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A classification of connected cubic vertex-transitive bi-Cayley graphs over semidihedral group*

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Abstract

A graph Γ is said to be a *bi-Cayley graph* over a group H if there exists a subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to H acting semiregularly on its vertex set with two orbits. In this paper, we give a complete classification of connected cubic vertex-transitive bi-Cayley graphs over semidihedral group. As a byproduct, we construct a family of vertex-transitive non-Cayley graphs.

Keywords: Semidihedral group, bi-Cayley graph, vertex-transitive.

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1 Introduction

Throughout this paper, all groups are assumed to be finite, and all graphs are assumed to be finite, connected, simple and undirected. For a graph Γ, let $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$ denote vertex set, edge set and arc set of Γ, respectively. A graph Γ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* if the full automorphism group $\mathrm{Aut}\,(\Gamma)$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectively. For other terminology related to group theory and graph theory not defined here, we refer the reader to [1, 11].

Let G be a group and S be a subset of G such that $S^{-1} = S$ and $1 \notin S$. Then the *Cayley graph* $\Gamma = \mathrm{Cay}(G, S)$ over G with respect to S is defined as the graph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$. Similarly, for a given group H, let R, L and S be subsets of H such that $R^{-1} = R$, $L^{-1} = L$ and $R \cup L$ does not contain the identity element of H. The *bi-Cayley graph* over H denoted by $\mathrm{BiCay}(H, R, L, S)$ is the graph having vertex set the union of the right part $H_0 = \{h_0 \mid h \in H\}$ and the left part $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the right edges $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$. When $|R| = |L| = s$, $\mathrm{BiCay}(H, R, L, S)$ is said to be an *s-type bi-Cayley graph*.

The triple (R, L, S) of three subsets R, L, S of a group H is called *bi-Cayley triple* if $R = R^{-1}, L = L^{-1}$ and $1 \in S$. Two bi-Cayley triples (R, L, S) and (R', L', S') of a group H are said to be *equivalent*, denoted by $(R, L, S) \equiv (R', L', S')$, if either $(R', L', S') = (R, L, S)^\alpha$ or $(R', L', S') = (L, R, S^{-1})^\alpha$ for some automorphism α of H. By Proposition 2.1(3)–(4), the bi-Cayley graphs corresponding to two equivalent bi-Cayley triples of the same group are isomorphic.

In the study of bi-Cayley graphs, a considerable attention was given to the following problem: for a given finite group H, classify bi-Cayley graphs over H with specific symmetry properties. For example, vertex-transitive (edge-transitive) generalized Petersen graphs had been classified in [4, 9]. Marušič and Pisanski in [6] classified all cubic arc-transitive bi-Cayley graphs over dihedral group. All tetravalent edge-transitive bicirculants (bi-Cayley group over cyclic group) were characterized in [5], a classification of cubic edge-transitive bi-Cayley graphs over inner abelian p-groups were presented in [10], and all cubic vertex-transitive bi-Cayley graphs over abelian groups were classified in [13]. Recently, Zhang and Zhou in [12] gave a classification of cubic edge-transitive bi-Cayley graphs over dihedral groups.

Motivated by the works listed above, in this paper, we shall investigate cubic bi-Cayley graphs over semidihedral groups. Recall that the *semidihedral group* of order $4n$ with n an even is defined as follows:

$$SD_{4n} = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{n-1} \rangle.$$

Note that all cubic bi-Cayley graphs over abelian groups have been classified in [13]. So we assume that $n \geq 4$.

The Petersen graph is a bi-Cayley graph over a cyclic group of order 5, and the Petersen graph is also a vertex-transitive non-Cayley graph. There are many research focusing on the classification of vertex-transitive non-Cayley graphs, see [3, 4, 7, 8]. By Magma, we found some examples of vertex-transitive non-Cayley graph Γ, where Γ is a cubic vertex-trasntive bi-Cayley graph over SD_{4n}. So another motivation for us to consider cubic vertex-transitive bi-Cayley graphs over SD_{4n} is to construct some kind of vertex-transitive non-Cayley graphs.

In [2] a classification of cubic edge-transitive bi-Cayley graphs over semidihedral group is given. For the completeness of the results, we list the main theorem in [2] in the following. (For the definition of $CQ(t, n)$, we refer the reader to [12])

Theorem 1.1 ([2, Theorem 1]). *Let Γ be a cubic connected bi-Cayley graph over semidihedral group SD_{4n}. Then Γ is edge-transitive if and only if (R, L, S) is equivalent to one of the following triples. Furthermore, all of the corresponding graphs are arc-transitive.*

(1) $(R, L, S) \equiv (\{b\}, \{ba^4\}, \{1, a\})$ with $n = 4$ and Γ is isomorphic to F032A.

(2) $(R, L, S) \equiv (\{b\}, \{ba^2\}, \{1, a\})$ with $n = 6$ and Γ is isomorphic to F048A.

(3) $(R, L, S) \equiv (\{b\}, \{ba^{2t}\}, \{1, a\})$, with t an odd, $3 \le t \le n-3$, $n \mid 2(t^2+t+1)$ and Γ is isomorphic to $CQ(t, n)$.

(4) $(R, L, S) \equiv (\{b, ba^2\}, \{a, a^{-1}\}, \{1\})$ with $n = 4$ and Γ is isomorphic to F032A.

(5) $(R, L, S) \equiv (\{b, ba^6\}, \{a, a^{-1}\}, \{1\})$ with $n = 10$ and Γ is isomorphic to F080A.

(6) $(R, L, S) \equiv (\{b, ba^2\}, \{a, a^{-1}\}, \{1\})$ with $n = 12$ and Γ is isomorphic to F096A.

In this paper, we determine all cubic vertex-transitive bi-Cayley graphs over semidihedral group SD_{4n}. The main results are in the following.

Theorem 1.2. *Let Γ be a 0-type cubic connected bi-Cayley graph over SD_{4n}. Then Γ is a Cayley graph and Γ is isomorphic to* $\mathrm{BiCay}(SD_{4n}, \emptyset, \emptyset, S)$*, where $S = \{1, a, b\}$, $\{1, ba, b\}$ or $\{1, ba, a\}$.*

Theorem 1.3. *Let Γ be a 1-type cubic connected bi-Cayley graph over SD_{4n}. Then Γ is a vertex-transitive graph if and only if one of the followings holds. Furthermore all of the corresponding graphs are Cayley graphs.*

(1) $(R, L, S) \equiv (\{b\}, \{ba^i\}, \{1, a^j\})$ *with i an even, j an odd and the greatest common divisor of i, j and n is equal to 1.*

(2) $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ *with* $(l-1)^2 \equiv 1, n-1 \pmod{2n}$.

Theorem 1.4. *Let Γ be a 2-type cubic connected bi-Cayley graph over SD_{4n}. Then Γ is a vertex-transitive graph if and only if one of the followings holds:*

(1) $(R, L, S) \equiv (\{b, ba^n\}, \{ba, ba^{n+1}\}, \{1\})$*;*

(2) $(R, L, S) \equiv (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ *with* $2l^2 \equiv 2 \pmod{2n}$*;*

(3) $(R, L, S) \equiv (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ *with* $2l^2 \equiv -2 \pmod{2n}$.

Furthermore, the graphs corresponding to (1) and (2) are Cayley graphs and the graph corresponding to (3) is a non-Cayley graph.

Theorem 1.1 gives a classification of cubic edge-transitive and arc-transitive bi-Cayley graphs over SD_{4n}. Theorems 1.2, 1.3 and 1.4 give a classification of cubic vertex-transitive bi-Cayley graphs over SD_{4n}. As a byproduct, we construct a family of vertex-transitive non-Cayley graphs which correspond to (3) in Theorem 1.4.

