if $\delta_1\delta_2 \in N$; $d(B_1,\, B_i) = 0$ or 4, provided $\delta_1\delta_2 \in S^*$; and $d(B_1,\, B_i) = 1$ or 3 if and only if $\delta_1\delta_2 = 0$.

Step 2: Show the existence of a H-cycle.

Let $Y_1 = Y[\mathbf{B}]$ be the subgraph of Y induced by $\mathbf{B}$. Then Y_1 is a Cayley graph on $\mathbb{Z}_{\frac{q-1}{2}}$. Since the valency of X is $q-1$, $d(B_1,\, B_\infty) = 2$, and $d(B_1,\, B_i) \leq 4$, it follows from

$$\frac{1}{4}(q-1-4-2) \geq \frac{1}{3} \cdot \frac{q-1}{2}.$$

Then we get a H-cycle, with the same arguments as in Step 3 of Lemma 3.4. □

Lemma 3.7. *Suppose that Δ is a non self-paired suborbit of length $q-1$. Then $X(G,\, \Delta)$ is hamiltonian.*

Proof. In this case, $\Delta = \{1,\, 0\}^H$ and $\Delta^* = \{-1,\, 0\}^H$. Let $X = X(\Delta \cup \Delta^*)$ and Y the quotient graph induced by $\mathbf{B} \cup \{B_\infty\}$. Then the neighborhood of $\{0,\, \infty\}$ is:

$$X_1(\{0,\, \infty\}) = \Delta \cup \Delta^* = \{\{0,\, \theta^k\},\, \{\infty,\, \theta^k\} \mid k \in \mathbb{F}_q\}.$$

By observing the vertices of block B_∞, we get $d(B_\infty) = q-1$, and since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$, $d(B_\infty,\, B_i) = 2$ for any $i = 1,\, 2,\, \cdots,\, \frac{q-1}{2}$. Since $\overline{u}'$ maps $\{\infty,\, 0\}$ to $\{0,\, 1\}$, we know that

$$X_1(\{0,\, 1\}) = \{\Delta,\, \Delta^*\}^{\overline{u}'} = \{\{0,\, \frac{\theta^k}{1+\theta^k}\},\, \{1,\, \frac{\theta^k}{1+\theta^k}\} \mid k \in \mathbb{F}_q\}.$$

A direct computation shows $d(B_1) = 2$. Moreover, $d(B_1, B_i)$ is exactly the number of union of solutions of the following two equations:

$$\{0, \frac{\theta^k}{1+\theta^k}\} = \{t+i, t\} \quad \text{and} \quad \{1, \frac{\theta^k}{1+\theta^k}\} = \{t+i, t\}.$$

Solving them, we get four solutions:

$$\theta^k = \frac{-i}{1+i}, t = -i; \quad \theta^k = \frac{i}{1-i}, t = 0;$$

$$\theta^k = \frac{1-i}{i}, t = 1-i; \quad \theta^k = \frac{-i-1}{i}, t = 1.$$

Therefore, $d(B_1, B_i) = 4$.

Since $\langle \bar{l} \rangle$ is regular on $\mathbf{B}$, $d(B_j, B_i) = d(B_1, B_{i'})$ for some i' and $d(B_i) = d(B_1)$. Then we conclude that $d(B_i, B_j) = 4$ and $d(B_i) = 2$. Thus the graph $Y \setminus \{B_\infty\}$ is a complete graph. As before, X is hamiltonian. □

3.2 $\mathrm{soc}(G) = \mathrm{PSL}(2, q)$ and $H = \mathbb{D}_{q+1}$

Let $G = \mathrm{PSL}(2, q)$ and $H = \mathbb{D}_{q+1}$. Consider the action of G on the set $[G : H]$ of cosets of H in G, see row 2 of Table 1. Then $n = \frac{q(q-1)}{2} = 2rs$. This implies that $q \equiv 1 (\mathrm{mod}\ 4)$ and both q and $\frac{q-1}{4}$ are primes. So $r = \frac{q-1}{4}$ and $s = q$. Set $\mathbb{F}_q^* = \langle \theta \rangle$ and $\sqrt{-1} = \theta^{\frac{q-1}{4}}$. In $\mathrm{GL}(2, q)$, we set

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, u' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, l = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$t(x, y) = \begin{pmatrix} x & y\theta \\ y & x \end{pmatrix}, t'(x, y) = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t(x, y) = \sqrt{-1}\begin{pmatrix} x & -y\theta \\ y & -x \end{pmatrix}, x \neq 0.$$

Then up to conjugacy, H may be chosen as

$$H = \{\overline{t(x, y)}, \overline{t'(x, y)} \mid x^2 - y^2\theta = 1\}.$$

Consider the action of $N_G(\langle \overline{u} \rangle) = \langle \overline{u} \rangle \rtimes \langle \bar{l} \rangle$ on the set of $\langle \overline{u} \rangle$-orbits (blocks) on $[G : H]$. Then $[G : H]$ can be divided into two parts, say $\mathbf{B}$ and $\mathbf{B}'$, where

$$\mathbf{B} = \{B_1, B_2, \cdots, B_{\frac{q-1}{4}}\}, \quad \mathbf{B}' = \{B_1', B_2', \cdots, B_{\frac{q-1}{4}}'\},$$

where $B_i = \{H\overline{u^j l^i} \mid j \in \mathbb{Z}_q\}$ and $B_i' = \{H\overline{tu^j l^i} \mid j \in \mathbb{Z}_q\}$, where $1 \le i \le \frac{q-1}{4}$.

