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Component (edge) connectivity of pancake graphs*

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Abstract

The l-component (edge) connectivity of a graph G, denoted by $c\kappa_l(G)$ $(c\lambda_l(G))$, is the minimum number of vertices (edges) whose removal from G results in a disconnected graph with at least l components. The pancake graph P_n is a popular underlying topology for distributed systems. In the paper, we determine the $c\kappa_l(P_n)$ and $c\lambda_l(P_n)$ for $3 \leq l \leq 5$.

Keywords: Component connectivity, component edge connectivity, pancake graphs, fault tolerance.

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1 Introduction

Multiprocessor systems are always built according to a graph which is called its interconnection network (network, for short). In a network, vertices correspond to processors, and edges correspond to communicating links between pairs of vertices. Since failures of processors and links are inevitable in multiprocessor systems, fault tolerance is an important issue in interconnection networks. Fault tolerance of interconnection networks becomes an essential problem and has been widely studied, such as, structure connectivity and substructure connectivity of hypercubes [20], extra connectivity of bubble sort star graphs [10], g-extra conditional diagnosability of hierarchical cubic networks [21], g-good-neighbor connectivity of graphs [25], conditional connectivity of Cayley graphs generated by unicyclic graphs [26].

Given a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links between processors. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices of G, if any, whose deletion disconnect G. The edge

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connectivity $\lambda(G)$ of a graph G is the minimum number of edges of G, if any, whose deletion disconnect G. The g-extra connectivity of G, denoted by $\kappa_g(G)$, is the minimum number of vertices whose removal separates G such that each component of the remaining graph has at least $g+1$ vertices.

The classic parameter is the connectivity $\kappa(G)$ and edge connectivity $\lambda(G)$. In general, the larger $\kappa(G)$ or $\lambda(G)$ is, the more stable the network is. The l-component connectivity of a graph was first introduced by Chartrand [8] and Sampathkumar [22], independently. Note that $c\kappa_2(G) = \kappa(G)$ and $c\lambda_2(G) = \lambda(G)$ for any graph if it is not a complete graph. Therefore, the l-component (edge) connectivity can be regarded as a generalization of the classical (edge) connectivity. The two parameters have been investigated in several interconnection networks. See for example [3, 7, 12, 16, 23, 27, 28, 29]. Recently, the relationship between extra connectivity and component connectivity of general networks has been investigated by Li et al. [14], while the relationship between extra edge connectivity and component edge connectivity of regular networks has been suggested by Hao et al. [15] and Guo et al. [13], independently.

The pancake graph, denoted by P_n, is one of alternative interconnection networks for multiprocessor systems, and it poses some attractive topological properties, such as $(n-1)$-regular, node-symmetric, bipartite and recursive [1]. The pancake graph has drawn considerable attention, such as, structure connectivity and substructure connectivity [6], super connectivity [19] and neighbor connectivity [9, 24] had been considered. For more examples, see [1, 5, 11, 17, 18, 30] and references therein.

The rest of the paper is organized as follows. Section 2 formally gives the definition of pancake graphs. In addition, we introduce some preliminary results. Section 3 determines the l-component connectivity of P_n for $l = 3, 4, 5$. Section 4 determines the l-component edge connectivity of P_n for $l = 3, 4, 5$. Concluding remarks are covered in Section 5.

2 Preliminaries

In this paper, graph-theoretical terminology and notation not defined here mostly follow [2].

For any two graphs G_1 and G_2, $G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$. For any sets A and B, $A - B = \{x : x \in A \text{ but } x \notin B\}$ and we sometimes write $A - B$ as $A \setminus B$ if $B \subseteq A$. For $X, Y \subseteq V(G)$, $[X, Y]$ represents the edge set of G in which one end is in X and the other is in Y. The distance of two vertices u and v in a graph G, denote by $dis_G(u, v)$, is the length of a shortest path between u and v in G. Set $N_G(u) = \{v : dis_G(u, v) = 1\}$, and set $N_G(U) = \bigcup_{u \in U} N_G(u) - U$. For any vertex v, denote by $E(v)$ the edges incident to v. A *k-cycle*, denoted by C_k, is a cycle on k vertices, and a *k-path* $u_1u_2...u_k$, is a path on k vertices. Let $\langle n\rangle = \{1, 2, \cdots, n\}$.

Definition 2.1 ([1])**.** The n-dimensional pancake graph is denoted by P_n. The vertex set $V(P_n) = \{u = u_1u_2\cdots u_n | u_i \in \langle n\rangle, u_i \neq u_j \text{ for } i \neq j\}$, the edge set $E(P_n) = \{uv | u = u_1u_2\cdots u_k\cdots u_n, v = u^k = u_ku_{k-1}\cdots u_2u_1u_{k+1}\cdots u_{n-1}u_n \text{ and } 2 \leq k \leq n\}$, where u^k denotes the unique k-neighbour of u, the edge uu^k is called k-edge.

Clearly, P_n consists of $(n-1)$ kinds of edges. The pancake graphs P_2, P_3 and P_4 are shown in Figure 1. The pancake graphs are Cayley graphs with having the hierarchical (recursive) structure. The removal of all n-edges from P_n results in n connected components $P_n^1, P_n^2, \cdots, P_n^n$, where P_n^i is the subgraph of P_n induced by $\{u = u_1u_2\cdots u_n \in V(P_n) :$

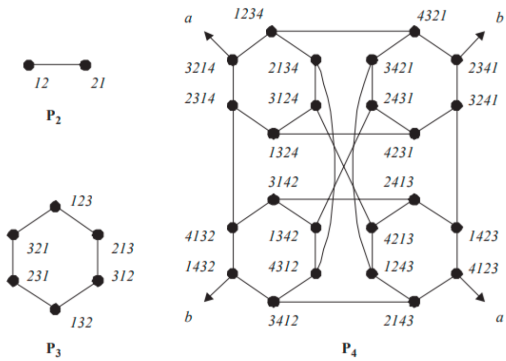

Figure 1: The pancake graphs P_2, P_3 and P_4.