2 Preliminary

In this section, we give two properties of bi-Cayley graph.

Proposition 2.1 ([14, Lemma 3.1]). *For a bi-Cayley graph* $\mathrm{BiCay}(H, R, L, S)$ *over* H*, the following hold.*

(1) *H is generated by $R \cup L \cup S$.*

(2) *Up to graph isomorphism, S can be chosen to contain the identity of H.*

(3) *For any automorphism α of H,* $\mathrm{BiCay}(H,\ R,\ L,\ S) \cong \mathrm{BiCay}(H,\ R^{\alpha},\ L^{\alpha},\ S^{\alpha})$.

(4) $\mathrm{BiCay}(H,\ R,\ L,\ S) \cong \mathrm{BiCay}(H,\ L,\ R,\ S^{-1})$.

Let $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$. For an automorphism α of H and $x, y, g \in H$, define two permutations of $V(\Gamma) = H_0 \cup H_1$ as follows:

$$\delta_{\alpha,x,y}\colon\ h_0 \mapsto (xh^{\alpha})_1,\ h_1 \mapsto (yh^{\alpha})_0,\ \forall h \in H,$$
$$\sigma_{\alpha,g}\colon\ h_0 \mapsto (h^{\alpha})_0,\ h_1 \mapsto (gh^{\alpha})_1,\ \forall h \in H.$$

Set

$$I = \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}\,(H)\ s.t.\ R^{\alpha} = x^{-1}Lx,\ L^{\alpha} = y^{-1}Ry,\ S^{\alpha} = y^{-1}S^{-1}x\},$$
$$F = \{\sigma_{\alpha,g} \mid \alpha \in \mathrm{Aut}\,(H)\ s.t.\ R^{\alpha} = R,\ L^{\alpha} = g^{-1}Lg,\ S^{\alpha} = g^{-1}S\}.$$

Proposition 2.2 ([14, Theorem 1.1]). *Let* $\Gamma = \mathrm{BiCay}(H,\ R,\ L,\ S)$ *be a bi-Cayley graph over the group H. Then* $N_{\mathrm{Aut}\,(\Gamma)}(R(H)) = R(H) \rtimes F$ *if* $I = \emptyset$ *and* $N_{\mathrm{Aut}\,(\Gamma)}(R(H)) = R(H)\langle F,\ \delta_{\alpha,x,y}\rangle$ *if* $I \neq \emptyset$ *and* $\delta_{\alpha,x,y} \in I$*. Furthermore, for any* $\delta_{\alpha,x,y} \in I$*, we have the following:*

(1) *$\langle R(H),\ \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$;*

(2) *if α has order 2 and $x = y = 1$, then Γ is isomorphic to the Cayley graph* $\mathrm{Cay}(\bar{H},\ R \cup \alpha S)$*, where* $\bar{H} = H \rtimes \langle\alpha\rangle$.

3 Proof of main theorems

In the beginning of this section, firstly we give some basic properties of SD_{4n} in the following lemma without proof which are needed in the proof of our main Theorems.

Lemma 3.1. *The following hold.*

(1) *$SD_{4n} = \langle a\rangle \cup b\langle a\rangle$. Where $b\langle a\rangle = \{ba^{2i}\} \cup \{ba^{2i+1}\}$ with $0 \leqslant i \leqslant n-1$, and furthermore every element of set $\{ba^{2i}\}$ has order 2 and every element of set $\{ba^{2i+1}\}$ has order 4.*

(2) $\mathrm{Aut}\,(SD_{4n})$ *is transitive on sets $\{ba^{2i}\}$ and $\{ba^{2i+1}\}$ with $0 \leqslant i \leqslant n-1$.*

(3) *If $SD_{4n} = \langle x, y\rangle$, then there exists $\alpha \in$* $\mathrm{Aut}\,(SD_{4n})$ *mapping $\{x, y\}$ to one of the following subsets:$\{a, b\}$, $\{ba, b\}$ $\{ba, a\}$.*

For any integers i, j satisfying $(i, 2n) = 1$ and j is even, we have $\langle a^i, ba^j\rangle = \langle a, b\rangle = SD_{4n}$ and the map

$$\psi_{i,j} \colon a \mapsto a^i, b \mapsto ba^j$$

can induce an automorphism of SD_{4n}. In the following of this section we shall use $\psi_{i,j}$ to denote the automorphism of SD_{4n} induced by the above map.

Proof of Theorem 1.2. Since Γ is a 0-type bi-Cayley graph, $R = L = \emptyset$. By Proposition 2.1(1) and (2), let $S = \{1, g, h\}$ with $SD_{4n} = \langle g, h\rangle$. By Lemma 3.1(3), S is equivalent to one of the following three subsets: $\{1, a, b\}, \{1, ba, b\}, \{1, ba, a\}$. It is easy to check that $\psi_{-1,0}$, $\psi_{n+1,0}$ and $\psi_{-1,n+2}$ are three automorphisms of SD_{4n} of order 2 such that $\{1, a, b\}^{\psi_{-1,0}} = \{1, a, b\}^{-1}$, $\{1, ba, b\}^{\psi_{n+1,0}} = \{1, ba, b\}^{-1}$ and $\{1, ba, a\}^{\psi_{-1,n+2}} = \{1, ba, a\}^{-1}$. By Proposition 2.2, Γ is a Cayley graph. □

In order to get the classification of 1-type graph Γ, we give the following Lemma.

Lemma 3.2 ([12, Proposition 5.1])**.** *Let $K = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle$ be a dihedral group of order $2n$, and $\Gamma_1 = \mathrm{BiCay}(K, \{b\}, \{ba^i\}, \{1, ba^j\})$ be a cubic bi-Cayley graph over K. If Γ_1 is vertex-transitive, then $(j, n) = 1$ and $j^2 \equiv \pm(j - i)^2 \pmod n$.*

Proof of Theorem 1.3. Assume that Γ is a 1-type cubic vertex-transitive bi-Cayley graph over SD_{4n}. By the definition of 1-type bi-Cayley graph, we can let $R = \{x\}$, $L = \{y\}$ and $S = \{1, z\}$. As $R = R^{-1}$ and $L = L^{-1}$, both x and y are involutions. Since Γ is connected, by Proposition 2.1(1), $SD_{4n} = \langle x, y, z\rangle$. We confirm that there is at least one of x and y is not contained in $\langle a\rangle$. If not, $x = y = a^n$ implies that $SD_{4n} = \langle a^n, z\rangle$ for some $z \in SD_{4n}$, a contradiction. Without loss of generality, assume that $x \in b\langle a^2\rangle$. By Lemma 3.1(2), $\mathrm{Aut}\,(SD_{4n})$ acts transitively on the set $\{ba^{2k} \mid 0 \le k \le n-1\}$. So we assume that $x = b$. Now $y = a^n$ or $y = ba^i$ for some even i.

Case 1: $y = a^n \in \langle a\rangle$.

In this case, $z = a^m$ or ba^m. Since $SD_{4n} = \langle x, y, z\rangle$, one has $(m, 2n) = 1$. The map $a^m \mapsto a, b \mapsto b$ can induce an automorphism α of SD_{4n}, such that $(R, L, S)^\alpha = (\{b\}, \{a^n\}, \{1, a\})$ or $(\{b\}, \{a^n\}, \{1, ba\})$.

If $(R, L, S) \equiv (\{b\}, \{a^n\}, \{1, a\})$, then $(1_0, 1_1, (a^n)_1, (a^n)_0, (a^{n+1})_1, a_1)$ is the unique 6-cycle passing through 1_0. On the other hand, there exists a 6-cycle $(1_1, (a^n)_1, (a^{n-1})_0, (a^{n-1})_1, (a^{-1})_1, (a^{-1})_0)$ passing through 1_1 but not passing through 1_0, contrary to the vertex-transitivity of Γ.

