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On adjacency and Laplacian cospectral switching non-isomorphic signed graphs*

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Abstract

Let $\Gamma = (G, \sigma)$ be a signed graph, where σ is the sign function on the edges of G. In this paper, we use the operation of partial transpose to obtain switching non-isomorphic Laplacian cospectral signed graphs. We will introduce a new operation on signed graphs. This operation will establish a relationship between the adjacency spectrum of one signed graph with the Laplacian spectrum of another signed graph. As an application, this new operation will be utilized to construct several pairs of switching non-isomorphic cospectral signed graphs. Finally, we construct integral signed graphs.

Keywords: Signed graph, partial transpose, cospectral signed graphs, Laplacian cospectral signed graphs, equienergetic signed graphs, integral signed graph.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. A signed graph is defined to be a pair $\Gamma = (G, \sigma)$, with $G = (V(G), E(G))$ as the underlying graph and $\sigma\colon E(G) \to \{-1, 1\}$ as the signing function. In this manuscript, bold lines denote positive edges, and dashed lines

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denote negative edges. Signed graphs are a generalization of graphs, since they are signed graphs with each edge positive. The sign of a cycle in a signed graph is defined to be the product of the signs of its edges. A signed cycle is said to be positive (resp. negative) if its sign is positive (resp. negative). A signed graph is said to be balanced if none of its cycles is negative, otherwise unbalanced.

In a signed graph $\Gamma = (G, \sigma)$, the degree of a vertex v is the same as its degree in the underlying graph G (denoted by $d_v(G)$). For a signed graph Γ with vertex set $V(G)$, let $X \subset V(G)$ be a nonempty set. Let Γ^X denote the signed graph obtained from Γ by reversing signs of edges between X and $V(G) - X$. Then, we say Γ^X is switching equivalent to Γ. Here, we note that the switching is an equivalence relation and preserves the eigenvalues of the adjacency and the Laplacian matrix including their multiplicities. A switching class is represented by a single signed graph.

The adjacency matrix of a signed graph Γ with vertex set $\{v_1, v_2, \ldots, v_n\}$, is the $n \times n$ matrix $A(\Gamma) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i, v_j), & \text{if there is an edge from } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

For a graph G, the Laplacian matrix is $L(G) = D(G) - A(G)$ and signless Laplacian matrix is $Q(G) = D(G) + A(G)$, where $A(G)$ and $D(G)$ are respectively the adjacency matrix and the diagonal matrix of vertex degrees of G. The Laplacian matrix of Γ is $L(\Gamma) = L(G, \sigma) = D(G) - A(\Gamma)$. Note that $L(G, +) = L(G)$ and $L(G, -) = Q(G)$. The characteristic polynomial $|xI - A(\Gamma)|$ and eigenvalues of the adjacency matrix $A(\Gamma)$ of Γ are denoted by $\phi_\Gamma(x)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. The characteristic polynomial $|xI - L(\Gamma)|$ and eigenvalues of the Laplacian matrix $L(\Gamma)$ of Γ are denoted by $\psi_\Gamma(x)$ and $\mu_1, \mu_2, \ldots, \mu_n$, respectively. For a graph G (resp. signed graph Γ), eigenvalues of its adjacency matrix and Laplacian matrix are called adjacency and Laplacian eigenvalues of G (resp. Γ). Clearly, $A(\Gamma)$ and $L(\Gamma)$ are real symmetric and so all their eigenvalues are real. Let the signed graph Γ of order n has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and let their respective multiplicities be $m_1, m_2, \ldots, m_k$. The adjacency spectrum of Γ is written as $Spec(\Gamma) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_k^{(m_k)}\}$. A signed graph is said to be an integral signed graph if its adjacency spectrum consists of integers only.

Given a graph G, its subdivision graph $S(G)$ is obtained from G by replacing each of its edge by a path of length 2, or, equivalently, by inserting an additional vertex into each edge of G. If two signed graphs have the same adjacency spectrum (resp. Laplacian spectrum), they are said to be cospectral (resp. Laplacian cospectral); otherwise, they are noncospectral (resp. Laplacian noncospectral). Any two switching isomorphic signed graphs are cospectral (resp. Laplacian cospectral). A signed graph is said to be determined by its adjacency spectrum if cospectral signed graphs are switching isomorphic. It is well-known that in general the adjacency spectrum does not determine the signed graph and this problem has attracted to identify, if any, switching non-isomorphic cospectral signed graphs for a given class of signed graphs. For open problems in signed graphs, we refer to [2].

The energy of a graph G is the sum of the absolute values of its adjacency eigenvalues. This concept was extended to signed graphs by Germina, Hameed and Zaslavsky [9]. The energy of a signed graph Γ with eigenvalues $x_1, x_2, \ldots, x_n$ is defined as $\mathcal{E}(\Gamma) = \sum_{j=1}^{n} |x_j|$. Two signed graphs of same order are said to be equienergetic if they have the same energy.