$u_n = i\}$. Clearly, P_n^i is isomorphic to the $(n-1)$-dimensional pancake graph P_{n-1} [17]. We call P_n^i a sub-pancake of P_n. For any vertex $u \in V(P_n^i)$, just one vertex in $N(u)$ is not in $V(P_n^i)$, we define this vertex to be *out-neighbor* of u. For $i \neq j \in \langle n \rangle$, an edge is called a *cross-edge* if its two terminal vertices are in P_n^i and P_n^j, respectively.

Lemma 2.2 ([1, 5, 17, 18]). *An n-dimensional pancake graph P_n has the following combinatorial properties.*

(1) *P_n has $n!$ vertices, $(n-1)n!/2$ edges, $(n-1)$-regular.*

(2) *The girth of P_n is 6 for $n \geq 3$. Let the 6-cycle be presented as $u_1u_2u_3u_4u_5u_6$. Then u_1u_2, u_3u_4, u_5u_6 are 2-edges and u_2u_3, u_4u_5, u_1u_6 are 3-edges.*

(3) *For any $i \neq j$, the number of cross edges between P_n^i and P_n^j is $(n-2)!$.*

Remark 2.3. One and the same path of length 2 cannot be contained in two 6-cycles.

Lemma 2.4 ([30]). *Let F be a set of faulty vertices in P_n with $|F| \leq 2n-4$ for $n \geq 5$. If $P_n - F$ is disconnected, then it has exactly two components, one of which is a singleton or a single edge.*

In [4], Chen and Tan proposed the family of interconnection networks SP_n. It is obviously that P_n is one of the network of SP_n.

Lemma 2.5 ([11]). *Let F be a set of faulty vertices in P_n with $|F| \leq 2n-5$ for $n \geq 3$. If $P_n - F$ is disconnected, then it has exactly two components, one of which is a singleton.*

Lemma 2.6 ([11, 30]). *Let F be a set of faulty vertices in P_n with $|F| \leq 3n-8$ for $n \geq 5$. If $P_n - F$ is disconnected, then it either has two components, one of which is a singleton or a single edge, or has three components, two of which are singletons.*

Lemma 2.7 ([11]). *Let F be a set of faulty vertices in P_n with $|F| \leq 4n-11$ for $n \geq 6$. If $P_n - F$ is disconnected, then $P_n - F$ satisfies one of the following conditions:*

(1) *$P_n - F$ has two components, one of which is a singleton or a single edge or a 3-path;*

(2) *$P_n - F$ has three components, two of which are singletons;*

(3) *$P_n - F$ has three components, two of which are a singleton and an edge, respectively;*

(4) *$P_n - F$ has four components, three of which are singletons.*

Lemma 2.8 ([4, 11]). *$\kappa_1(P_n) = 2n - 4$ for $n \geq 3$, $\kappa_2(P_n) = 3n - 7$ for $n \geq 5$ and $\kappa_3(P_n) = 4n - 10$ for $n \geq 6$.*

Hereafter, we suppose that F is a vertex cut or an edge cut in P_n. For each $i \in \langle n \rangle$, let $F_i = F \cap V(P_n^i)$ or $F_i = F \cap E(P_n^i)$, and $f_i = |F_i|$. Let $I = \{i \in \langle n \rangle | f_i \geq n - 2\}$, $P_n^I = \bigcup_{i \in I} P_n^i$, $F_I = \bigcup_{i \in I} F_i$, and let $J = \langle n \rangle \setminus I$, $P_n^J = \bigcup_{j \in J} P_n^j$, $F_J = \bigcup_{j \in I} f_j$. Also, we Let H be the union of smaller components of $P_n - F$ and let $c(H)$ be the number of components of H.

3 The component connectivity of P_n

Lemma 3.1. *Let S be an independent set of $V(P_n)$ for $n \geq 4$. Then the following assertions hold.*

(1) *If $|S| = 2$, then $|N(S)| \geq 2n - 3$.*

(2) *If $|S| = 3$, then $|N(S)| \geq 3n - 6$.*

(3) *If $|S| = 4$, then $|N(S)| \geq 4n - 8$.*

Proof. For (1), let $S = \{v_1, v_2\}$. By Lemma 2.2, P_n contains no 4-cycle. Thus, v_1 and v_2 have at most one common neighbor, and $|N(S)| = |N(v_1)| + |N(v_2)| - |N(v_1) \cap N(v_2)| \geq 2n - 3$.

For (2), let $S = \{v_1, v_2, v_3\}$. By Lemma 2.2, P_n contains 6-cycle, there exists at most three common neighbors among these three singletons. Thus, we have $|N(S)| \geq \sum_{i=1}^{3} |N(v_i)| - 3 \geq 3n - 6$.

For (3), let $S = \{v_1, v_2, v_3, v_4\}$. Since P_n contains 8-cycle, and in order to make these four singletons contain as many common vertices as possible, we may assume that the 8-cycle is presented as $v_1u_1v_2u_2v_3u_3v_4u_4$. Then there exists four common neighbors among these four singletons. If there exists five common neighbors among these four singletons, then it forms a cycle of length less than 6 or two 6-cycles with common 2-path, contradicting Lemma 2.2(2) and Remark 2.3, respectively. Thus, we have $|N(S)| \geq \sum_{i=1}^{4} |N(v_i)| - 4 \geq 4n - 8$. □

The following remark provides instances that attain the bounds for the assertions of Lemma 3.1.