Suppose that $(R, L, S) \equiv (\{b\}, \{a^n\}, \{1, ba\})$. Then there is a 5-cycle $(1_1, (a^n)_1, (a^n)_0, (ba^{n+1})_1, (ba^{n+1})_0)$ passing through 1_1 but not passing through 1_0. On the other hand, $(1_0, 1_1, (a^n)_1, (ba)_0, (ba)_1)$ and $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (ba)_1)$ are all 5-cycles passing through 1_0, and these also passing through 1_1, contrary to the vertex-transitivity of Γ.

Case 2: $y = ba^i \in b\langle a^2\rangle$ for some even i.

In this case, $z = a^j$ or ba^j for some odd j.

Subcase 2.1: $z = a^j \in \langle a\rangle$ for some odd j.

In this case, it is easy to check that $\psi_{-1,i}$ is an automorphism of SD_{4n} of order 2 such that $R^{\psi_{-1,i}} = L$, $L^{\psi_{-1,i}} = R$ and $S^{\psi_{-1,i}} = S^{-1}$. By Proposition 2.2(2), Γ is a Cayley graph. If $(j, 2n) \neq 1$, then $\langle a\rangle = \langle a^i, a^j\rangle$. So the greatest common divisor of i, j and n is equal to 1. This is the graph corresponding to (1) in the theorem.

Subcase 2.2: $z = ba^j \in b\langle a\rangle$ for some odd j.

Suppose that $j = \frac{n}{2}$ with odd $\frac{n}{2}$. By the connectivity of Γ, $\langle a\rangle = \langle a^{\frac{n}{2}}, a^i\rangle$, and hence $\langle a^i\rangle = \langle a^2\rangle$ or $\langle a^4\rangle$. By Proposition 2.1(3), we may assume that $a^i = a^2$ or a^4. Now Γ is isomorphic to a bi-Cayley graph with $(R, L, S) \equiv (\{b\}, \{ba^2\}, \{1, ba^{\frac{n}{2}}\})$ or $(R, L, S) \equiv (\{b\}, \{ba^4\}, \{1, ba^{\frac{n}{2}}\})$. In the above two cases, we can find that there are two 6-cycles passing through 1_0, namely $(1_0, b_0, b_1, (a^{\frac{3n}{2}})_0, (ba^{\frac{3n}{2}})_0, 1_1)$ and $(1_0, b_0, (a^{\frac{n}{2}})_1, (a^{\frac{n}{2}})_0, (ba^{\frac{n}{2}})_0, (ba^{\frac{n}{2}})_1)$. On the other hand, $(1_1, 1_0, b_0, b_1, (a^{\frac{3n}{2}})_0, (ba^{\frac{3n}{2}})_0)$ is the unique 6-cycle passing through 1_1, contrary to the vertex-transitivity of Γ. So, we may assume that $z \neq ba^{\frac{n}{2}}$.

Suppose that $(j, 2n) \neq 1$. Since $\langle a\rangle = \langle a^i, a^j\rangle$ with i an even and j an odd, the greatest common divisor of i, j and n is equal to 1. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 4 respectively. Since $j \not\equiv \frac{n}{2} \pmod{2n}$, one can see that $(1_0, 1_1, (ba^{n+j})_0, (ba^{n+j})_1, (a^n)_0, (a^n)_1, (ba^j)_0, (ba^j)_1)$ is the unique 8-cycle passing through 1_0 and it is also the unique 8-cycle passing through 1_1 too.

Since $(a^n)_0$ and $(a^n)_1$ are the unique vertices that have the longest distance from 1_0 and 1_1 in the 8-cycle, respectively, $\{1_0, (a^n)_0\}$ and $\{1_1, (a^n)_1\}$ are blocks of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$. Let

$$\mathcal{C}_0 = \{\{1_0, (a^n)_0\}^{R(h)} \mid h \in SD_{4n}\}, \quad \mathcal{C}_1 = \{\{1_1, (a^n)_1\}^{R(h)} \mid h \in SD_{4n}\}$$

and $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. Then $\mathcal{C}$ is a complete block system of $\mathrm{Aut}(\Gamma)$.

Let $\Gamma_{\mathcal{C}}$ be the quotient graph. Let $N = \langle R(a^n)\rangle$ and let K be the kernel of $\mathrm{Aut}(\Gamma)$ acting on $\mathcal{C}$. Now one can show that $K = N$ and $\Gamma_{\mathcal{C}}$ has valence 3 and $K = N$ is semiregular. Since $R(SD_{4n})$ acts on $V(\Gamma)$ semiregularly with two orbits, $R(SD_{4n})/N$ acts on $\mathcal{C}$ semiregularly with two orbits $\mathcal{C}_0$ and $\mathcal{C}_1$. So the quotient graph $\Gamma_{\mathcal{C}}$ is a bi-Cayley graph over $R(SD_{4n})/N$. Let $\overline{SD_{4n}} = R(SD_{4n})/N$ and let $\bar{h} = hN$ for any $h \in R(SD_{4n})$. Now $\overline{SD_{4n}} = \langle \bar{a}, \bar{b} \mid \bar{a}^n = \bar{b}^2 = (\bar{a}\bar{b})^2 = \bar{1}\rangle \cong D_{2n}$. Also we can assume that

$$V(\Gamma_{\mathcal{C}}) = \{\bar{h}_0 \mid \bar{h} \in \overline{SD_{4n}}\} \cup \{\bar{h}_1 \mid \bar{h} \in \overline{SD_{4n}}\}.$$

Note that for any $\bar{h} \in \overline{SD_{4n}}$

$$N_{\Gamma_{\mathcal{C}}}(\bar{h}_0) = \{\overline{bh}_0, \bar{h}_1, \overline{ba^j h}_1\}, \quad N_{\Gamma_{\mathcal{C}}}(\bar{h}_1) = \{\overline{ba^i h}_1, \bar{h}_0, \overline{ba^{n+j} h}_0\}.$$

So we may view $\Gamma_{\mathcal{C}}$ as the bi-Cayley graph $\mathrm{BiCay}(\overline{SD_{4n}}, \{\bar{b}\}, \{\overline{ba^i}\}, \{\bar{1}, \overline{ba^j}\})$. Since Γ is vertex-transive, the quotient graph $\Gamma_{\mathcal{C}}$ is also vertex-transitive. Since $\Gamma_{\mathcal{C}}$ is a 1-type cubic vertex-transitive bidihedrant, $\Gamma_{\mathcal{C}}$ is a Cayley graph by [12, Proposition 5.1]. Also by Lemma 3.2, $(j, n) = 1$ and $j^2 \equiv \pm(j-i)^2 \pmod{n}$. This implies that $(j, 2n) = 1$, which is contrary to $(j, 2n) \neq 1$.

So we can assume that $(j, 2n) = 1$. The map $a^j \mapsto a$, $b \mapsto b$ can induce an automorphism β of SD_{4n}, such that $(R, L, S)^\beta \equiv (\{b\}, \{ba^l\}, \{1, ba\})$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 4, respectively. Considering possible 8-cycles containing 1_0 and 1_1, we see that if $l \not\equiv 2, n, n+2, \frac{n}{2}+1 \pmod{2n}$, then $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is the unique 8-cycle passing through 1_0 and it is the unique 8-cycle passing through 1_1 too. In this case, any automorphism of Γ sends spokes to spokes. Furthermore, $\{1_0, (a^n)_0\}$ and $\{1_1, (a^n)_1\}$ are blocks of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$ and by a similar way, one can see that $(l-1)^2 \equiv \pm 1 \pmod{n}$. So $(l-1)^2 \equiv \pm 1 \pmod{2n}$, $(l-1)^2 \equiv n+1 \pmod{2n}$ or $(l-1)^2 \equiv n-1 \pmod{2n}$. It is easy to find that $l \equiv 2, n, n+2 \pmod{2n}$ also satisfy $(l-1)^2 \equiv \pm 1 \pmod{2n}$. In the following, we divide the proof into four subcases.