Lemma 3.8. *Suppose $q \equiv 1 (\mathrm{mod}\ 4)$. Then for G acting on $[G : H]$,*

(1) *there are $\frac{q-3}{2}$ suborbits of length $\frac{q+1}{2}$, while $\frac{q-1}{4}$ of them $\{H\overline{l^i t}H \mid 1 \le i \le \frac{q-1}{4}\}$ are self-paired and $\frac{q-5}{4}$ of them $\{H\overline{l^i}H \mid 1 \le i \le \frac{q-1}{4}\}$ are non-self-paired suborbits;*

(2) *there are $\frac{q-1}{4}$ suborbits of length $q+1$, with the form $H\overline{u^i}H$, where $i^2 \in S^* \cap (4\theta + N)$. All of them are self-paired.*

Proof. Since $q+1 \equiv 3 (\mathrm{mod}\ 4)$, for any $g \in G$, $H \cap H^g$ is either $\mathbb{Z}_2$ or 1, so every suborbit is of length either $\frac{q+1}{2}$ or $q+1$.

(1) $|\Delta| = \frac{q+1}{2}$

Let $\Delta = HgH$ be a suborbit of length $\frac{q+1}{2}$. Then $H^g \cap H \cong Z_2$ and so α^g is an involution of H, where $\alpha = \overline{l^{\frac{q-1}{4}}} \in H$. Then $\alpha^g = \alpha^h$ for some $h \in H$, and so $gh^{-1} \in C_G(\alpha) = \langle \bar{l}, \bar{t} \rangle$. Since $HgH = Hgh^{-1}H$, we may choose $h = 1$ so that $g \in C_G(\alpha)$. Set $g = \overline{l^i}$ or $\overline{l^i t}$ for some i. Moreover, direct computations show that for any two distinct elements $g_1, g_2 \in C_G(\alpha) = \langle \bar{l}, \bar{t} \rangle$, $Hg_1H = Hg_2H$ if and only if $g_1 = g_2\alpha$. Therefore, we have $\frac{q-1}{2}$ suborbits of length $\frac{q+1}{2}$. In particular, $HgH = Hg^{-1}H$ if and only if either $g^2 = 1$ or $g^{-1} = g\alpha$, where the second case gives $g \in H$. So we get $\frac{q-1}{4}$ self-paired suborbits HgH where g is non-central involution in $C_G(\alpha)$, noting $Hg\alpha H = HgH$. So the remaining $\frac{q-5}{4}$ suborbits of length $\frac{q+1}{2}$ are non self-paired.

(2) $|\Delta| = q+1$

Let first consider the suborbits $D = H\overline{u^i}H$ where $i \in \mathbb{Z}_q^*$. From the arguments in (1), we know that $|\Delta| = q+1$. Since $H\overline{u^i}H = H\alpha\overline{u^i}\alpha H = H\overline{u^{-i}}H$, Δ is self-paired. Set $g = \overline{u^i}$.

Suppose that $H^g \cap H = Z_2$, that is

$$\overline{u^{-i}t'(x_1, y_1)u^i} \in H,$$

which implies $2x_1 - iy_1 = 0$. Insetting it in $x_1^2 - y_1^2\theta = 1$, we get

$$i^2 = 4\theta + 4x_1^{-2} \in S^* \cap (4\theta + S^*).$$

Therefore, Δ is of length $q+1$ if and only if $i^2 \in S^* \cap (4\theta + N)$. By Proposition 2.1, $|S^* \cap (4\theta + N)| = \frac{q-1}{4}$. Check that $H\overline{u^i}H = H\overline{u^j}H$ if and only if $i = \pm j$. Therefore, we get $\frac{q-1}{4}$ suborbits of length $q+1$.

Since $1 + \frac{q-3}{2}\frac{q+1}{2} + \frac{q-1}{4}(q+1) = \frac{q(q-1)}{2} = |[G : H]|$, we already find all suborbits. □

In what follows we deal with all cases of suborbits Δ in Lemma 3.8, separately.

Lemma 3.9. *Suppose that Δ is a self-paired suborbit of length $\frac{q+1}{2}$. Then $X(G, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G, \Delta)$, where Δ is self-paired and of length $\frac{q+1}{2}$. From the last lemma, $\Delta = H\overline{l^k t}H$ for some k. Note $\frac{q-1}{4} = r$ is a prime, the two smallest values for q are 13 and 29. One may find a H-cycle by Magma for $q = 13$ and 29. So let $q \neq 13, 29$. First we give a remark.

Remark: Suppose we may get two facts: ① for any $B' \in \mathbf{B}'$, $d(H, B') = 0, 2$ or 4; ② $d(H, \cup_{B' \in \mathbf{B}'} B') \geq 5$. Then H is adjacent to at least two blocks B'_i, B'_j in $\mathbf{B}$ such that $d(H, B'_i) = 2$ or 4. Let Y be the block graph. Then Y is a bipartite graph of order $2r$, where $r = \frac{q-1}{4}$ is a prime. Note that $H \in B_r$. Since $\langle \bar{l} \rangle / \langle \bar{l}^r \rangle$ acts regularly on both $\mathbf{B}$ and $\mathbf{B}'$, we may set $B_i'^d = B'_j$ for some $d \in \langle \bar{l} \rangle / \langle \bar{l}^r \rangle$. Then we get a H-cycle of Y:

$$B'_i, B_r, B_i'^d, B_r^d, B_i'^{d^2}, \cdots, B_r^{d^{r-1}}, B'_i.$$

Then by Proposition 2.4, we may find a H-cycle for $X(G, \Delta)$.

