Johnson graphs are Hamilton-connected

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Abstract

We prove that the Johnson graphs are Hamilton-connected and apply this to obtain that another family of graphs is Hamilton-connected.

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1 Main Result

The Johnson graph $J(n, k)$, $0 \leq k \leq n$, is defined by letting the vertices correspond to the $k$-subsets of an $n$-set, where two vertices are adjacent if and only if the corresponding $k$-subsets have exactly $k - 1$ elements in common. A graph is Hamilton-connected if for any pair of distinct vertices $u, v$ there is a Hamilton path whose terminal vertices are $u$ and $v$. The graph with a single vertex is trivially Hamilton-connected.

In a recent paper [1], I needed a certain graph to be Hamilton-connected. This graph, defined below, contains vertex-disjoint Johnson graphs. The result I needed is embodied in the corollary below.

Theorem 1.1. The Johnson graph $J(n, k)$ is Hamilton-connected for all $n \geq 1$.

Proof. For ease of exposition, instead of talking about the vertex corresponding to a subset, we shall simply treat the subsets as if they are vertices so that we use equality notation between vertices and sets. The graphs $J(n, k)$ and $J(n, n - k)$ are isomorphic via the mapping that takes a $k$-subset to its complement.

The graphs $J(n, 0)$ and $J(n, n)$, $n \geq 1$, are isomorphic to the single vertex $K_1$ and trivially Hamilton-connected. The graphs $J(n, 1)$ and $J(n, n - 1), n \geq 1$, are isomorphic to the complete graph $K_n$. Complete graphs certainly are Hamilton-connected.

We proceed by double induction and when considering $J(n, k)$, the induction hypotheses are: $J(m, k')$ is Hamilton-connected whenever $k' < k$ and $m \geq k'$, or $J(m, k)$ is Hamilton-connected.
Hamilton-connected whenever \( m < n \) and \( m \geq k \). As noted above, \( J(n, 1) \) is Hamilton-connected for all \( n \geq 1 \). For a fixed \( k \) we start with \( J(k, k) \) and then proceed by going from \( J(n - 1, k) \) to \( J(n, k) \). Thus, the induction hypotheses make sense.

If \( k \leq n \leq 2k - 1 \), then \( n - k < k \) so that \( J(n, n - k) \) is Hamilton-connected by hypothesis. This, in turn, implies that \( J(n, k) \) is Hamilton-connected because \( J(n, k) \) and \( J(n, n - k) \) are isomorphic. Thus, it follows that \( J(n, k) \) is Hamilton-connected for all \( n \) satisfying \( k \leq n \leq 2k - 1 \).

For the remaining cases we need to actually show how to find appropriate Hamilton paths. The symmetric group \( S_n \) acts in the obvious way on the vertex set of \( J(n, k) \). This action is transitive so that it suffices to find a Hamilton path from the vertex \( u = \{1, 2, 3, \ldots, k\} \) to any other vertex. Let \( v = \{a_1, a_2, \ldots, a_k\} \) be an arbitrary vertex.

If there is an element \( x \) of \( \{1, 2, \ldots, n\} \) belonging to neither of the sets, we may relabel elements so that \( n \) is missing from both sets. Thus, both \( k \)-sets are subsets of \( \{1, 2, \ldots, n - 1\} \). By induction there is a Hamilton path from \( u \) to \( v \) in \( J(n - 1, k) \). Because the vertices that are adjacent along that path also are adjacent in \( J(n, k) \), let \( P' \) be the corresponding path from \( u \) to \( v \) in \( J(n, k) \). The path \( P' \) contains all the vertices corresponding to \( k \)-subsets that do not contain \( n \).

Let \( w_1 = \{y_1, y_2, \ldots, y_{k-1}, y_k\} \) and \( w_2 = \{y_1, y_2, \ldots, y_{k-1}, z_k\} \) be two adjacent vertices on \( P' \). The vertex \( w_1 \) is adjacent to the vertex \( w_3 = \{y_2, \ldots, y_{k-1}, y_k, n\} \), and the vertex \( w_2 \) is adjacent to the vertex \( w_4 = \{y_2, \ldots, y_{k-1}, z_k, n\} \).

The subgraph \( X \) induced by \( J(n, k) \) on all the subsets containing \( n \) clearly is isomorphic to \( J(n - 1, k - 1) \). Thus, there is a path from \( w_3 \) to \( w_4 \) spanning all the vertices of \( X \). Thus, remove the edge of \( P' \) between \( w_1 \) and \( w_2 \), add the edges \( w_1w_3 \) and \( w_2w_4 \), and then add the path from \( w_3 \) to \( w_4 \) spanning \( X \). The resulting path is a Hamilton path in \( J(n, k) \) with \( u \) and \( v \) as terminal vertices.