Subsubcase 2.2.1: $(R, L, S) \equiv (\{b\}, \{ba^{\frac{n}{2}+1}\}, \{1, ba\})$ with $\frac{n}{2}$ an odd.

In this case, we can find that there are two 6-cycles passing through 1_1, namely $(1_1, (ba^{\frac{n}{2}+1})_1, (ba^{\frac{n}{2}+1})_0, (a^{\frac{3n}{2}})_1, (ba)_1, 1_0)$ and $(1_1, (ba^{\frac{n}{2}+1})_1, (a^{\frac{n}{2}})_0, (a^{\frac{n}{2}})_1, (ba^{n+1})_1, (ba^{n+1})_0)$. On the other hand, $(1_1, (ba^{\frac{n}{2}+1})_1, (ba^{\frac{n}{2}+1})_0, (a^{\frac{3n}{2}})_1, (ba)_1, 1_0)$ is the unique 6-cycle passing through 1_0, contrary to the vertex-transitivity of Γ.

Subsubcase 2.2.2: $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ with $(l-1)^2 \equiv \pm 1 \pmod{2n}$.

If $(l-1)^2 \equiv -1 \pmod{2n}$, then $l^2 - 2l + 2 \equiv 0 \pmod{2n}$ implies that $n | (\frac{l^2}{2} - l + 1)$. Since l is even, $\frac{l^2}{2} - l + 1$ is odd. This implies that n is also odd, a contradiction.

Assume that $(l-1)^2 \equiv 1 \pmod{2n}$. Then $((l-1)^2, 2n) = 1$, and furthermore $(n+1-l, 2n) = 1$. It is easy to check that $\psi_{n+1-l,l}$ is an automorphism of SD_{4n} of order 2 such that $R^{\psi_{n+1-l,l}} = L$, $L^{\psi_{n+1-l,l}} = R$ and $S^{\psi_{n+1-l,l}} = S^{-1}$. By Proposition 2.2(2), Γ is a Cayley graph. This is the graph of type (2) in the theorem.

Subsubcase 2.2.3: $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ with $(l-1)^2 \equiv n + 1 (\mod 2n)$.

Suppose that Γ is vertex-transitive. Then there is an automorphism ω_2 of Γ such that ${1_0}^{\omega_2} = 1_1$. Note that ${1_1}^{\omega_2} = 1_0$ or ${1_1}^{\omega_2} = (ba^{n+1})_0$.

Suppose that ${1_1}^{\omega_2} = 1_0$. Then ${b_0}^{\omega_2} = (ba^l)_1$ and $({ba^l}_1)^{\omega_2} = b_0$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 5, respectively. It is easy to find that both $(ba^l)_0$ and $(a^{l-1})_0$ are adjacent with $(ba^l)_1$, furthermore $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (ba^l)_0\}$ or $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (a^{l-1})_0\}$.

Assume that $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (ba^l)_0\}$. Then ${(a^{n-1})_1}^{\omega_2} = (ba^l)_0$ and ${(a^{n-1})_0}^{\omega_2} = (a^{n+l-1})_1$. There is a unique 10-cycle passing through vertexes $1_1, 1_0, b_0, (a^{n-1})_1$ and $(a^{n-1})_0$, that is $(1_1, 1_0, b_0, (a^{n-1})_1, (a^{n-1})_0, (ba^n)_1, (ba^n)_0, (a^n)_0, (ba^{n+1})_1, (ba^{n+1})_0)$. On the other hand, there are two 10-cycles passing through vertexes $1_0, 1_1, (ba^l)_1, (ba^l)_0$ and $(a^{n+l-1})_1$, that are $(1_0, 1_1, (ba^l)_1, (ba^l)_0, (a^{n+l-1})_1, (a^{n+l-1})_0, (ba^{n+l-1})_0, (ba^{n+l-1})_1, (a^{n-1})_1, b_0)$ and $(1_0, 1_1, (ba^l)_1, (ba^l)_0, (a^{n+l-1})_1, (a^{n+l-1})_0, (ba^{n+l})_1, (a^n)_1, (ba)_0, (ba)_1)$, a contradiction. So $\{b_0, (a^{n-1})_1\}^{\omega_2} = \{(ba^l)_1, (a^{l-1})_0\}$, which implies ${(a^{n-1})_1}^{\omega_2} = (a^{l-1})_0$. By a similar reason, one can show that $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{b_0, (a^{n-1})_1\}$, and hence ${(a^{l-1})_0}^{\omega_2} = (a^{n-1})_1$. Therefore

$$1_0 \overset{\omega_2}{\text{---}} 1_1 \overset{\omega_2}{\text{---}} 1_0, \quad b_0 \overset{\omega_2}{\text{---}} (ba^l)_1 \overset{\omega_2}{\text{---}} b_0, \quad (a^{l-1})_0 \overset{\omega_2}{\text{---}} (a^{n-1})_1 \overset{\omega_2}{\text{---}} (a^{l-1})_0;$$

Now let us consider the subgraphs of Γ induced by the vertices at distance from $(a^{l-1})_0$ and $(a^{n-1})_1$ at most 5, respectively. We can get

$$(a^{n-1})_0 \overset{\omega_2}{\text{---}} (a^{l-1})_1 \overset{\omega_2}{\text{---}} (a^{n-1})_0,$$

$$(ba^{n-1})_0 \overset{\omega_2}{\text{---}} (ba^{2l-1})_1 \overset{\omega_2}{\text{---}} (ba^{n-1})_0,$$

$$(a^{2(l-1)})_0 \overset{\omega_2}{\text{---}} (a^{2(n-1)})_1 \overset{\omega_2}{\text{---}} (a^{2(l-1)})_0;$$

By a similar way, we can get ${(a^{k(n-1)})_1}^{\omega_2} = (a^{k(l-1)})_0$ for any k. By inserting $k = l-1$, we have ${(a^{(l-1)(n-1)})_1}^{\omega_2} = {(a^{n+1-l})_1}^{\omega_2} = (a^{n+1})_0$. On the other hand, $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is the unique 8-cycle passing through 1_0 and 1_1 implies that $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is mapped to $(1_1, 1_0, (ba)_1, (ba)_0, (a^n)_1, (a^n)_0, (ba^{n+1})_1, (ba^{n+1})_0)$ by ω_2. So ${(ba^{n+1})_1}^{\omega_2} = (ba)_0$, furthermore ${(a^{n+1-l})_1}^{\omega_2} = a_0$, a contradiction.

Suppose that ${1_1}^{\omega_2} = (ba^{n+1})_0$. Then ${b_0}^{\omega_2} = (ba^l)_1$ and $({ba^l}_1)^{\omega_2} = (a^{n+1})_0$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 5, respectively. Since both $(a^{n+1})_1$ and $(ba^{n+2})_1$ are adjacent with $(a^{n+1})_0$, we have $\{{ba^l}_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (a^{n+1})_1\}$ or $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (ba^{n+2})_1\}$.