Remark 3.2. Let $x = 123 \cdots n$, $y = (x^2)^3$, $z = (y^2)^3$, $w = (y^2)^n$, $o = (w^2)^3$. Clearly, $\{x, y, z\}$ is an independent set of P_n and $\{x, y, z\}$ lie on a 6-cycle in the subgraph of P_n, $\{x, y, w, o\}$ is an independent set of P_n and $\{x, y, w, o\}$ lie on a 8-cycle in the subgraph of P_n. Clearly, if $S = \{x, y\}$, then $|N(S)| = 2n - 3$. Since $P_n - F$ has three components, we have $c\kappa_3(P_n) \leq 2n - 3$. Similarly, if $S = \{x, y, z\}$, then $|N(S)| = 3n - 6$. Since $P_n - F$ has four components, we have $c\kappa_4(P_n) \leq 3n - 6$. Also, if $S = \{x, y, w, o\}$, then $|N(S)| = 4n - 8$. Since $P_n - F$ has five components, we have $c\kappa_5(P_n) \leq 4n - 8$.

Theorem 3.3. *For $n \geq 3$, $c\kappa_3(P_n) = 2n - 3$.*

Proof. It is true if $n = 3$. From Remark 3.2, we obtain the upper bound $c\kappa_3(P_n) \leq 2n-3$ for $n \geq 4$. It suffices to show $c\kappa_3(P_n) \geq 2n-3$. Suppose on the contrary that there is a vertex cut F with $|F| \leq 2n-4$, and $P_n - F$ has at least three components.

We first consider that $n = 4$. Since $|F| \leq 4$, it is clear that $|I| \leq 2$. If $|I| = 1$, let $I = \{i\}$, then $f_i \in \{2, 3, 4\}$. If $|I| = 2$, let $I = \{i, j\}$, then $f_i = f_j = 2$. No matter which case, it's not hard to prove that $P_4 - F$ has at most two components, a contradiction. We now consider that $n \geq 5$. By Lemma 2.4, $P_n - F$ has exactly two components, a contradiction.

Thus, $c\kappa_3(P_n) \geq 2n-3$. □

Next, we give a lemma which is used by Theorem 3.5 and 3.6.

Lemma 3.4. *For $n \geq 5$, if $|I| \leq 3$, then $P_n^J - F_J$ is connected.*

Proof. By the definition of J, we have $|J| = n - |I| \geq n - 3 \geq 2$ for $n \geq 5$ and $f_j \leq n-3$ for $j \in J$. Since each subgraph P_n^j is isomorphic to P_{n-1}, by Lemma 2.2, we have $\kappa(P_n^j) = n-2$. Thus, for each $j \in J$, $P_n^j - F_j$ is connected. For distinct $j, k \in J$, by Lemma 2.2, the number of cross edges between P_n^j and P_n^k is $(n-2)!$, since $(n-2)! > 2(n-3)$ for $n \geq 5$, we have $P_n^j - F_j$ is connected to $P_n^k - F_k$. Therefore, $P_n^J - F_J$ is connected. □

Theorem 3.5. *For $n \geq 4$, $c\kappa_4(P_n) = 3n-6$.*

Proof. Remark 3.2 acquires the upper bound $c\kappa_4(P_n) \leq 3n-6$ for $n \geq 4$. It suffices to show $c\kappa_4(P_n) \geq 3n-6$. Suppose that there is a vertex cut F with $|F| \leq 3n-7$, and $P_n - F$ has at least four components.

We first consider that $n = 4$. By Theorem 3.3, $c\kappa_3(P_4) = 5$ and $|F| \leq 3n-7 = 5$, we have know $P_n - F$ has at most three components, a contradiction.

Next, Let $n \geq 5$. Lemma 2.6 shows that the removal of a vertex cut with no more that $3n-8$ vertices in P_n results in a disconnected graph with at most three components, a contradiction. To complete the proof, we need to show result holds when $|F| = 3n-7$. Partition P_n into n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$ of P_{n-1} along dimension n. Recall that $I = \{i \in \langle n \rangle : f_i \geq n-2\}$. Since $|F| = 3n-7$, it is clear that $|I| \leq 2$. By Lemma 3.4, $P_n^J - F_J$ is connected. If $|I| = 0$, then $P_n - F = P_n^J - F_J$ is connected, a contradiction. Consider the following cases.

Case 1: $|I| = 1$. Let $I = \{i\}$.

Case 1.1: $n-2 \leq f_i \leq 3(n-1)-8$.

Since each subgraph P_n^i is isomorphic to P_{n-1}, by Lemma 2.6, $P_n^i - F_i$ has at most three components, and all small components contain at most two vertices in total. Since $(n-1)!-2 > 3n-7$ for $n \geq 5$, the large component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. This implies that $|V(H)| \leq 2$. It is clear that $c(H) \leq |V(H)| \leq 2$, a contradiction.

Case 1.2: $3n-10 \leq f_i \leq 3n-7$.

In this case, we have $F_J = |F| - f_i \leq (3n-7) - (3n-10) = 3$. Since every vertex of H has exactly one out-neighbor, we have $|V(H)| \leq 3$. If $|V(H)| = 3$, then $c(H) \leq 2$. Otherwise, $c(H) = 3$ and it implies that H is a set of three singletons. By Lemma 3.1, we have $|N_{P_n}(V(H))| \geq 3n-6 > 3n-7 = |F|$, a contradiction. If $|V(H)| \leq 2$, it is clear that $c(H) \leq |V(H)| \leq 2$, a contradiction.