Now we continue to prove the lemma. Clearly, the neighborhood of H is:

$$X_1(H) = \Delta = H\overline{l^k t}H = \{H\overline{l^k tt(x_1, y_1)} \mid x_1^2 - y_1^2\theta = 1\}.$$

The vertex $H\overline{l^k tt(x_1, y_1)}$ is contained in B'_i if and only if

$$H\overline{l^k tt(x_1, y_1)} = H\overline{tu^j l^i}, \text{ for some } j,$$

if and only if

$$\overline{l^k tt(x_1, y_1)}(\overline{tu^j l^i})^{-1} \in H,$$

if and only if one of the following two systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\varepsilon, \eta) = (1, -1)$ and $(-1, 1)$:

$$\begin{cases} y_1 j\theta^{2k} & = x_1(\theta^{2k+2i} - \varepsilon), \\ y_1(\theta^{2i+2} + \eta\theta^{2k}) & = x_1\theta j, \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.12}$$

Every such system has the same solutions with

$$\begin{cases} y_1^2 = \frac{\theta^{2k+1}}{\eta\theta^{4k}+\theta^2\varepsilon}\theta^{2i} - \frac{\varepsilon\theta}{\eta\theta^{4k}+\theta^2\varepsilon}, & (i) \\ y_1^2 = \theta^{-1}x_1^2 - \theta^{-1}, & (ii) \\ j \ = (\theta^{2i} - \varepsilon\theta^{-2k})\frac{x_1}{y_1}. & (iii) \end{cases} \tag{3.13}$$

From (iii), we know that given a solution for x_1^2, y_1^2 and i, we have two values of j, that is $\pm j$. Then the possible values for $d(H, B'_i)$ is $0, 2$ or 4, noting we have two choices for (ε, η), showing fact ①.

Set $b = -\theta^{-1}$, $a_1 = \frac{\theta^{2k+1}}{\eta\theta^{4k}+\theta^2\varepsilon}$ and $a_2 = -\frac{\varepsilon\theta}{\eta\theta^{4k}+\theta^2\varepsilon}$. Then $a_1, a_2 \neq 0$ and $a_2 \neq b$. From (i) and (ii), we get that either

$$y_1^2 \in S^* \cap (S^* + a_2) \cap (N + b) \text{ if } a_1 \in S^* \quad \text{or} \quad y_1^2 \in S^* \cap (N + a_2) \cap (N + b) \text{ if } a_1 \in N.$$

By using Lemma 3.3, we get that the number of solutions for y_1^2 is at least $\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$, which implies that the number of solutions for j, i, x_1, y_1 is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$, for given (ε, η). In other words, $d(H, \cup_{B' \in \mathbf{B}'} B')$ is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor$. Moreover, $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q}) \rfloor \geq 5$, showing fact ②. □

Lemma 3.10. *Suppose that* Δ *is a non self-paired suborbit of length* $\frac{q+1}{2}$*. Then* $X(G, \Delta \cup \Delta^*)$ *is hamiltonian.*

Proof. Let $X = X(G, \Delta \cup \Delta^*)$, where Δ is non self-paired and of length $\frac{q+1}{2}$. From Lemma 3.8, $\Delta = H\overline{l^k}H$ and $\Delta^* = H\overline{l^{-k}}H$ for some integer k. Note $\frac{q-1}{4} = r$ is a prime, the three smallest values for q are 13, 29 and 53. One may find a H-cycle by Magma for $q = 13, 29$ and 53. So let $q \neq 13, 29, 53$.

From the remark in last lemma, it suffices to show two facts: (i) for any $B' \in \mathbf{B}'$, $d(H, B') = 0, 2, 4, 6$ or 8; (ii) $d(H, \cup_{B' \in \mathbf{B}'} B') \geq 9$.

Check that the neighborhood of H is:

$$X_1(H) = \Delta \cup \Delta^* = \{H\overline{l^k t(x_1,\, y_1)},\, H\overline{l^{-k} t(x_1,\, y_1)} \mid x_1^2 - y_1^2\theta = 1\}.$$

The vertex $H\overline{l^k t(x_1,\, y_1)}$ and $H\overline{l^{-k} t(x_1,\, y_1)}$ are contained in B_i' if and only if either

$$H\overline{l^k t(x_1,\, y_1)} = H\overline{tu^j l^i},\ \text{or}\ H\overline{l^{-k} t(x_1,\, y_1)} = H\overline{tu^j l^i},\ \text{for some}\ j$$

if and only if either

$$\overline{l^k t(x_1,\, y_1)}(\overline{tu^j l^i})^{-1} \in H,\ \text{or}\ \overline{l^{-k} t(x_1,\, y_1)}(\overline{tu^j l^i})^{-1} \in H$$

if and only if one of the following four systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\varepsilon,\, \eta) = (1,\, -1),\, (1,\, 1),\, (-1,\, -1)$ or $(-1,\, 1)$:

$$\begin{cases} y_1(\theta^{i+\epsilon k+1} - \eta\theta^{-i-\varepsilon k}) & = x_1 j\theta^{\varepsilon k - i}, \\ y_1 j\theta^{-\varepsilon k - i+1} & = x_1(\theta^{i-\epsilon k+1} - \eta\theta^{-i+\varepsilon k}), \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.14}$$

Every such system has the same solutions with

$$\begin{cases} y_1^2 = \frac{\theta^{2i}\theta}{\eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}} - \frac{\eta\theta^{2\varepsilon k}}{\eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}}, & (i) \\ y_1^2 = \theta^{-1}x_1^2 - \theta^{-1}, & (ii) \\ j\ = \frac{\theta^{i+\varepsilon k+1} - \eta\theta^{-i-\varepsilon k}}{\theta^{\varepsilon k - i}} \frac{y_1}{x_1}. & (iii) \end{cases} \tag{3.15}$$

From (iii), we know that given a solution for x_1^2, y_1^2 and i, we have two values of j, that is $\pm j$. Then the possible values for $d(H,\, B_i')$ is $0, 2, 4, 6$ or 8, noting we have four choices for $(\varepsilon,\, \eta)$, showing fact (i).