Assume that $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (ba^{n+2})_1\}$. Then ${(a^{l-1})_0}^{\omega_2} = (ba^{n+2})_1$ and ${(a^{l-1})_1}^{\omega_2} = (ba^{n+2})_0$. There is a unique 10-cycle passing through vertexes $1_0, 1_1, (ba^l)_1, (a^{l-1})_0$ and $(a^{l-1})_1$, that is $(1_0, 1_1, (ba^l)_1, (a^{l-1})_0, (a^{l-1})_1, (ba^{n+l})_0, (ba^{n+l})_1, (a^n)_1, (ba)_0, (ba)_1)$. On the other hand, there are two 10-cycles passing through vertexes $1_1, (ba^{n+1})_0, (a^{n+1})_0, (ba^{n+2})_1$ and $(ba^{n+2})_0$, that are $(1_1, (ba^{n+1})_0, (a^{n+1})_0, (ba^{n+2})_1, (ba^{n+2})_0, a_1, a_0, (ba)_0, (ba)_1, 1_0)$ and $(1_1, (ba^{n+1})_0, (a^{n+1})_0, (ba^{n+2})_1, (ba^{n+2})_0, a_1, (ba^{l+1})_1, (a^l)_0, (ba^l)_0, (ba^l)_1)$, a contradiction. So $\{(ba^l)_1, (a^{l-1})_0\}^{\omega_2} = \{(a^{n+1})_0, (a^{n+1})_1\}$, and hence ${(a^{l-1})_0}^{\omega_2} = (a^{n+1})_1$. Therefore ω_2 maps the path $P_1\colon 1_0, 1_1, (ba^l)_1, (a^{l-1})_0$ to the path $Q_1\colon 1_1, (ba^{n+1})_0, (a^{n+1})_0, (a^{n+1})_1$ in this order, namely ${1_0}^{\omega_2} = 1_1$, ${1_1}^{\omega_2} = (ba^{n+1})_0$, ${(ba^l)_1}^{\omega_2} = (a^{n+1})_0$ and ${(a^{l-1})_0}^{\omega_2} = (a^{n+1})_1$. Similarly, we consider the subgraphs of Γ induced by the vertices at distance from $(a^{l-1})_0$ and $(a^{n+1})_1$ at most 5, respectively. One can show that ω_2 maps the path $P_2\colon (a^{l-1})_0, (a^{l-1})_1, (ba^{2l-1})_1, (a^{2(l-1)})_0$ to the path $Q_2\colon (a^{n+1})_1, (ba^2)_0, (a^2)_0, (a^2)_1$ in this order. By a similar way, we can get ${(a^{k(l-1)})_0}^{\omega_2} = (a^{k(n+1)})_1$ for any k. By inserting $k = l-1$, we have ${(a^{(l-1)(l-1)})_0}^{\omega_2} = {(a^{n+1})_0}^{\omega_2} = (a^{(l-1)(n+1)})_1 = (a^{n+l-1})_1$. On the other hand, $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is the unique 8-cycle passing through 1_0 and 1_1 implies that $(1_0, 1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1)$ is mapped to $(1_1, (ba^{n+1})_0, (ba^{n+1})_1, (a^n)_0, (a^n)_1, (ba)_0, (ba)_1, 1_0)$ by ω_2. So ${(ba^{n+1})_0}^{\omega_2} = (ba^{n+1})_1$, which implies ${(a^{n+1})_0}^{\omega_2} = (a^{n+1-l})_1$, a contradiction. Therefore Γ is not vertex-transitive, a contradiction.

Subsubcase 2.2.4: $(R, L, S) \equiv (\{b\}, \{ba^l\}, \{1, ba\})$ with $(l-1)^2 \equiv n-1 \pmod{2n}$.

Note that $n \equiv 2 \pmod 4$ in this case. One can check that the map

$$\begin{array}{llllll} \omega_3\colon & (a^k)_0 & \mapsto & (a^{k(1-l)})_1, & (a^k)_1 & \mapsto & (ba^{n+1+k(1-l)})_0, \\ & (ba^k)_0 & \mapsto & (ba^{k(1-l)+l})_1, & (ba^k)_1 & \mapsto & (a^{(k-1)(1-l)})_0, \end{array}$$

with $0 \le k < 2n$ is a permutation on $V(\Gamma)$ with order 8. Furthermore, for any $0 \le k < 2n$, we have

$$N_\Gamma((a^k)_0)^{\omega_3} = \{(ba^{n+1+k(1-l)})_0, (a^{k(1-l)})_0, (ba^{k(1-l)+l})_1\} = N_\Gamma((a^{k(1-l)})_1),$$

$$N_\Gamma((a^k)_1)^{\omega_3} = \{(a^{k(1-l)})_1, (ba^{n+1+k(1-l)})_1, (a^{n+1+k(1-l)})_0\} = N_\Gamma((ba^{n+1+k(1-l)})_0),$$

$$N_\Gamma((ba^k)_0)^{\omega_3} = \{(ba^{k(1-l)+l})_0, (a^{(k-1)(1-l)})_0, (a^{k(1-l)})_1\} = N_\Gamma((ba^{k(1-l)+l})_1),$$

$$N_\Gamma((ba^k)_1)^{\omega_3} = \{(a^{(k-1)(1-l)})_1, (ba^{k(1-l)+l})_1, (ba^{(k-1)(1-l)})_0\} = N_\Gamma((a^{(k-1)(1-l)})_0).$$

So ω_3 induces an automorphism of Γ of order 8. Denote $H_{01} = \{(a^k)_0\}$, $H_{02} = \{(ba^k)_0\}$, and $H_{11} = \{(a^k)_1\}$, $H_{12} = \{(ba^k)_1\}$ with $0 \le k < 2n$. We have the following:

$$H_{01} \xrightarrow{\omega_3} H_{11} \xrightarrow{\omega_3} H_{02} \xrightarrow{\omega_3} H_{12} \xrightarrow{\omega_3} H_{01}$$

Note that $\langle R(a)\rangle$ acts transitively on the sets $H_{01}, H_{02}, H_{11}, H_{12}$, respectively. So $M_2 = \langle R(a), \omega_3\rangle$ is a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$. By calculation, ${\omega_3}^4 = R(a^n)$, ${\omega_3}^{-1}R(a)\omega_3 = R(a)^{1-l}$. So $M_2 = \langle R(a)\rangle\langle\omega_3\rangle$ and furthermore $|M_2| = 8n$. Therefore M_2 acts regularly on $V(\Gamma)$, and hence Γ is a Cayley graph of type (2) in the theorem. □

Proof of Theorem 1.4. Let Γ be a 2-type bi-Cayley graph and let $R = \{x_1, x_2\}$, $L = \{y_1, y_2\}$ and $S = \{1\}$.

Firstly, we assume that all of x_1, x_2, y_1 and y_2 belong to $b\langle a\rangle$. By the structure of SD_{4n}, if their orders are the same, then $SD_{4n} \neq \langle x_1, x_2, y_1, y_2\rangle$. So without loss of generality, we can assume that x_1, x_2 have order 2 and y_1, y_2 have order 4. By Lemma 3.1 (2), $\mathrm{Aut}\,(SD_{4n})$ acts transitively on set $\{ba^{2i}\}$ with $0 \leqslant i \leqslant n-1$, we can let $x_1 = b$, $x_2 = ba^{2t}$, $y_1 = ba^s$ and $y_2 = ba^{n+s}$. So $(R, L, S) \equiv (\{b, ba^{2t}\}, \{ba^s, ba^{n+s}\}, \{1\})$ with s an odd. It is easy to find that $(1_1, (ba^s)_1, (a^n)_1, (ba^{n+s})_1)$ is the unique 4-cycle passing through 1_1. The vertex-transitivity of Γ implies that there is a unique 4-cycle passing through 1_0. Considering possible 4-cycles containing 1_0, we have $(a^{2t})_0 = (a^{-2t})_0$, and hence $n = 2t$. Noticing that $\langle a\rangle = \langle a^s, a^n\rangle$, we get $(s, 2n) = 1$. So the map $f\colon\ a^s \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^f \equiv (\{b, ba^n\}, \{ba, ba^{n+1}\}, \{1\})$. Hence, we can assume that $(R, L, S) = (\{b, ba^n\}, \{ba, ba^{n+1}\}, \{1\})$.