Case 2: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Since $|F| = 3n - 7$, we have $n - 2 \leq f_i \leq f_j \leq (3n - 7) - (n - 2) = 2n - 5$. If $f_i = 2n - 5$, then $f_i + f_j = 2(2n - 5) = 4n - 10 > 3n - 7$ for $n \geq 5$. Thus, it requires that $f_i \leq 2n - 6$.

Case 2.1: $n - 2 \leq f_i \leq f_j \leq 2n - 6$.

For $l \in \{i, j\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.4, $P_n^l - F_l$ has at most two components, one of which is a singleton or an edge. Since $(n-1)! - (n-2)! - 2 > 3n - 7$ for $n \geq 5$ and $l \in \{i, j\}$, then the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. It implies that $c(H) \leq 2$, a contradiction.

Case 2.2: $f_j = 2n - 5$, and $f_i = n - 2$.

Since $|F| = 3n - 7$, we have $F_J = |F| - f_i - f_j = 0$. Thus, at most two vertices in $P_n^i \cup P_n^j - (F_i \cup F_j)$ cannot connect with $P_n^J - F_J$ in $P_n - F$, and the two vertices form an edge. Thus, $c(H) \leq 1$, a contradiction. □

Theorem 3.6. *For $n \geq 6$, $c\kappa_5(P_n) = 4n - 8$.*

Proof. Remark 3.2 acquires the upper bound $c\kappa_5(P_n) \leq 4n - 8$. It suffices to show $c\kappa_5(P_n) \geq 4n - 8$.

Suppose that there is a vertex cut F with $|F| \leq 4n - 9$, and $P_n - F$ has at least five components. Lemma 2.7 shows that the removal of a vertex cut with no more that $4n - 11$ vertices in P_n results in a disconnected graph with at most four components, a contradiction. To complete the proof, we need to show result holds when $4n - 10 \leq |F| \leq 4n - 9$. Partition P_n into n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$ of P_{n-1} along dimension n. Recall that $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$. Since $|F| \leq 4n - 9$, it is clear that $|I| \leq 3$. By Lemma 3.4, $P_n^J - F_J$ is connected. If $|I| = 0$, then $P_n - F = P_n^J - F_J$ is connected, a contradiction.

Consider the following cases.

Case 1: $|I| = 1$. Let $I = \{i\}$.

Case 1.1: $n - 2 \leq f_i \leq 4(n - 1) - 11$.

Since each subgraph P_n^i is isomorphic to P_{n-1}, by Lemma 2.7, $P_n^i - F_i$ has at most four components, and all small components contain at most three vertices in total. Since $(n-1)! - 3 > 4n - 9$ for $n \geq 6$, the large component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. This implies that $|V(H)| \leq 3$. It is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 1.2: $4n - 14 \leq f_i \leq 4n - 9$.

In this case, we have $F_J = |F| - f_i \leq (4n - 9) - (4n - 14) = 5$. Since every vertex of H has exactly one out-neighbor, we have $|V(H)| \leq 5$. If $|V(H)| = 5$, then $c(H) \leq 3$. Otherwise, $c(H) \geq 4$ and it implies that H contains five singletons or three singletons together with an edge. In the former case, let $H = H_0 \cup \{x\}$, where H_0 is a set of four singletons and x is a singleton. By Lemma 3.1(3), we have $|N_{P_n}(V(H_0))| \geq 4n - 8$. Clearly, $|N_{P_n}(V(H))| = |N_{P_n}(V(H_0))| + |N_{P_n}(x)| - |N_{P_n}(V(H)) \cap N_{P_n}(x)| \geq 4n - 8 + (n - 1) - 4 = 5n - 13 > 4n - 9 \geq |F|$ for $n \geq 6$, a contradiction. In the latter case, let $H = H_0 \cup \{u, v\}$, where H_0 is a set of three singletons and uv is an edge. Then, we have $|N_{P_n}(V(H_0))| \geq 3n - 6$ by Lemma 3.1(2) and $|N_{P_n}(\{u, v\})| = 2n - 4$ by Lemma 2.8. Also, the girth of P_n is 6 and it follows that $|N_{P_n}(V(H)) \cap N_{P_n}(\{u, v\})| \leq 3$. Thus $|N_{P_n}(V(H))| = |N_{P_n}(V(H_0))| + |N_{P_n}(\{u, v\})| - |N_{P_n}(V(H)) \cap N_{P_n}(\{u, v\})| \geq 3n - 6 + (2n - 4) - 3 = 5n - 13 > 4n - 9 \geq |F|$ for $n \geq 6$, a contradiction. If $|V(H)| = 4$, then $c(H) \leq 3$. Otherwise, H contains four singletons. By Lemma 3.1(3), we have $|N_{P_n}(V(H))| \geq 4n-8 > 4n-9 \geq |F|$, a contradiction. Also, if $|V(H)| \leq 3$, it is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 2: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Since $|F| \leq 4n-9$, we have $n-2 \leq f_i \leq f_j \leq (4n-9)-(n-2) = 3n-7$. If $f_i \geq 3n-10$, then $f_i + f_j \geq 2(3n-10) = 6n-20 > 4n-9$ for $n \geq 6$. Thus, it requires that $f_i \leq 3n-11$.

Case 2.1: $n-2 \leq f_i \leq f_j \leq 3n-11$.

For each $l \in \{i, j\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.6, $P_n^l - F_l$ has at most three components and all smaller components contain at most two vertices in total. Since $(n-1)! - (n-2)! - 2 > 4n-9$ for $n \geq 6$, the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Thus, $|V(H)| \leq 2|I| = 4$. If $|V(H)| = 4$, then $c(H) \leq 3$. Otherwise, H contains four singletons. By Lemma 3.1(3), we have $|N_{P_n}(V(H))| \geq 4n-8 > 4n-9 \geq |F|$, a contradiction. Also, if $|V(H)| \leq 3$, it is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 2.2: $3n-10 \leq f_j \leq 3n-7$, and $n-2 \leq f_i \leq 4n-9-(3n-10) = n+1$.