Set $b = -\theta^{-1}$, $a_1 = \eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}$ and $a_2 = -\frac{\eta\theta^{2\varepsilon k}}{\eta\theta\theta^{2\varepsilon k} - \eta\theta\theta^{-2\varepsilon k}}$. Then $a_1, a_2 \neq 0$ and $a_2 \neq b$. From (i) and (ii), we get that either

$$y_1^2 \in S^* \cap (N + a_2) \cap (N + b)\ \text{if}\ a_1 \in S^* \quad \text{or} \quad y_1^2 \in S^* \cap (S^* + a_2) \cap (N + b)\ \text{if}\ a_1 \in N.$$

By using Lemma 3.3, we get that the number of solutions for y_1^2 is at least $\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q})\rfloor$, which implies that the number of solutions for j, i, x_1, y_1 is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q})\rfloor$, for given $(\varepsilon,\, \eta)$. In other words, $d(H,\, \cup_{B' \in \mathbf{B}'} B')$ is at least $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q})\rfloor$. Moreover, $2\lfloor \frac{1}{8}(q - 11 - 2\sqrt{q})\rfloor \geq 9$, showing fact (ii). □

Lemma 3.11. *Suppose that Δ is a self-paired suborbit of length $q + 1$. Then $X(G,\, \Delta)$ is hamiltonian.*

Proof. Let $X = X(G,\, \Delta)$, where Δ is self-paired and of length $q + 1$. From Lemma 3.8, $\Delta = H\overline{u^k}H$ for some integer k. Note $\frac{q-1}{4} = r$ is a prime.

If we may get two facts: (i) for any $B' \in \mathbf{B}'$, $d(H,\, B') = 0, 2$ or 4; (ii) for any $B' \in \mathbf{B}'$, $d(H,\, B') = 0, 2$ or 4, then every vertex in block graph has the valency at least $\frac{(q+1)-2}{4} = \frac{q-1}{4} = \frac{1}{2}\frac{q-1}{2}$. So Y contains a H-cycle. Since $d(B_i,\, B_j')$ is even, this cycle can lift a H-cycle for $X(G,\, \Delta)$ by Proposition 2.4.

In fact, check that the neighborhood of H is:

$$X_1(H) = \Delta = H\overline{u^k}H = \{H\overline{u^k t(x_1,\, y_1)}, H\overline{u^k t'(x_1,\, y_1)} \mid x_1^2 - y_1^2\theta = 1\}.$$

By observing the neighbor, one can see these neighbors contained in $B_{\frac{q-1}{4}}$ are just $H\overline{u^k}$ and $H\overline{u^{-k}}$, which implies $d(B_{\frac{q-1}{4}}) = 2$. The vertex $H\overline{u^k t(x_1, y_1)}$ and $H\overline{u^k t'(x_1, y_1)}$ are contained in B_i if and only if either:

$$H\overline{u^k t(x_1, y_1)} = H\overline{u^j l^i}, \text{ or } H\overline{u^k t'(x_1, y_1)} = H\overline{u^j l^i}$$

if and only if either:

$$\overline{u^k t(x_1, y_1)}(\overline{u^j l^i})^{-1} \in H, \text{ or } \overline{u^k t'(x_1, y_1)}(\overline{u^j l^i})^{-1} \in H$$

if and only if one of the following systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\epsilon, \eta, \gamma, \delta) = (-1, 1, -1, 1)$, $(1, -1, 1, -1)$, $(-1, -1, -1, -1)$ or $(1, 1, 1, 1)$:

$$\begin{cases} \epsilon\theta^{-i} y_1 j & = (x_1 + ky_1)\theta^{-i} - \eta x_1 \theta^i, \\ \gamma(x_1 + ky_1)\theta^{-i} j & = y_1\theta^{-i}\theta - \delta\theta^i(y_1\theta + kx_1), \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.16}$$

This system has the same solutions with

$$j = \frac{(\theta^{-i} - \eta\theta^i)x_1}{\varepsilon\theta^{-i} y_1} + k\varepsilon^{-1} \text{ where } \delta\varepsilon\theta^{2i} = \gamma(k^2 y_1^2 + 2kx_1 y_1 + 1).$$

Calculating the equation $\delta\varepsilon\theta^{2i} = \gamma(k^2 y_1^2 + 2kx_1 y_1 + 1)$ we could get

$$(4k^2\theta - k^4)u^2 + (2k^2 + 2\delta\varepsilon\gamma\theta^{2i}k^2)u - (\delta\varepsilon\theta^{2i} - \gamma)^2 = 0,$$

where $u = y_1^2$. Since the product of the two solutions is $\frac{-(\delta\varepsilon\theta^{2i}-\gamma)^2}{4k^2\theta - k^4}$, a non-square (as $4\theta - k^2 \in N$), there exists at most one solution for $u = y_1^2$. It is easy to see that there are two solutions for j. Since there are just two different equations for $\delta\varepsilon\theta^{2i} = \gamma(k^2 y_1^2 + 2kx_1 y_1 + 1)$, there are at most 4 solutions for j, that is $d(H, B_i) = 0$, 2 or 4, showing fact (i).