Let $\Sigma = \mathrm{Cay}(D_{8n}, \{d, dc, dc^{2n}\})$ where $D_{8n} = \langle c, d \mid c^{4n} = d^2 = 1, dcd = c^{-1}\rangle$. Define a map from $V(\Gamma)$ to $V(\Sigma)$ as follows:

$$\begin{array}{llllllll} \phi\colon & (a^r)_0 & \mapsto & c^{2r}, & (a^r)_1 & \mapsto & dc^{2r+1}, \\ & (ba^r)_0 & \mapsto & dc^{2r}, & (ba^r)_1 & \mapsto & c^{2r-1}, \end{array}$$

with $0 \leq r \leq 2n-1$. Furthermore, for any $r \in \mathbb{Z}_{2n}$, we have

$$\begin{aligned} N_\Gamma((a^r)_0)^\phi &= \{(a^r)_1, (ba^r)_0, (ba^{n+r})_0\}^\phi = \{dc^{2r+1}, dc^{2r}, dc^{2n+2r}\} = N_\Sigma(c^{2r}), \\ N_\Gamma((a^r)_1)^\phi &= \{(a^r)_0, (ba^{r+1})_1, (ba^{n+r+1})_1\}^\phi = \{c^{2r}, c^{2r+1}, c^{2n+2r+1}\} \\ &= N_\Sigma(dc^{2r+1}), \\ N_\Gamma((ba^r)_0)^\phi &= \{(ba^r)_1, (a^r)_0, (a^{n+r})_0\}^\phi = \{c^{2r-1}, c^{2r}, c^{2n+2r}\} = N_\Sigma(dc^{2r}), \\ N_\Gamma((ba^r)_1)^\phi &= \{(ba^r)_0, (a^{n+r-1})_1, (a^{r-1})_1\}^\phi = \{dc^{2r}, dc^{2n+2r-1}, dc^{2r-1}\} \\ &= N_\Sigma(c^{2r-1}). \end{aligned}$$

It follows that ϕ is an isomorphism from Γ to Σ. Then Γ is a Cayley graph over a dihedral group. This is the graph of type (1) in the theorem.

In the following, we assume that there is at least one of x_1, x_2, y_1, y_2 belongs to $\langle a\rangle$. Without loss of generality, let $x_1 \in \langle a\rangle$. We divide the proof into two cases:

Case 1: $|\langle x_1\rangle| = 2$.

In this case, $x_1 = a^n$. The condition $R = R^{-1}$ implies that x_2 is also an element of order 2, and hence $x_2 = ba^{2i_1}$ for some integer i_1. Since there is an automorphism of SD_{4n} sending a and ba^{2i_1} to a and b, respectively, we can assume $x_2 = b$ up to equivalence. If $y_1 \in \langle a\rangle$ and $y_2 \notin \langle a\rangle$, then $y_1 \neq {y_2}^{-1}$. So $L = L^{-1}$ implies that $y_1 = a^n$ and $y_2 = ba^{2i_2}$. Then $SD_{4n} \neq \langle b, ba^{2i_2}, a^n\rangle$, contrary to the connectivity of Γ. Similarly, if $y_2 \in \langle a\rangle$ and $y_1 \notin \langle a\rangle$, then one can get a similar contradiction. So we consider the following two subcases:

Subcase 1.1: Both y_1 and y_2 belong to $b\langle a\rangle$.

In this case, both y_1 and y_2 have order 4 by the connectivity of Γ and the structure of SD_{4n}. Let $y_1 = ba^{l_1}$ and $y_2 = ba^{n+l_1}$ for some odd integer l_1. Now it holds that $SD_{4n} = \langle b, a^n, a^{l_1}\rangle$. This implies that $(l_1, 2n) = 1$. So the map $d\colon\ a^{l_1} \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^d \equiv (\{a^n, b\}, \{ba, ba^{n+1}\}, \{1\})$. Hence, we can assume that $(R, L, S) \equiv (\{a^n, b\}, \{ba, ba^{n+1}\}, \{1\})$ up to equivalence. Let us consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 3, respectively. There are two 7-cycles passing through 1_0 but not passing through 1_1 that is $(1_0, b_0, b_1, (a^{n-1})_1, (ba^n)_1, (ba^n)_0, (a^n)_0)$ and $(1_0, b_0, b_1, (a^{-1})_1, (ba^n)_1, (ba^n)_0, (a^n)_0)$. On the other hand, $(1_1, (ba)_1, (ba)_0, a_0, (a^{n+1})_0, (ba^{n+1})_0, (ba^{n+1})_1)$ is the unique 7-cycle passing through 1_1 but not passing through 1_0, contrary to the vertex-transitivity of Γ.

Subcase 1.2: Both y_1 and y_2 belong to $\langle a\rangle$.

In this case, we can let $y_1 = a^{i_3}, y_2 = a^{-i_3}$. Now it holds that $SD_{4n} = \langle b, a^n, a^{i_3}\rangle$. This implies that $(i_3, 2n) = 1$. So the map $e\colon\ a^{i_3} \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^e \equiv (\{a^n, b\}, \{a, a^{-1}\}, \{1\})$. Hence we can assume that $(R, L, S) \equiv (\{a^n, b\}, \{a, a^{-1}\}, \{1\})$. It is easy to find that there is a unique 4-cycle $(1_0, b_0, (ba^n)_0, (a^n)_0)$ passing through 1_0. The vertex-transitivity of Γ implies that there is a unique 4-cycle passing through 1_1. Considering possible 4-cycles containing 1_1, we have $(a^2)_1 = (a^{-2})_1$, and hence $n = 2$, contrary to $n \geq 4$.

Case 2: $|\langle x_1\rangle| \neq 2$

In this case, we can let $x_1 = a^i, x_2 = a^{-i}$. By the connectivity of Γ and the structure of SD_{4n}, if $y_1 \in \langle a\rangle$, then $y_2 \notin \langle a\rangle$ and $y_1 \neq {y_2}^{-1}$. By the condition $L = L^{-1}$, we can assume that $L = \{a^n, b\}$ up to equivalence. In this case, Γ is isomorphic to a graph of subcase 1.2, by Proposition 2.1(4). So we can assume that both y_1 and y_2 belong to $b\langle a\rangle$. We divide the proof into the following two subcases:

Subcase 2.1: The orders of y_1 and y_2 are 2.