$P_n^i - F_i$ has at most two components, one of which is a singleton. If $f_j = 3n-10$, by Theorem 3.5, $P_n^j - F_j$ has at most three components. Then $F_J = |F| - f_i - f_j \leq 4n-9-(n-2)-(3n-10) = 3$. If $3n-9 \leq f_j \leq 3n-7$, then $F_J = |F| - f_i - f_j \leq 4n-9-(n-2)-(3n-9) = 2$. No matter which case, $c(H) \leq 3$, a contradiction.

Case 3: $|I| = 3$.

Let $I = \{i, j, k\}$. Without loss of generality, assume $f_i \leq f_j \leq f_k$. Since $|F| \leq 4n-9$, we have $n-2 \leq f_i \leq f_j \leq f_k \leq (4n-9)-2(n-2) = 2n-5$. If $f_i \geq 2n-6$, then $f_i + f_j + f_k \geq 3(2n-6) = 6n-18 > 4n-9$ for $n \geq 6$. Thus, it requires that $f_i \leq 2n-7$. If $f_j \geq 2n-6$, then $f_i + f_j + f_k \geq n-2+2(2n-6) = 5n-14 > 4n-9$ for $n \geq 6$. Thus, it requires that $f_j \leq 2n-7$.

Case 3.1: $n-2 \leq f_i \leq f_j \leq f_k \leq 2n-7$.

For each $l \in \{i, j, k\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.5, $P_n^l - F_l$ has at most two components, one of which is a singleton. Since $(n-1)! - 2(n-2)! - 1 > 4n-9$ for $n \geq 6$, the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Thus, $|V(H)| \leq 3|I| = 3$. It is clear that $c(H) \leq |V(H)| \leq 3$, a contradiction.

Case 3.2: $n-2 \leq f_i \leq f_j \leq 2n-7 < f_k \leq 2n-5$.

For each $l \in \{i, j\}$, if $P_n^l - F_l$ is disconnected, by Lemma 2.5, $P_n^l - F_l$ has at most two components, one of which is a singleton. By a similar argument as Case 3.1, the large component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Since $|f_k| \leq 2n-5 \leq 3(n-1)-8$ for $n \geq 6$, by Lemma 2.6, either $P_n^k - F_k$ is connected or $P_n^k - F_k$ has at most three components and all smaller components contain at most two vertices in total. Since $(n-1)! - 2(n-2)! - 2 > 4n-9 \geq |F|$ for $n \geq 6$, the large component of $P_n^k - F_k$ is connected to $P_n^J - F_J$. Thus, $|V(H)| \leq 4$. Then, an argument similar to Case 2.1 shows that $c(H) \leq 3$, a contradiction. □

4 The edge component connectivity of P_n

Theorem 4.1. *For $n \geq 3$, $c\lambda_3(P_n) = 2n-3$.*

Proof. Take an edge $e = xy$ and $F = E(x) \cup E(y)$. Then $|F| = 2n-3$ and $P_n - F$ has at least three components. Hence $c\lambda_3(P_n) \leq 2n-3$. It suffices to show $c\lambda_3(P_n) \geq 2n-3$.

We consider an inductive proof as follows. The statement of theorem holds for $n = 3$. We assume that the result holds for P_{n-1}, and prove that it also holds for P_n, where $n \geq 4$. Suppose that there is an edge set F with $|F| \leq 2n - 4$, and $P_n - F$ has at least three components. Consider n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$. Since $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$, and $|F| \leq 2n - 4$, it is clear that $|I| \leq 2$.

Consider the following cases.

Case 1: $|I| = 0$.

Each $P_n^i - F_i$ is connected for $i \in \langle n \rangle$. For distinct $i, j \in \langle n \rangle$, by Lemma 2.2, the number of cross edges between P_n^i and P_n^j is $(n-2)!$. Since $(2n-4) < 3(n-2)!$ for $n \geq 4$, there are at most two $[P_n^i, P_n^j]$'s which are contained in F for distinct $i, j \in \langle n \rangle$. Thus $P_n - F$ is connected, a contradiction.

Case 2: $|I| = 1$. Let $I = \{i\}$.

Case 2.1: $n - 2 \leq f_i \leq 2(n-1) - 4$.

If each $P_n^i - F_i$ is connected for $i \in \langle n \rangle$, since $(2n-4) - (n-2) < 2(n-2)!$ for $n \geq 4$, then there is at most one $[P_n^i, P_n^j]$ which is contained in F for distinct $i, j \in \langle n \rangle$. Thus, $P_n - F$ is connected, a contradiction. Hence, there exists i such that $P_n^i - F_i$ is not connected. By the inductive hypothesis, $P_n^i - F_i$ has at most two components.

Since $(2n-4) - (n-2) < 2(n-2)!$ for $n \geq 4$, there is at most one $[P_n^j, P_n^k]$ which is contained in F for distinct $j, k \in \langle n \rangle \setminus \{i\}$. Thus $P_n^J - F_J$ is connected. Furthermore, $|[P_n^i, P_n^J - F_J]| = (n-1)! > 2n-4-(n-2)$ for $n \geq 4$. At least one component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. Hence $P_n - F$ has at most two components, a contradiction.

Case 2.2: $2n - 5 \leq f_i \leq 2n - 4$.

In this case, we have $|F| - f_i \leq (2n-4) - (2n-5) = 1$. Then $P_n^J - F_J$ is connected. Note that at most one vertex of $P_n^i - F_i$ is disconnected to $P_n^J - F_J$. Hence $P_n - F$ has at most two components, a contradiction.

