Also available at http://amc.imfm.si
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 6 (2013) 21–23

Johnson graphs are Hamilton-connected

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Received 15 December 2011, accepted 15 January 2012, published online 1 June 2012

Abstract

We prove that the Johnson graphs are Hamilton-connected and apply this to obtain that another family of graphs is Hamilton-connected.

Keywords: Hamilton path, Johnson graph, Hamilton-connected.

Math. Subj. Class.: 05C70

1 Main Result

The *Johnson graph* $J(n,k)$, $0 \leq k \leq n$, is defined by letting the vertices correspond to the k-subsets of an n-set, where two vertices are adjacent if and only if the corresponding k-subsets have exactly $k-1$ elements in common. A graph is *Hamilton-connected* if for any pair of distinct vertices u, v there is a Hamilton path whose terminal vertices are u and v. The graph with a single vertex is trivially Hamilton-connected.

In a recent paper [1], I needed a certain graph to be Hamilton-connected. This graph, defined below, contains vertex-disjoint Johnson graphs. The result I needed is embodied in the corollary below.

Theorem 1.1. *The Johnson graph $J(n,k)$ is Hamilton-connected for all $n \geq 1$.*

Proof. For ease of exposition, instead of talking about the vertex corresponding to a subset, we shall simply treat the subsets as if they are vertices so that we use equality notation between vertices and sets. The graphs $J(n,k)$ and $J(n,n-k)$ are isomorphic via the mapping that takes a k-subset to its complement.

The graphs $J(n,0)$ and $J(n,n)$, $n \geq 1$, are isomorphic to the single vertex K_1 and trivially Hamilton-connected. The graphs $J(n,1)$ and $J(n,n-1)$, $n \geq 1$, are isomorphic to the complete graph K_n. Complete graphs certainly are Hamilton-connected.

We proceed by double induction and when considering $J(n,k)$, the induction hypotheses are: $J(m,k')$ is Hamilton-connected whenever $k' < k$ and $m \geq k'$, or $J(m,k)$ is

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Hamilton-connected whenever $m < n$ and $m \geq k$. As noted above, $J(n, 1)$ is Hamilton-connected for all $n \geq 1$. For a fixed k we start with $J(k, k)$ and then proceed by going from $J(n-1, k)$ to $J(n, k)$. Thus, the induction hypotheses make sense.

If $k \leq n \leq 2k-1$, then $n-k < k$ so that $J(n, n-k)$ is Hamilton-connected by hypothesis. This, in turn, implies that $J(n, k)$ is Hamilton-connected because $J(n, k)$ and $J(n, n-k)$ are isomorphic. Thus, it follows that $J(n, k)$ is Hamilton-connected for all n satisfying $k \leq n \leq 2k-1$.

For the remaining cases we need to actually show how to find appropriate Hamilton paths. The symmetric group S_n acts in the obvious way on the vertex set of $J(n, k)$. This action is transitive so that it suffices to find a Hamilton path from the vertex $u = \{1, 2, 3, \ldots, k\}$ to any other vertex. Let $v = \{a_1, a_2, \ldots, a_k\}$ be an arbitrary vertex.

If there is an element x of $\{1, 2, \ldots, n\}$ belonging to neither of the sets, we may relabel elements so that n is missing from both sets. Thus, both k-sets are subsets of $\{1, 2, \ldots, n-1\}$. By induction there is a Hamilton path from u to v in $J(n-1, k)$. Because the vertices that are adjacent along that path also are adjacent in $J(n, k)$, let P' be the corresponding path from u to v in $J(n, k)$. The path P' contains all the vertices corresponding to k-subsets that do not contain n.

Let $w_1 = \{y_1, y_2, \ldots, y_{k-1}, y_k\}$ and $w_2 = \{y_1, y_2, \ldots, y_{k-1}, z_k\}$ be two adjacent vertices on P'. The vertex w_1 is adjacent to the vertex $w_3 = \{y_2, \ldots, y_{k-1}, y_k, n\}$, and the vertex w_2 is adjacent to the vertex $w_4 = \{y_2, \ldots, y_{k-1}, z_k, n\}$.

The subgraph X induced by $J(n, k)$ on all the subsets containing n clearly is isomorphic to $J(n-1, k-1)$. Thus, there is a path from w_3 to w_4 spanning all the vertices of X. Thus, remove the edge of P' between w_1 and w_2, add the edges w_1w_3 and w_2w_4, and then add the path from w_3 to w_4 spanning X. The resulting path is a Hamilton path in $J(n, k)$ with u and v as terminal vertices.