By Lemma 3.1(2), Aut (SD_{4n}) acts transitively on the set $\{ba^{2i}\}$ for $0 \leqslant i \leqslant n-1$, we can let $y_1 = b$, $y_2 = ba^{2j}$. So $(R, L, S) \equiv (\{a^i, a^{-i}\}, \{b, ba^{2j}\}, \{1\})$. The vertex-transitivity of Γ implies that there must exist an automorphism α in $\mathrm{Aut}(\Gamma)$ such that $1_0^\alpha = 1_1$. We consider the subgraphs of Γ induced by the vertices at distance from 1_0 and 1_1 at most 5, respectively. There are six 10-cycles passing through edge $\{1_0, (a^i)_0\}$, $\{1_0, (a^{-i})_0\}$, $\{1_1, b_1\}$ and $\{1_1, (ba^{2j})_1\}$ separately. In the same time, there are eight 10-cycles passing through edge $\{1_0, 1_1\}$. This implies that $\{1_0, (a^i)_0\}^\alpha \neq \{1_0, 1_1\}$. Hence $\{1_0, (a^i)_0\}^\alpha = \{1_1, b_1\}$ or $\{1_1, (ba^{2j})_1\}$ and $\{1_0, (a^{-i})_0\}^\alpha = \{1_1, b_1\}$ or $\{1_1, (ba^{2j})_1\}$. So ${(a^i)_0}^\alpha = b_1$ or $(ba^{2j})_1$ and ${(a^{-i})_0}^\alpha = b_1$ or $(ba^{2j})_1$. Without loss of generality, let ${(a^i)_0}^\alpha = b_1$. We consider the subgraphs of Γ induced by the vertices at distance from $(a^i)_0$ and b_1 at most 5, respectively. Similarly, we can find that there exists six 10-cycles passing through edge $\{(a^i)_0, (a^{2i})_0\}$ and $\{b_1, (a^{-2j})_1\}$ separately. There are eight 10-cycles passing through edge $\{(a^i)_0, (a^i)_1\}$ and $\{b_1, b_0\}$ separately. So by the vertex-transitivity of Γ, $\{(a^i)_0, (a^{2i})_0\}^\alpha = \{b_1, (a^{-2j})_1\}$, and hence ${(a^{2i})_0}^\alpha = (a^{-2j})_1$. In a similar way, one can see that the cycle $(1_0, (a^i)_0, (a^{2i})_0, (a^{3i})_0, \cdots, (a^{-i})_0)$ is mapped to the cycle $(1_1, b_1, (a^{-2j})_1, (ba^{-2j})_1, \cdots, (ba^{2j})_1)$ by α. So the lengthes of the cycles $(1_0, (a^i)_0, (a^{2i})_0, (a^{3i})_0, \cdots, (a^{-i})_0)$ and $(1_1, b_1, (a^{-2j})_1, (ba^{-2j})_1, \cdots, (ba^{2j})_1)$ are the same, and hence $\langle a^i\rangle = \langle a^j\rangle$. Furthermore, $SD_{4n} = \langle a^i, a^{2j}, b\rangle = \langle a^i, b\rangle$ implies that $(i, 2n) = (j, 2n) = 1$. It is easy to find that the map $g\,:\ a^i \mapsto a, b \mapsto b$ can induce an automorphism of SD_{4n} such that $(R, L, S)^g \equiv (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ with $(l, 2n) = 1$. So we can assume that $(R, L, S) = (\{a, a^{-1}\}, \{b, ba^{2l}\}, \{1\})$ with $(l, 2n) = 1$ up to equivalence.

Now the cycle $(1_0, a_0, (a^2)_0, (a^3)_0, \cdots, (a^{-1})_0)$ is mapped to the cycle $(1_1, b_1, (a^{-2l})_1, (ba^{-2l})_1, \cdots, (a^{2l})_1, (ba^{2l})_1)$ by α. So we can get

$$ {(a^{2k})_0}^\alpha = (a^{-2kl})_1, {(a^{2k+1})_0}^\alpha = (ba^{-2kl})_1 \qquad \text{(I)} $$

where $0 \leq k \leq n-1$. Note that $(a^{2k})_0$ and $(a^{2k+1})_0$ are adjacent with $(a^{2k})_1$ and $(a^{2k+1})_1$, respectively, and $(a^{-2kl})_1$ and $(ba^{-2kl})_1$ are adjacent with $(a^{-2kl})_0$ and $(ba^{-2kl})_0$, respectively. So we can get

$$(a^{2k})_1{}^{\alpha} = (a^{-2kl})_0, (a^{2k+1})_1{}^{\alpha} = (ba^{-2kl})_0 \tag{II}$$

where $0 \leq k \leq n-1$. By (I) and (II), it is easy to get

$$\alpha\colon\ (a^{2k})_0 \mapsto (a^{-2kl})_1 \mapsto (a^{2kl^2})_0. \tag{III}$$

Since $(a^{2k})_0$ is adjacent with $(a^{2k+1})_0$, we have $(a^{2k+1})_0{}^{\alpha^2} = (a^{2kl^2\pm1})_0$. Similarly, by $\{(a^{2k+1})_0, (a^{2k+2})_0\}^{\alpha^2} = \{(a^{2kl^2\pm1})_0, (a^{2kl^2+2l^2})_0\}$, it holds that $(a^{2kl^2+2l^2})_0 = (a^{2kl^2\pm2})_0$ or $(a^{2kl^2+2l^2})_0 = (a^{2kl^2})_0$. If $(a^{2kl^2+2l^2})_0 = (a^{2kl^2})_0$, then $2l^2 \equiv 0 \pmod{2n}$, which implies that $n|l^2$, contrary to l is odd. So $(a^{2kl^2+2l^2})_0 = (a^{2kl^2\pm2})_0$, and hence, $2l^2 \equiv \pm2 \pmod{2n}$.

If $2l^2 \equiv 2 \pmod{2n}$, then the map

$$\begin{array}{llllll}
\beta\colon & (a^{2k})_0 & \mapsto & (a^{-2kl})_1, & (a^{2k})_1 & \mapsto & (a^{-2kl})_0,\\
& (a^{2k+1})_0 & \mapsto & (ba^{-2kl})_1, & (a^{2k+1})_1 & \mapsto & (ba^{-2kl})_0,\\
& (ba^{2k})_0 & \mapsto & (a^{-2kl+1})_1, & (ba^{2k})_1 & \mapsto & (a^{-2kl+1})_0,\\
& (ba^{2k+1})_0 & \mapsto & (ba^{n+1-2kl})_1, & (ba^{2k+1})_1 & \mapsto & (ba^{n+1-2kl})_0,
\end{array}$$

with $0 \leq k < n$ is a permutation on $V(\Gamma)$ with order 2. Furthermore, for any $0 \leq k < n$ we have

$$\begin{aligned}
&N_\Gamma((a^{2k})_0)^\beta = \{(ba^{-2kl})_1, (ba^{2(1-k)l})_1, (a^{-2kl})_0\} = N_\Gamma((a^{-2kl})_1),\\
&N_\Gamma((a^{2k})_1)^\beta = \{(a^{-2kl+1})_0, (a^{-2kl-1})_0, (a^{-2kl})_1\} = N_\Gamma((a^{-2kl})_0),\\
&N_\Gamma((a^{2k+1})_0)^\beta = \{((a^{-2kl})_1, (a^{-2(k+1)l})_1, (ba^{-2kl})_0\} = N_\Gamma((ba^{-2kl})_1),\\
&N_\Gamma((a^{2k+1})_1)^\beta = \{(ba^{n-1-2kl})_0, (ba^{n+1-2kl})_0, (ba^{-2kl})_1\} = N_\Gamma((ba^{-2kl})_0),\\
&N_\Gamma((ba^{2k})_0)^\beta = \{(ba^{-2kl+1})_1, (ba^{2(1-k)l+1})_1, (a^{-2kl+1})_0\} = N_\Gamma((a^{-2kl+1})_1),\\
&N_\Gamma((ba^{2k})_1)^\beta = \{(a^{-2kl+2})_0, (a^{-2kl})_0, (a^{-2kl+1})_1\} = N_\Gamma((a^{-2kl+1})_0)\\
&N_\Gamma((ba^{2k+1})_0)^\beta = \{(a^{n+1-2(k+1)l})_1, (a^{n+1-2kl})_1, (ba^{n+1-2kl})_0\} = N_\Gamma((ba^{n+1-2kl})_1),\\
&N_\Gamma((ba^{2k+1})_1)^\beta = \{(ba^{-2kl})_0, (ba^{2-2kl})_0, (ba^{n+1-2kl})_1\} = N_\Gamma((ba^{n+1-2kl})_0).
\end{aligned}$$