Case 3: $|I| = 2$.

Let $I = \{i, j\}$. Then $f_i = f_j = n - 2$ and $|F| - f_i - f_j = 0$. Thus $P_n^l - F_l$ has at most two components for any $l \in \{i, j\}$ and $P_n^J - F_J$ is connected. And so either any component of $P_n^l - F_l$ is connected to $P_n^J - F_J$ or two singletons are connected and the other component of $P_n^l - F_l$ is connected to $P_n^J - F_J$ if both $P_n^i - F_i$ and $P_n^j - F_j$ have a singleton, respectively. Thus $P_n - F$ has at most two components, a contradiction. □

Theorem 4.2. *For* $n \geq 3$, $c\lambda_4(P_n) = 3n - 5$.

Proof. Take a 3-path xyz and $F = E(x) \cup E(y) \cup E(z)$. Then $|F| = 3n-5$ and $P_n - F$ has at least four components. Hence $c\lambda_4(P_n) \leq 3n-5$. It suffices to show $c\lambda_4(P_n) \geq 3n-5$.

We consider an inductive proof as follows. The statement of theorem holds for $n = 3$. We assume that the result holds for P_{n-1}, and prove that it also holds for P_n, where $n \geq 4$. Suppose that there is an edge set F with $|F| \leq 3n - 6$, and $P_n - F$ has at least four components. Consider n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$. Since $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$, and $|F| \leq 3n - 6$, it is clear that $|I| \leq 3$.

Consider the following cases.

Case 1: $|I| = 0$.

Similar to the proof of Case 1 of Theorem 4.1, we can show that $P_n - F$ is connected for $n \geq 5$, a contradiction. Consider that $n = 4$. Since $(4-2)! = 2$ and $|F| \leq 3n - 6 = 6$, there are at most three $[P_4^i, P_4^j]$'s which are contained in F. Hence $P_4 - F$ has at most two components, a contradiction.

Case 2: $|I| = 1$. Let $I = \{i\}$.

Case 2.1: $n - 2 \leq f_i \leq 3(n-1) - 6$.

Similar to the proof of Case 2.1 of Theorem 4.1, we can show that $P_n - F$ has at most three components for $n \geq 5$, a contradiction. Consider that $n = 4$. Then $2 \leq f_i \leq 3$. If $f_i = 2$, then $P_4^i - F_i$ has at most two components, and $|F| - f_i \leq (3n-6) - 2 = 4$. It is not hard to prove that $P_4 - F$ has at most two components, a contradiction. If $f_i = 3$, then $P_4^i - F_i$ has at most three components, and $|F| - f_i \leq (3n-6) - 3 = 3$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction.

Case 2.2: $3n - 8 \leq f_i \leq 3n - 6$.

In this case, we have $|F| - f_i \leq (3n-6) - (3n-8) = 2$. Furthermore, $P_n^J - F_J$ is connected. Note that at most two vertices of $P_n^i - F_i$ are disconnected to $P_n^J - F_J$. Hence $P_n - F$ has at most three components, a contradiction.

Case 3: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Then $f_j \leq 3n - 6 - (n-2) = 2n - 4$.

Case 3.1: $n - 2 \leq f_j \leq 2(n-1) - 4$.

In this case, we have $n - 2 \leq f_i \leq f_j \leq 2(n-1) - 4$. By Theorem 4.1, both $P_n^i - F_i$ and $P_n^j - F_j$ have at most two components.

Consider that $n = 4$. Then $f_i = f_j = 2$, implying that $P_4^l - F_l$ has at most two components for $l \in \{i, j\}$, and $|F| - f_i - f_j \leq (3n-6) - 2 - 2 = 2$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction.

Consider that $n \geq 5$. Since $|[P_n^k, P_n^l]| = (n-2)! > 3n - 6 - 2(n-2)$ for $n \geq 5$ and $k, l \in \langle n \rangle \setminus \{i, j\}$, $P_n^J - F_J$ is connected. Furthermore, $|[P_n^i, P_n^J - F_J]| = (n-1)! - (n-2)! > 3n - 6 - 2(n-2)$ for $n \geq 5$. At least one component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. Similarly, at least one component of $P_n^j - F_j$ is connected to $P_n^J - F_J$. Hence $P_n - F$ has at most three components, a contradiction.

Case 3.2: $f_j = 2n - 5$.

Then $n - 2 \leq f_i \leq 3n - 6 - (2n-5) = n - 1$.

If $f_i = n - 2$, $P_n^i - F_i$ has at most two components. Then $|F| - f_i - f_j \leq 1$, and so $P_n^J - F_J$ is connected. Thus $P_n - F$ has at most three components, a contradiction. If $f_i = n - 1$, then $|F| - f_i - f_j = 0$ and $P_n^J - F_J$ is connected. Thus $P_n - F$ has at most three components, a contradiction.

Case 3.3: $f_j = 2n - 4$.

Then $f_i = n - 2$ and $|F| - f_i - f_j = 0$. Thus $P_n^i - F_i$ has at most two components and $P_n^J - F_J$ is connected. Thus $P_n - F$ has at most two components, a contradiction.

Case 4: $|I| = 3$.

Let $I = \{i, j, k\}$. Then $f_i = f_j = f_k = n - 2$ and $|F| - f_i - f_j - f_k = 0$. Thus $P_n^l - F_l$ has at most two components for any $l \in \{i, j, k\}$. Thus $P_n - F$ has at most two components, a contradiction. □

Theorem 4.3. *For $n \geq 3$, $c\lambda_5(P_n) = 4n - 7$.*

Proof. Take a 4-path $xyzw$ and $F = E(x) \cup E(y) \cup E(z) \cup E(w)$. Then $|F| = 4n - 7$ and $P_n - F$ has at least five components. Hence $c\lambda_5(P_n) \leq 4n - 7$. It suffices to show $c\lambda_5(P_n) \geq 4n - 7$.