The vertex $H\overline{u^k t(x_1, y_1)}$ and $H\overline{u^k t'(x_1, y_1)}$ are contained in B_i' if and only if either

$$H\overline{u^k t(x_1, y_1)} = H\overline{tu^j l^i} \text{ or } H\overline{u^k t'(x_1, y_1)} = H\overline{tu^j l^i}$$

if and only if either

$$\overline{u^k t(x_1, y_1)}(\overline{tu^j l^i})^{-1} \in H \text{ or } \overline{u^k t'(x_1, y_1)}(\overline{tu^j l^i})^{-1} \in H$$

if and only if one of the following systems of equations with unknowns j, i, x_1 and y_1 has solutions corresponding to $(\epsilon, \eta, \gamma, \delta) = (1, -1, 1, -1)$, $(1, 1, 1, 1)$, $(-1, -1, -1, -1)$ or $(-1, 1, -1, 1)$:

$$\begin{cases} -(x_1 + ky_1)\theta^{-i} j & = \eta y_1 \theta^{-i} - \epsilon(y_1\theta + kx_1)\theta^{i-1} \\ -y_1\theta^{-i}\theta j & = \delta(x_1 + ky_1)\theta^{-i} - \gamma x_1 \theta^{i-1}\theta \\ x_1^2 - y_1^2\theta & = 1. \end{cases} \tag{3.17}$$

This system has the same solutions with

$$j = \frac{\delta\theta^{-i} - \gamma\theta^{i}\theta}{-\theta^{-i}\theta}\frac{x_1}{y_1} - \frac{k\delta}{\theta} \text{ where } \gamma\theta^{2i}\theta = \delta(k^2y_1^2 + 2kx_1y_1 + 1).$$

Calculating the equation $\gamma\theta^{2i}\theta = \delta(k^2y_1^2 + 2kx_1y_1 + 1)$ we could get

$$(4k^2\theta - k^4)u^2 + (2k^2 + 2\delta\gamma k^2\theta^{2i}\theta)u - (-\gamma\theta^{2i}\theta + \delta)^2 = 0,$$

where $u = y_1^2$. Since the product of the two solutions is $\frac{-(-\gamma\theta^{2i}\theta+\delta)^2}{4k^2\theta-k^4}$, a non-square (as $4\theta - k^2 \in N$), there exists at most one solution for $u = y_1^2$ and it is easy to see there are two solutions for j. Since there are just two different equations for $\gamma\theta^{2i} = \delta(k^2y_1^2 + 2kx_1y_1 + 1)$, there are at most 4 solutions for j, that is $d(B_{\frac{q-1}{4}}, B_i') = 0,\ 2$ or 4, showing fact (ii). □

3.3 Groups in Table 2

In this subsection, we shall deal with the groups in Table 2, separately.

Lemma 3.12. *Let G be one of groups in rows 1 and 2 of Table 2. Then every orbital graph of G contains a Hamilton cycle.*

Proof. Let $T = \mathrm{PSL}(m, q)$ where $m = 4$ or 5. It suffices to consider the group T. We shall deal with two cases: $m = 4$ and $m = 5$, separately.

Case 1: $m = 4$.

Let Ω be the set of 2-dim. subspaces of a space V of dimension 4. Then $n = \frac{(q^4-1)(q^3-1)}{(q-1)(q^2-1)} = (q^2 + q + 1)(q^2 + 1)$, where $s = q^2 + q + 1$ and $r = \frac{q^2+1}{2}$ are two primes. Consider a subspace W_0 of dimension $d(W_0) = 2$. Then T has two nontrivial suborbits relative to W_0:

$$\Delta_1 = \{W \in \Omega \mid d(W \cap W_0) = 1\} \quad \text{and} \quad \Delta_2 = \{W \in \Omega \mid d(W \cap W_0) = 0\},$$

where $r_1 := |\Delta_1| = \frac{q^4-q}{q^2-q} = \frac{q^3-1}{q-1}$ and $r_2 := |\Delta_2| = n - 1 - r_1$. Since $r_2 \geq \frac{n}{2}$, the corresponding orbital graph $\Gamma(T, \Delta_2)$ has a H-cycle.

Now we are considering $X(T, \Delta_1)$. Take a projective point $\langle\alpha\rangle$ and extend it into a base $\alpha, \alpha_1, \alpha_2, \alpha_3$ of V. Let $\Sigma(\alpha)$ be the set of all 2-dim. subspaces containing α. Then $|\Sigma(\alpha)| = q^2 + q + 1$. Since $\langle\alpha_1, \alpha_2, \alpha_3\rangle$ contains exactly $q^2 + q + 1$ points and for any two distinct points β, β' in $\langle\alpha_1, \alpha_2, \alpha_3\rangle$, $\langle\alpha, \beta\rangle \neq \langle\alpha, \beta'\rangle$, one may see

$$\Sigma(\alpha) = \{\langle\alpha, \beta\rangle \mid \beta \in \langle\alpha_1, \alpha_2, \alpha_3\rangle\}.$$

Let $\langle h\rangle$ be the Singer subgroup of $\mathrm{PSL}(3, q)$ and $\beta \in \langle\alpha_1, \alpha_2, \alpha_3\rangle$. Since $s = q^2 + q + 1$ is a prime, $\langle\beta\rangle, \langle\beta^h\rangle, \langle\beta^{h^2}\rangle, \cdots, \langle\beta^{h^{s-1}}\rangle$ are all the projective points of $\langle\alpha_1, \alpha_2, \alpha_3\rangle$. Denote $\beta^{h^i} = \beta_i$. Since the subgraph induced by $\Sigma(\alpha)$ is a complete graph, we may consider a H-cycle of the subgraph, say

$$\langle\alpha, \beta_0\rangle, \langle\alpha, \beta_1\rangle, \langle\alpha, \beta_2\rangle, \cdots, \langle\alpha, \beta_{s-2}\rangle, \langle\alpha, \beta_{s-1}\rangle, \langle\alpha, \beta_0\rangle,$$

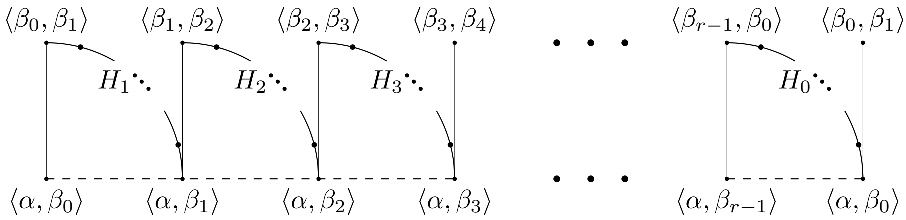

Figure 1:

where $\beta_i \in \langle \alpha_1,\, \alpha_2,\, \alpha_3\rangle$ and $s = q^2 + q + 1$.