If $n > 2k$, then there always is an element x missing both subsets and the preceding argument establishes that $J(n, k)$ is Hamilton-connected. If $n = 2k$, there is exactly one subset that fails the criterion, namely, the complement of $\{1, 2, \ldots, k\}$. So we need to find a Hamilton path in $J(2k, k)$ from u to its complement.

Consider the k-subsets of $\{1, 2, \ldots, 2k\}$ not containing the element $2k$. The subgraph induced by $J(2k, k)$ on this collection of subsets is isomorphic to the graph $J(2k-1, k)$. It is Hamilton-connected by induction so that there is a Hamilton path from u to $w = \{1, 2, \ldots, k-1, 2k-1\}$. Let P be the copy of this path in $J(2k, k)$.

Now consider all the k-subsets of $\{1, 2, \ldots, 2k\}$ that contain the element $2k$. The subgraph Y' induced on this collection of sets is isomorphic to $J(2k-1, k-1)$ so that it has a spanning path from $\{1, 2, \ldots, k-1, 2k\}$ to $\{k+1, k+2, \ldots, 2k\}$. Because the intermediate terminal vertices on the two paths are adjacent, we have a Hamilton path in $J(2k, k)$ from u to its complement. This completes the proof. $\square$

The corollary below is the real target of this short paper. We need to define a particular graph first. Let $A = \{a_1, a_2, \ldots, a_m\}$ be a non-empty subset of $\{0, 1, 2, \ldots, n\}$ such that the elements are listed in the order $a_1 < a_2 < \cdots < a_m$. We define the graph $QJ(n, A)$ in the following way. For each $a_i \in A$, we include a copy of the Johnson graph $J(n, a_i)$. Thus far the Johnson graphs are vertex-disjoint. We then insert edges between $J(n, a_i)$ and $J(n, a_{i+1})$, for each i, using set inclusion, that is, we join an a_i-subset S_1 and an a_{i+1}-subset S_2 if S_1 is contained in S_2. The graph $QJ(n, A)$ can be pictured as having levels made up of Johnson graphs with edges between successive levels based on set inclusion.

Corollary 1.2. *The graph $QJ(n, A)$ is Hamilton-connected for all $n \geq 1$.*

Proof. If A is a singleton set, then $QJ(n, A)$ is a Johnson graph and the result follows from Theorem 1.1. Hence, we assume that A has at least two elements. Suppose that u and v are two vertices of $QJ(n, A)$ lying at different levels, where u has cardinality a_i, v has cardinality a_j, and $a_i < a_j$. Construct a path starting at u that spans the vertices at level a_i and terminates at an arbitrary vertex u_i at level a_i.

Choose a neighbor u_{i+1} of u_i at level a_{i+1} making certain it is distinct from v if $j = i + 1$. Then add the edge from u_i to u_{i+1} followed by a path spanning the vertices at level a_{i+1}. If v happens to lie at this level make certain the path terminates at v. Otherwise, the path can terminate at any vertex at level a_{i+1}.

It is obvious how to continue this until we have a path starting at u, terminating at v, and spanning all the vertices at levels a_i, a_{i+1} up through level a_j. If this happens to be all the levels of $QJ(n, A)$, then we have found a Hamilton path joining u and v. If we are missing levels, we then continue as follows.

If there are missing levels above level a_j, then remove an edge xy of the current path at level a_j and take distinct neighbors x' and y' of x and y, respectively, at level a_{j+1}. Then extend to a larger path by taking a path joining x' and y' spanning all the vertices at level a_{j+1}. If x and y don't have distinct neighbors at level a_{j+1}, then $a_{j+1} = n$ and the level is the singleton vertex $w = \{1, 2, 3, \ldots, n\}$ which is adjacent to everything at level a_j so that we replace the edge xy of the path with the 2-path xwy.

It is obvious how to continue adding the vertices one level at a time until we finish with the top level. We also can do the analogous extension with the levels below a_i until we achieve a Hamilton path in $Q(n, A)$ that has u and v as terminal vertices.

If u and v are at the same level. Then we start with a path spanning level a_i that has u and v as terminal vertices. We then extend the path through the other levels as above. This completes the proof. □

Corollary 1.2 allows us to process a variety of collections of sets of different cardinalities where we can move from one set to another either by a revolving door operation or restricted inclusions. This is what was required in [1].

References

[1] B. Alspach, Hamilton paths in Cayley graphs on Coxeter groups: I, preprint.