So β induces an automorphism of Γ of order 2. Denote $H_{01} = \{(a^{2i})_0\}$, $H_{02} = \{(a^{2i+1})_0\}$, $H_{03} = \{(ba^{2i})_0\}$, $H_{04} = \{(ba^{2i+1})_0\}$, $H_{11} = \{(a^{2i})_1\}$, $H_{12} = \{(a^{2i+1})_1\}$, $H_{13} = \{(ba^{2i})_1\}$, $H_{14} = \{(ba^{2i+1})_1\}$, with $0 \leq i < n$. Let $H_0 = H_{01} \bigcup H_{02} \bigcup H_{03} \bigcup H_{04}$; $H_1 = H_{11} \bigcup H_{12} \bigcup H_{13} \bigcup H_{14}$; We have the following:

$H_{01} \overset{\beta}{—} H_{11} \overset{\mathrm{R(b)}}{—} H_{13} \overset{\beta}{—} H_{02} \overset{\mathrm{R(b)}}{—} H_{04} \overset{\beta}{—} H_{14} \overset{\mathrm{R(b)}}{—} H_{12} \overset{\beta}{—} H_{03} \overset{\mathrm{R(b)}}{—} H_{01}$.
$R(a^2)$ acts transitively on H_{ij} where $i = 0, 1$; $j = 1, 2, 3, 4$; So $M = \langle R(a^2), R(b), \beta\rangle$ is a vertex-transitive subgroup of Γ.

By calculation, $\beta R(a^2)\beta = R(a^{-2l}) \in \langle R(a^2)\rangle$, so β normalizes $\langle R(a^2)\rangle$. Noticing that $R(b)$ also normalizes $\langle R(a^2)\rangle$, we see that $\langle R(a^2)\rangle \trianglelefteq M$. We consider the group $\overline{M/\langle R(a^2)\rangle} = \langle\overline{\beta}, \overline{R(b)}\rangle = \langle\overline{\beta R(b)}, \overline{R(b)}\rangle$. By calculation $(\beta R(b))^4 = R(a^{-2l-2}) \in \langle R(a^2)\rangle$ and $R(b)^2 = 1$ imply that $\overline{\beta R(b)}^4 = \overline{R(b)}^2 = \overline{1}$. It is easy to find that $\overline{R(b)}^{-1}\overline{\beta R(b)}\,\overline{R(b)} = \overline{\beta R(b)}^{-1}$. So $\overline{M} \simeq D_8$, and $|M| = 8n$. Therefore M acts regularly on $V(\Gamma)$. So Γ is a Cayley graph. This is the graph of type (2) in the theorem.

Let $2l^2 \equiv -2 \pmod{2n}$. Now the map

$$\gamma\colon \begin{array}{llllll} (a^{2k})_0 & \mapsto & (a^{-2kl})_1, & (a^{2k})_1 & \mapsto & (a^{-2kl})_0, \\ (a^{2k+1})_0 & \mapsto & (ba^{-2kl})_1, & (a^{2k+1})_1 & \mapsto & (ba^{-2kl})_0, \\ (ba^{2k})_0 & \mapsto & (a^{-2kl-1})_1, & (ba^{2k})_1 & \mapsto & (a^{-2kl-1})_0, \\ (ba^{2k+1})_0 & \mapsto & (ba^{n-1-2kl})_1, & (ba^{2k+1})_1 & \mapsto & (ba^{n-1-2kl})_0, \end{array}$$

with $0 \le k < n$ is a permutation on $V(\Gamma)$. Furthermore, for any $0 \le k < n$ we have

$$N_\Gamma((a^{2k})_0)^\gamma = \{(ba^{-2kl})_1, (ba^{2(1-k)l})_1, (a^{-2kl})_0\} = N_\Gamma((a^{-2kl})_1),$$
$$N_\Gamma((a^{2k})_1)^\gamma = \{(a^{-2kl+1})_0, (a^{-2kl-1})_0, (a^{-2kl})_1\} = N_\Gamma((a^{-2kl})_0),$$
$$N_\Gamma((a^{2k+1})_0)^\gamma = \{((a^{-2kl})_1, (a^{-2(k+1)l})_1, (ba^{-2kl})_0\} = N_\Gamma((ba^{-2kl})_1),$$
$$N_\Gamma((a^{2k+1})_1)^\gamma = \{(ba^{n-1-2kl})_0, (ba^{n+1-2kl})_0, (ba^{-2kl})_1\} = N_\Gamma((ba^{-2kl})_0),$$
$$N_\Gamma((ba^{2k})_0)^\gamma = \{(ba^{-2kl-1})_1, (ba^{2(1-k)l-1})_1, (a^{-2kl-1})_0\} = N_\Gamma((a^{-2kl-1})_1),$$
$$N_\Gamma((ba^{2k})_1)^\gamma = \{(a^{-2kl-2})_0, (a^{-2kl})_0, (a^{-2kl-1})_1\} = N_\Gamma((a^{-2kl-1})_0)$$
$$N_\Gamma((ba^{2k+1})_0)^\gamma = \{(a^{n-1-2kl})_1, (a^{n-1-2(k+1)l})_1, (ba^{n-1-2kl})_0\} = N_\Gamma((ba^{n-1-2kl})_1),$$
$$N_\Gamma((ba^{2k+1})_1)^\gamma = \{(ba^{-2kl})_0, (ba^{-2-2kl})_0, (ba^{n-1-2kl})_1\} = N_\Gamma((ba^{n-1-2kl})_0).$$

Therefore γ induces an automorphism of Γ mapping 1_0 to 1_1. So Γ is a vertex-transitive graph. This is the graph of type (3) in the theorem. Now we aim to show that Γ is a non-Cayley graph. Let H be a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$. Then there exists an automorphism $\varphi \in H$ such that ${1_0}^\varphi = 1_1$. It is easy to find that ${1_1}^\varphi = 1_0$. So $\varphi^2 \in H_{1_0}$. Similar with proof of (III), we can get ${(a^{2k})_0}^{\varphi^2} = (a^{2kl^2})_0 = (a^{-2k})_0$. Since $(a^{2k+1})_0$ is adjacent with $(a^{2k})_0$ and $(a^{2k+2})_0$, we have ${(a^{2k+1})_0}^{\varphi^2} = (a^{-2k-1})_0$, where $0 \le k \le n-1$. The condition ${(a^{2k+1})_0}^{\varphi^2} = (a^{-2k-1})_0$ implies that ${a_0}^{\varphi^2} = a_0^{-1}$. So $\varphi^2 \ne 1$ and $|H_{1_0}| \ge 2$. This implies that H does not act regularly on $V(\Gamma)$. Since we choose an arbitrary vertex-transitive subgroup H, Γ is a non-Cayley graph.

Subcase 2.2: The orders of y_1 and y_2 are 4.

By Lemma 3.1(2), $\mathrm{Aut}(SD_{4n})$ acts transitively on the set $\{ba^{2i+1}\}$ for $0 \leqslant i \leqslant n-1$, we can let $y_1 = ba$, $y_2 = ba^{n+1}$. The condition $SD_{4n} = \langle a^i, ba\rangle$ implies that $(i, 2n) = 1$. So we can assume $(R, L, S) = (\{a, a^{-1}\}, \{ba, ba^{n+1}\}, \{1\})$ up to equivalence. It is easy to find that $(1_1, (ba)_1, (a^n)_1, (ba^{n+1})_1)$ is the unique 4-cycle passing through 1_1. The vertex-transitivity of Γ implies that there is a unique 4-cycle through 1_0. Considering possible 4-cycles containing 1_0, we have $(a^2)_0 = (a^{-2})_0$, and hence $n = 2$, contrary to $n \ge 4$. The proof is now complete. □

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