Set

$$A = \{\langle \beta_i,\, \beta_{i+1}\rangle,\, \langle \beta_{s-1},\, \beta_0\rangle | i = 0,\, 1,\, \cdots,\, s-2\} = \{\langle \beta,\, \beta^h\rangle^{h^i} | i = 0,\, 1,\, \cdots,\, s-1\},$$

$$X_i = \Sigma(\beta_i) \setminus (\bigcup_{j=1}^{i-1} \Sigma(\beta_j) \bigcup A),\ i \in \{1,\, 2,\, \cdots,\, s-1\},$$

$$X_0 = \Sigma(\beta_0) \setminus (\bigcup_{j=1}^{s-1} \Sigma(\beta_j) \bigcup A).$$

Since every 2-subspace $\langle \eta, \gamma\rangle$ can be expressed as $\langle \eta,\, \gamma - \frac{b_0}{a_0}\eta\rangle$, where $\eta = a_0\alpha + a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$ and $\gamma = b_0\alpha + b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3$, every 2-subspace of V is contained in $(\bigcup_{i=0}^{s-1} X_i) \bigcup A$. Moreover, from the definition, we know that $X_0,\, X_1,\, \cdots,\, X_{s-1},\, A$ are mutually disjoint.

Now we are ready to find a H-cycle for $X(T,\, \Delta_1)$. For $i = 0,\, 1,\, \cdots,\, r-2$, consider a H-path H_{i+1} in the subgraph induced by $X_{i+1} \bigcup \{\langle \beta_i,\, \beta_{i+1}\rangle\}$ with the starting vertex $\langle \beta_i,\, \beta_{i+1}\rangle$ and the ending vertex $\langle \alpha,\, \beta_{i+1}\rangle$. Consider H_0 in the subgraph induced by $X_0 \bigcup \{\langle \beta_{s-1},\, \beta_0\rangle\}$ with the starting vertex $\langle \beta_{s-1},\, \beta_0\rangle$ and the ending vertex $\langle \alpha,\, \beta_0\rangle$. Then by replacing every arc $(\langle \alpha,\, \beta_i\rangle, \langle \alpha,\, \beta_{i+1}\rangle)$ by the path $(\langle \alpha,\, \beta_i\rangle,\, H_{i+1})$ and the arc $(\langle \alpha,\, \beta_{s-1}\rangle,\, \langle \alpha,\, \beta_0\rangle)$ by the path $(\langle \alpha,\, \beta_{s-1}\rangle,\, H_0)$, we get a cycle:

$$\langle \alpha,\, \beta_0\rangle,\, H_1,\, H_2,\, \cdots,\, H_{s-1},\, H_0,$$

which is clearly a H-cycle of $X(T,\, \Delta_1)$, as shown in Figure 1.

Case 2: $m = 5$.

Let Ω be the set of 2-dim. subspaces of V. Then

$$n = |\Omega| = \frac{(q^5-1)(q^4-1)}{(q-1)(q^2-1)} = (q^4 + \cdots + 1)(q^2+1) = 2rs.$$

Then $s = q^4 + \cdots + 1$ is a prime and $r = \frac{q^2+1}{2}$ are two prime. Let $S = \langle h\rangle$ be a Singer subgroup of PSL$(5, q)$, where $|S| = s$. Take a projective point α. Then $\alpha,\, \alpha^h,\, \cdots,\, \alpha^{h^{s-1}}$ are all the projective points. Set $W_i = \langle \alpha,\, \alpha^{h^i}\rangle$ where $i = 1,\, 2,\, \cdots,\, s-1$. Then G has two nontrivial suborbits relative to W_1:

$$\Delta_1 = \{W \in \Omega | d(W \cap W_1) = 1\} \quad \text{and} \quad \Delta_2 = \{W \in \Omega | d(W \cap W_1) = 0\},$$

where

$$r_1 := |\Delta_1| = (\tfrac{q^4}{q-1} - 1)(q+1) = q(q+1)(q^2+q+1),$$
$$r_2 := |\Delta_2| = \tfrac{(q^5-q^2)(q^5-q^3)}{(q^2-1)(q^2-q)} = q^4(q^2+q+1).$$

Since $r_2 \geq \frac{n}{2}$, the corresponding orbital graph $X(T, \Delta_2)$ has a H-cycle.