We consider an inductive proof as follows. The statement of theorem holds for $n = 3$. We assume that the result holds for P_{n-1}, and prove that it also holds for P_n, where $n \geq 4$. Suppose that there is an edge set F with $|F| \leq 4n - 8$, and $P_n - F$ has at least five components. Consider n disjoint copies $P_n^1, P_n^2, \cdots, P_n^n$. Since $I = \{i \in \langle n \rangle : f_i \geq n - 2\}$, and $|F| \leq 4n - 8$, it is clear that $|I| \leq 4$.

Consider the following cases.

Case 1: $|I| = 0$.

Similar to the proof of Case 1 of Theorem 4.2, we can show that $P_n - F$ is connected for $n \geq 5$ and $P_4 - F$ has at most two components, a contradiction.

Case 2: $|I| = 1$. Let $I = \{i\}$.

Case 2.1: $n - 2 \leq f_i \leq 4(n-1) - 8$.

Similar to the proof of Case 2.1 of Theorem 4.1, we can show that $P_n - F$ has at most three components for $n \geq 5$, a contradiction. Consider that $n = 4$. Then $2 \leq f_i \leq 4$ and $(4-2)! = 2$. If $f_i = 2$, then $P_4^i - F_i$ has at most two components, and $|F| - f_i \leq (4n - 8) - 2 = 6$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction. If $f_i = 3$, then $P_4^i - F_i$ has at most three components, and $|F| - f_i \leq (4n - 8) - 3 = 5$. It is not hard to prove that $P_4 - F$ has at most three components, a contradiction. If $f_i = 4$, then $P_4^i - F_i$ has at most four components, three of which are singletons, and $|F| - f_i \leq (4n - 8) - 4 = 4$. It is not hard to prove that $P_4 - F$ has at most four components, a contradiction.

Case 2.2: $4n - 11 \leq f_i \leq 4n - 8$.

In this case, we have $|F| - f_i \leq (4n - 8) - (4n - 11) = 3$. Since $3 < 2(n-2)!$ for $n \geq 4$, there is at most one $[P_n^j, P_n^k]$ which is contained in F for $j, k \in \langle n \rangle \setminus \{i\}$, and so $P_n^J - F_J$ is connected. Note that at most three vertices of $P_n^i - F_i$ are disconnected to $P_n^J - F_J$. Hence $P_n - F$ has at most four components, a contradiction.

Case 3: $|I| = 2$.

Let $I = \{i, j\}$. Without loss of generality, assume $f_i \leq f_j$. Then $f_j \leq 4n - 8 - (n - 2) = 3n - 6$.

Case 3.1: $n - 2 \leq f_j \leq 2(n-1) - 4$.

Similar to the proof of Case 3.1 of Theorem 4.2, we can show that $P_n - F$ has at most three components, a contradiction.

Case 3.2: $2n - 5 \leq f_j \leq 3n - 9$.

Consider that $n = 4$. Then $f_j = 3$ and $2 \leq f_i \leq 3$. Then $P_4^j - F_j$ has at most three components. If $f_i = 2$, then $P_4^i - F_i$ has at most two components, and $|F| - f_i - f_j \leq (4n - 8) - 2 - 3 = 3$. It is not hard to prove that $P_4 - F$ has at most four components, a contradiction. If $f_i = 3$, then $P_4^i - F_i$ has at most three components, and $|F| - f_i - f_j \leq (4n - 8) - 3 - 3 = 2$. Suppose either $P_4^i - F_i$ or $P_4^j - F_j$ contains no singleton, it is not hard to prove that $P_4 - F$ has at most three components, a contradiction. Suppose both $P_4^i - F_i$ and $P_4^j - F_j$ contain singletons, then $P_4 - F$ has at most four components, a contradiction. Otherwise, $P_4 - F$ have five components, four of which are singletons. If two singletons of P_4^l are not an edge of P_4^l for $l \in \{i, j\}$, then $f_l \geq 4$, a contradiction. Thus, two singletons form an edge of P_4^i and the other two singletons form an edge of P_4^j, implying that the four singletons form a 4-cycle, contradicting Lemma 2.2(2).

Consider that $n \geq 5$. If $f_i \geq 2n-3$, then $f_i + f_j \geq 2(2n-3) > 4n-8 \geq |F|$, a contradiction. Thus $n-2 \leq f_i \leq 2n-4$. Note that $2n-5 \leq f_j \leq 3n-9$. By Theorem 4.2, $P_n^j - F_j$ has at most three components. Since $|[P_n^k, P_n^l]| = (n-2)! > 4n-8-(n-2)-(2n-5)$ for $k, l \in \langle n \rangle \setminus \{i, j\}$, $P_n^J - F_J$ is connected. Furthermore, $|[P_n^i, P_n^J - F_J]| = (n-1)! - (n-2)! > 4n-8-(n-2)-(2n-5)$. At least one component of $P_n^i - F_i$ is connected to $P_n^J - F_J$. Similarly, at least one component of $P_n^j - F_j$ is connected to $P_n^J - F_J$.