If \( n > 2k \), then there always is an element \( x \) missing both subsets and the preceding argument establishes that \( J(n, k) \) is Hamilton-connected. If \( n = 2k \), there is exactly one subset that fails the criterion, namely, the complement of \( \{1, 2, \ldots, k\} \). So we need to find a Hamilton path in \( J(2k, k) \) from \( u \) to its complement.

Consider the \( k \)-subsets of \( \{1, 2, \ldots, 2k\} \) not containing the element \( 2k \). The subgraph induced by \( J(2k, k) \) on this collection of subsets is isomorphic to the graph \( J(2k - 1, k) \). It is Hamilton-connected by induction so that there is a Hamilton path from \( u \) to \( w = \{1, 2, \ldots, k - 1, 2k - 1\} \). Let \( P \) be the copy of this path in \( J(2k, k) \).

Now consider all the \( k \)-subsets of \( \{1, 2, \ldots, 2k\} \) that contain the element \( 2k \). The subgraph \( Y' \) induced on this collection of sets is isomorphic to \( J(2k - 1, k - 1) \) so that it has a spanning path from \( \{1, 2, \ldots, k - 1, 2k\} \) to \( \{k + 1, k + 2, \ldots, 2k\} \). Because the intermediate terminal vertices on the two paths are adjacent, we have a Hamilton path in \( J(2k, k) \) from \( u \) to its complement. This completes the proof.

The corollary below is the real target of this short paper. We need to define a particular graph first. Let \( A = \{a_1, a_2, \ldots, a_m\} \) be a non-empty subset of \( \{0, 1, 2, \ldots, n\} \) such that the elements are listed in the order \( a_1 < a_2 < \cdots < a_m \). We define the graph \( QJ(n, A) \) in the following way. For each \( a_i \in A \), we include a copy of the Johnson graph \( J(n, a_i) \). Thus far the Johnson graphs are vertex-disjoint. We then insert edges between \( J(n, a_i) \) and \( J(n, a_{i+1}) \), for each \( i \), using set inclusion, that is, we join an \( a_i \)-subset \( S_1 \) and an \( a_{i+1} \)-subset \( S_2 \) if \( S_1 \) is contained in \( S_2 \). The graph \( QJ(n, A) \) can be pictured as having levels made up of Johnson graphs with edges between successive levels based on set inclusion.
**Corollary 1.2.** The graph $QJ(n, A)$ is Hamilton-connected for all $n \geq 1$.

**Proof.** If $A$ is a singleton set, then $QJ(n, A)$ is a Johnson graph and the result follows from Theorem 1.1. Hence, we assume that $A$ has at least two elements. Suppose that $u$ and $v$ are two vertices of $QJ(n, A)$ lying at different levels, where $u$ has cardinality $a_i$, $v$ has cardinality $a_j$, and $a_i < a_j$. Construct a path starting at $u$ that spans the vertices at level $a_i$ and terminates at an arbitrary vertex $u_i$ at level $a_i$.

Choose a neighbor $u_{i+1}$ of $u_i$ at level $a_{i+1}$ making certain it is distinct from $v$ if $j = i + 1$. Then add the edge from $u_i$ to $u_{i+1}$ followed by a path spanning the vertices at level $a_{i+1}$. If $v$ happens to lie at this level make certain the path terminates at $v$. Otherwise, the path can terminate at any vertex at level $a_{i+1}$.

It is obvious how to continue this until we have a path starting at $u$, terminating at $v$, and spanning all the vertices at levels $a_i, a_{i+1}$ up through level $a_j$. If this happens to be all the levels of $QJ(n, A)$, then we have found a Hamilton path joining $u$ and $v$. If we are missing levels, we then continue as follows.

If there are missing levels above level $a_j$, then remove an edge $xy$ of the current path at level $a_j$ and take distinct neighbors $x'$ and $y'$ of $x$ and $y$, respectively, at level $a_{j+1}$. Then extend to a larger path by taking a path joining $x'$ and $y'$ spanning all the vertices at level $a_{j+1}$. If $x$ and $y$ don’t have distinct neighbors at level $a_{j+1}$, then $a_{j+1} = n$ and the level is the singleton vertex $w = \{1, 2, 3, \ldots, n\}$ which is adjacent to everything at level $a_j$ so that we replace the edge $xy$ of the path with the 2-path $xwy$.

It is obvious how to continue adding the vertices one level at a time until we finish with the top level. We also can do the analogous extension with the levels below $a_i$ until we achieve a Hamilton path in $Q(n, A)$ that has $u$ and $v$ as terminal vertices.

If $u$ and $v$ are at the same level. Then we start with a path spanning level $a_i$ that has $u$ and $v$ as terminal vertices. We then extend the path through the other levels as above. This completes the proof.

Corollary 1.2 allows us to process a variety of collections of sets of different cardinalities where we can move from one set to another either by a revolving door operation or restricted inclusions. This is what was required in [1].

**References**