Now we are considering $X(T, \Delta_1)$. Let S_i be the path

$$W_i,\, W_i^{h^i},\, W_i^{h^{2i}},\, W_i^{h^{3i}},\, \cdots,\, W_i^{h^{(s-1)i}}.$$

Since $\langle h^i \rangle$ acts nontrivially on W_i and it is of order a prime s, $\langle h^i \rangle$ moves W_i. Since every 2-subspace must be contained in some clique and either $|S_i \cap S_j| = 0$ or $S_i = S_j$ for any two distinct cliques S_i and S_j, we could pick up q^2+1 distinct cliques which cover all 2-dim. subspaces, denoted by $W_{\mu_1}, W_{\mu_2}, \cdots, W_{\mu_{q^2+1}}$. Then we can get a H-cycle of $X(T, \Delta_1)$: $W_{\mu_1}, W_{\mu_1}^{h^{\mu_1}}, W_{\mu_1}^{h^{2\mu_1}}, \cdots, W_{\mu_1}^{h^{(s-1)\mu_1}}, W_{\mu_2}, W_{\mu_2}^{h^{\mu_2}}, W_{\mu_2}^{h^{2\mu_2}}, W_{\mu_2}^{h^{3\mu_2}}, \cdots, W_{\mu_{q^2+1}}^{h^{(s-1)\mu_{q^2+1}}}, W_{\mu_1}$. □

Lemma 3.13. *Every orbital graph of* $G = \mathrm{P\Omega}^-(2m, q)$ *in row* 3 *of* Table 2 *is hamiltonian.*

Proof. Let $G = \mathrm{P\Omega}^-(2m, q)$ act on n totally singular (t.s.) 1-spaces, where $n = \frac{(q^m+1)(q^{m-1}-1)}{q-1} = 2rs$ and $m = 2^{2^l}$. Then $m-1$ is a prime. Since $m - 1 = (2^{2^{l-1}} - 1)(2^{2^{l-1}}+1)$, we get $2^{2^{l-1}} - 1 = 1$, which implies $l = 1$ and then $m = 4$. Now $r =: \frac{q^3-1}{q-1}$ is a prime. Let Ω be the set of all t.s.1-spaces. Recall that $\mathrm{SO}^-(8, q) \leq \mathrm{GL}(8, q)$ and $|\mathrm{GL}(8, q)| = q^{28}\Pi_{i=1}^{8}(q^i - 1)$. To describe $\mathrm{SO}^-(8, q)$, take a symmetric bilinear form, given by the following matrix:

$$J = \begin{pmatrix} 0 & E_3 & 0 \\ E_3 & 0 & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix}, \quad t \in N.$$

Let $\langle A \rangle$ be a Singer subgroup of $\mathrm{GL}(3, q)$, $C = A^{q-1}$ and $D = (C^{-1})'$, where C' denotes the transpose of C. Set $B = C \bigoplus (C')^{-1} \bigoplus E_2$, the block diagonal matrix. Then we have $BJB' = J$, which means $B \in \mathrm{SO}^-(8, q)$. Since $\overline{B}$ is of prime order, $\overline{B} \in (\mathrm{PSO}^-(8, q))' = \mathrm{P\Omega}^-(8, q)$. Set $S = \langle \overline{B} \rangle$ and $\alpha = (1, 0, \cdots, 0)$. Then there are two nontrivial suborbits for the action of $G_{\langle \alpha \rangle}$ relative to $\langle \alpha \rangle$, see [22]:

$$\Delta_1 = \{\langle \beta \rangle \in \Omega \setminus \{\langle a \rangle\} \mid (\alpha, \beta) = 0\} \quad \text{and} \quad \Delta_2 = \{\langle \beta \rangle \in \Omega \setminus \{\langle a \rangle\} \mid (\alpha, \beta) \neq 0\},$$

where $|\Delta_1| = q^5 + q^4 + q^2 + q$ and $|\Delta_2| = q^6$. Since $|\Delta_2| \geq \frac{1}{2}n$, we only need to consider $X(G, \Delta_1)$.

Noting that S acts semiregularly on Ω, we consider the block graph $\overline{X}$ induced by S-orbits, where $V(\overline{X}) = q^4 + 1$. For any $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Omega$, where $\gamma_1 = (c_1, c_2, c_3)$, $\gamma_2 = (c_4, c_5, c_6)$ and $\gamma_3 = (c_7, c_8)$, we have $\gamma \overline{B}^i J\alpha' = 0$ if and only if $\gamma_2 D^i \alpha' = 0$, that is $\gamma_2 D^i = (0, c_5', c_6')$ for some c_5', c_6'. Since $\langle C \rangle$ (and so $\langle D \rangle$) is regular on nonzero 1-spaces, we know that α has $q+1$ (resp. q^2+q) neighbors in the block γ^S if $\gamma \notin \alpha^S$ (resp. $\gamma \in \alpha^S$). From $((q^5 + q^4 + q^2 + q) - (q^2 + q))/(q + 1) = q^4$ we know that $\overline{X}$ is a complete graph. By Propsosition 2.4, $X(G, \Delta_1)$ is hamiltonian. □

Lemma 3.14. *Every orbital graph of* $G = \mathrm{P\Omega}^+(2m, q)$ *in row* 4 *of* Table 2 *is hamiltonian.*

Proof. Let $G = \mathrm{P}\Omega^+(2m, q)$ act on n totally singular 1-spaces, where the degree $n = \frac{(q^m-1)(q^{m-1}+1)}{q-1} = 2rs$, $m = 2^{2^l} + 1$, and $s = \frac{q^m-1}{q-1}$ and $r = \frac{q^{m-1}+1}{2}$ are primes. Let Ω be the set of all totally singular 1-spaces. Recall that $\mathrm{SO}^+(2m, q) \le \mathrm{GL}(2m, q)$. To describe $\mathrm{SO}^+(2m, q)$, take a symmetric bilinear form, given by the following matrix:

$$J = \begin{pmatrix} 0 & E_m \\ E_m & 0 \end{pmatrix}.$$

Let $\langle A\rangle$ be a Singer subgroup of $\mathrm{GL}(m, q)$, $C = A^{q-1}$ and $D = (C^{-1})'$, where C' denotes the transpose of C. Set $B = C \bigoplus (C')^{-1}$. Then we have $BJB' = J$, which means $B \in \mathrm{SO}^+(2m, q)$. Since B is of prime order, $\overline{B} \in (\mathrm{PSO}^+(m, q))' = \mathrm{P}\Omega^+(m, q)$. Set $S = \langle \overline{B}\rangle$ and $\alpha = (1, 0, \cdots, 0)$. Then there are two nontrivial suborbits for the action of $G_{\langle\alpha\rangle}$ relative to $\langle\alpha\rangle$, see By [22]:

$$\Delta_1 = \{\langle\beta\rangle \in \Omega \setminus \{\langle a\rangle\} \bigm| (\alpha, \beta) = 0\} \quad \text{and} \quad \Delta_2 = \{\langle\beta\rangle \in \Omega \setminus \{\langle a\rangle\} \bigm| (\alpha, \beta) \neq 0\},$$

where $|\Delta_1| = \frac{(q^{m-1}+q)(q^{m-1}-1)}{q-1}$ and $|\Delta_2| = q^{2m-2}$. Since $|\Delta_2| \ge \frac{1}{2}n$, we only need to consider $X(G, \Delta_1)$.

Noting that S acts semiregularly on Ω, we consider the block graph $\overline{X}$ induced by S-orbits, where $V(\overline{X}) = q^{m-1} + 1$. For any $\gamma = (\gamma_1, \gamma_2) \in \Omega$, we have $\gamma \overline{s}^i J\alpha' = 0$ if and only if $\gamma_2 D^i \alpha' = 0$, which implies that the first coordinate of $\gamma_2 D^i$ is 0. Since $\langle C\rangle$ (and so $\langle D\rangle$) is regular on nonzero 1-spaces, we know that α has $\frac{q^{m-1}-1}{q-1}$ (resp. $\frac{q^m-1}{q-1} - 1$) neighbors in the block γ^S if $\gamma \notin \alpha^S$ (resp. $\gamma \in \alpha^S$). From $(\frac{(q^{m-1}+q)(q^{m-1}-1)}{q-1} - (\frac{q^m-1}{q-1} - 1))/(\frac{q^{m-1}-1}{q-1}) = q^{m-1}$ we know that $\overline{X}$ is a complete graph. By Proposition 2.4, $X(G, \Delta_1)$ is hamiltonian. □

Lemma 3.15. *Vertex-transitive graphs arising from the action of* A_c *on* 2*-subsets given in row* 6 *of* Table 2 *are hamiltonian.*

Proof. Let $\Omega = \{\alpha_1, \alpha_2, \cdots, \alpha_c\}$, where $c \ge 5$. Then we only have the following two orbital graphs:

(1) Two subsets are adjacent if and only if they intersect at a single point. In this case, the orbital graph is the Johnson graph $J(c, 2)$. Then we may get a H-cycle as the following way:

first pick up a cycle of c vertices, say $\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}, \{\alpha_3, \alpha_4\}, \cdots, \{\alpha_{c-1}, \alpha_c\}, \{\alpha_c, \alpha_1\}, \{\alpha_1, \alpha_2\}$; then

replace the edge $\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}$ by any H-path of all 2-subsets containing α_2, with the starting vertex $\{\alpha_1, \alpha_2\}$ and the ending vertex $\{\alpha_2, \alpha_3\}$; then

replace the edge $\{\alpha_2, \alpha_3\}, \{\alpha_3, \alpha_4\}$ by any H-path of all 2-subsets containing α_3, with the starting vertex $\{\alpha_2, \alpha_3\}$ and the ending vertex $\{\alpha_3, \alpha_4\}$; then for $5 \le i \le c$,

replace the edge $\{\alpha_{i-2}, \alpha_{i-1}\}, \{\alpha_{i-1}, \alpha_i\}$ by any H-path of all 2-subsets containing α_{i-1} but removing $\{\{\alpha_2, \alpha_{i-1}\}, \{\alpha_3, \alpha_{i-1}\}, \cdots, \{\alpha_{i-3}, \alpha_{i-1}\}\}$, with the starting vertex $\{\alpha_{i-2}, \alpha_{i-1}\}$ and the ending vertex $\{\alpha_{i-1}, \alpha_i\}$.

(2) Two subsets are adjacent if and only if they have no intersecting point. Then the orbital graph is the Kneser graph $K(c, 2)$. If $c \geq 8$, then the degree of the graph is more than $\frac{n}{2}$ and so it is hamiltonian, where n is the order of the graph. For the cases when $c \leq 7$, we do it just by Magma. □

Lemma 3.16. *Let G be one of the groups listed in row $5, 7 - 10$ of* Table 2*. Then every orbital graph of G is hamiltonian.*

Proof. Using Magma, we compute the suborbits for these groups and show that every corresponding orbital graph is hamiltonian.

(1) The action of $\mathrm{PSL}(3, 5).2$ on the flags has three nontrivial suborbits, with the respective length $10, 50$ and 125;

(2) The action of M_{11} on the cosets of a subgroup isomorphic to S_5 has three nontrivial suborbits, with the respective length $15, 20$ and 30;

(3) The action of M_{12} on the cosets of a subgroup isomorphic to $M_{10} : 2$ has two nontrivial suborbits, with the respective length 20 and 45;

(4) The action of M_{23} on the cosets of a subgroup isomorphic to A_8 has three nontrivial suborbits, with the respective length $15, 210$ and 280;

(5) The action of J_1 on the cosets of a subgroup isomorphic to $\mathrm{PSL}(2, 11)$ has four nontrivial suborbits, with the respective length $11, 12, 110$ and 132. □

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