If $n-2 \leq f_i \leq 2n-6$, by Theorem 4.1, $P_n^i - F_i$ has at most two components. Hence $P_n - F$ has at most four components, a contradiction. If $f_i = 2n-5$, and assume first that $f_j = 2n-5$. Then $|F| - f_i - f_j = 4n-8-(2n-5)-(2n-5) = 2$. By Theorem 4.1, we have $P_n^l - F_l$ has three components for $l \in \{i, j\}$. Similar to the case of $f_i = 3$ in the first paragraph of Case 3.2, $P_n - F$ has at most four components, a contradiction. Now assume that $2n-4 \leq f_j \leq 2n-3$, then $|F| - f_i - f_j \leq 4n-8-(2n-5)-(2n-4) = 1$. It is not hard to prove that $P_n - F$ has at most three components, a contradiction. If $f_i = 2n-4$, then $f_j = f_i = 2n-4$ and $|F| - f_i - f_j = 4n-8-2(2n-4) = 0$. By Theorem 4.2, $P_n^i - F_i$ has at most three components. Hence $P_n - F$ has at most three components, a contradiction.

Case 3.3: $f_j = 3n-8$.

It follows that $n-2 \leq f_i \leq n$. If $f_i = n-2$, by Theorem 4.1, $P_n^i - F_i$ has at most two components. Then $|F| - f_i - f_j \leq 4n-8-(3n-8)-(n-2) = 2$. It is not hard to prove that $P_n - F$ has at most four components, a contradiction. If $f_i = n-1$, by Theorem 4.2, $n-1 < 3(n-1)-5$ for $n \geq 4$, then $P_n^i - F_i$ has at most three components, and $|F| - f_i - f_j \leq 4n-8-(3n-8)-(n-1) = 1$. Then $P_n - F$ has at most four components, a contradiction. If $f_i = n$, then $|F| - f_i - f_j = 0$. By Theorem 4.2, both $P_n^i - F_i$ and $P_n^j - F_j$ have at most four components. Then $P_n - F$ has at most four components, a contradiction.

Case 3.4: $f_j = 3n-7$.

Similar to the proof of Case 3.3 of Theorem 4.3, we can show that $P_n - F$ has at most three components, a contradiction.

Case 3.5: $f_j = 3n-6$.

Then $f_i = n-2$ and $|F| - f_i - f_j = 0$. By Theorem 4.1, $P_n^i - F_i$ has at most two components. Thus $P_n - F$ has at most two components, a contradiction.

Case 4: $|I| = 3$.

Let $I = \{i, j, k\}$. Without loss of generality, assume $f_i \leq f_j \leq f_k$. Then $f_k \leq 4n-8-2(n-2) = 2n-4$. Consider $n = 4$. Then $f_i = 2, f_j = 2, f_k = 2$, or $f_i = 2, f_j = 2, f_k = 3$, or $f_i = 2, f_j = 2, f_k = 4$, or $f_i = 2, f_j = 3, f_k = 3$. No matter which case, it's not hard to prove that $P_4 - F$ has at most four components, a contradiction.

Next, we consider $n \geq 5$. If $f_j \geq 2n-5$, then $f_i + f_j + f_k \geq n-2+2(2n-5) = 5n-12 > 4n-8 \geq |F|$ for $n \geq 5$, a contradiction. Then $f_j \leq 2n-6$.

Case 4.1: $n-2 \leq f_i \leq f_j \leq f_k \leq 2n-6$.

By Theorem 4.1, $P_n^l - F_l$ has at most two components for any $l \in \{i, j, k\}$. Since $|[P_n^x, P_n^y]| = (n-2)! > 4n-8-3(n-2)$ for $n \geq 5$ and $x, y \in \langle n \rangle \setminus \{i, j, k\}$, $P_n^J - F_J$ is connected. Furthermore, $|[P_n^l, P_n^J - F_J]| = (n-1)! - 2(n-2)! > 4n-8-3(n-2)$ for $n \geq 5$. At least one component of $P_n^l - F_l$ is connected to $P_n^J - F_J$. Hence $P_n - F$ has at most four components, a contradiction.

Case 4.2: $n-2 \leq f_i \leq f_j \leq 2n-6 < f_k \leq 2n-4$.

By Theorem 4.1, $P_n^l - F_l$ has at most two components for any $l \in \{i, j\}$, and by Theorem 4.2, $P_n^k - F_k$ has at most three components. Since $|F| - f_i - f_j - f_k \leq 4n - 8 - 2(n-2) - (2n-5) = 1$, $P_n^J - F_J$ is connected, and at most four vertices of $P_n^i - F_i$, $P_n^j - F_j$ and $P_n^k - F_k$ are disconnected to $P_n^J - F_J$. If four vertices of $P_n^i - F_i$, $P_n^j - F_j$ and $P_n^k - F_k$ are disconnected to $P_n^J - F_J$, then two of which forms an edge, and then $P_n - F$ has at most four components, a contradiction. Otherwise, $P_n - F$ has at most four components, a contradiction.

Case 5: $|I| = 4$.

Let $I = \{i, j, k, p\}$. Then $f_i = f_j = f_k = f_p = n-2$ and $|F| - f_i - f_j - f_k - f_p = 0$. Thus $P_n^l - F_l$ has at most two components for any $l \in \{i, j, k, p\}$. Thus $P_n - F$ has at most three components, a contradiction. □

5 Concluding remarks

In this paper, we study the l-component (edge) connectivity of P_n for $3 \leq l \leq 5$. We have know that the l-component connectivity of P_n are $c\kappa_3(P_n) = 2n - 3$ for $n \geq 3$, $c\kappa_4(P_n) = 3n - 6$ for $n \geq 4$, $c\kappa_5(P_n) = 4n - 8$ for $n \geq 6$. Also, for $n \geq 3$, we have know the l-component edge connectivity of P_n are $c\lambda_3(P_n) = 2n - 3$, $c\lambda_4(P_n) = 3n - 5$, $c\lambda_5(P_n) = 4n - 7$. We study the larger component (edge) connectivity of P_n in the future work.

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