Harary [12] pioneered the use of signed graphs in connection with the study of social balance theory. Signed graphs have been intensively explored in a variety of fields such as group theory, topological graph theory and classical root system. The reader is referred to [17] for a complete bibliography on signed graphs.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results which will be used in the sequel. In Section 3, we define the concept of partial transpose in signed graphs and use it to obtain switching non-isomorphic Laplacian cospectral signed graphs. In Section 4, we introduce a new operation on signed graphs and this will be utilized to construct switching non-isomorphic cospectral signed graphs, noncospectral equienergetic signed graphs and integral signed graphs.

2 Preliminaries

In this section, we recall some previously established results which will be required in the subsequent sections.

Definition 2.1 ([6]). Let $P = (p_{ij}) \in M_{m\times n}(\mathbb{R})$ and $Q \in M_{p\times q}(\mathbb{R})$. The Kronecker product of P and Q, denoted by $P \otimes Q$, is defined as

$$P \otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q & \dots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \dots & p_{2n}Q \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}Q & p_{m2}Q & \dots & p_{mn}Q \end{pmatrix}.$$

Lemma 2.2 ([6]). *Let P, $Q \in M_n(\mathbb{R})$ be two square matrices of order n. Let λ be an eigenvalue of matrix P with corresponding eigenvector x and μ be an eigenvalue of matrix Q with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $P\otimes Q$ with corresponding eigenvector $x \otimes y$.*

The Cartesian product (or sum) of two signed graphs $\Gamma_1 = (V(G_1), E(G_1), \sigma_1)$ and $\Gamma_2 = (V(G_2), E(G_2), \sigma_2)$, denoted by $\Gamma_1 \times \Gamma_2$, is the signed graph $(V(G_1)\times V(G_2), E, \sigma)$, where the edge set is that of the Cartesian product of underlying unsigned graphs and the sign function is defined by

$$\sigma((u_i, v_j), (u_k, v_l)) = \begin{cases} \sigma_1(u_i, u_k), & \text{if } j = l, \\ \sigma_2(v_j, v_l), & \text{if } i = k. \end{cases}$$

The Kronecker product (or conjunction) of two signed graphs $\Gamma_1 = (V(G_1), E(G_1), \sigma_1)$ and $\Gamma_2 = (V(G_2), E(G_2), \sigma_2)$, denoted by $\Gamma_1 \otimes \Gamma_2$, is the signed graph $(V(G_1) \times V(G_2), E, \sigma)$, where the edge set is that of the Kronecker product of underlying unsigned graphs and the sign function is defined by $\sigma((u_i, v_j), (u_k, v_l)) = \sigma_1(u_i, u_k)\sigma_2(v_j, v_l)$.

Lemma 2.3 ([9]). *Let Γ_1 and Γ_2 be two signed graphs with respective eigenvalues $x_1, x_2, \dots, x_{n_1}$ and $y_1, y_2, \dots, y_{n_2}$. Then*

(i) *the eigenvalues of $\Gamma_1 \times \Gamma_2$ are $x_i + y_j$, for all $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$,*

(ii) *the eigenvalues of $\Gamma_1 \otimes \Gamma_2$ are x_iy_j, for all $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

Lemma 2.4 ([4]). *Let Γ be an unbalanced signed graph with at least one edge, whose spectrum is symmetric about the origin, having eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$. Then $\Gamma \times K_2$ and $\Gamma \otimes K_2$, where K_2 is a complete signed graph on 2 vertices, are unbalanced, noncospectral and equienergetic if and only if $|\xi_j| \geq 1$, for all $j = 1, 2, \ldots, n$.*

Lemma 2.5 ([14]). *Let $P(\bullet)$ be a given polynomial. If μ is an eigenvalue of $A \in M_n$, while y is an associated eigenvector, then $P(\mu)$ is an eigenvalue of the matrix $P(A)$ and y is an eigenvector associated with $P(\mu)$.*

Lemma 2.6 ([4]). *Let Γ be a signed graph of order n. Then the following statements are equivalent.*

(i) *The spectrum of Γ is symmetric about the origin,*

(ii) $\phi_\Gamma(x) = x^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k b_{2k} x^{n-2k}$, *where b_{2k} are non negative integers for all $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$,*

(iii) Γ *and* $-\Gamma$ *are cospectral, where* $-\Gamma$ *is the signed graph obtained by negating sign of each edge of* Γ.

Lemma 2.7 ([16]). *For infinitely many n, there exists a family of 2^k pairwise nonisomorphic Laplacian integral, Laplacian cospectral graphs on n vertices, where $k > \frac{n}{(2log_2(n))}$.*

3 Constructing Laplacian cospectral non-isomorphic signed graphs

Dutta [8] constructed large families of non-isomorphic signless Laplacian cospectral graphs using partial transpose on graphs. In this section, we define the partial transpose in signed graphs. Let $\Gamma = (G, \sigma)$ be a signed graph on $2n$ vertices with vertex set $V(G) = V_1 \cup V_2$, such that $V_1 \cap V_2 = \emptyset$, and $V_1 = \{u_1, u_2, \ldots u_n\}$, $V_2 = \{v_1, v_2, \ldots v_n\}$. We denote by $\langle V_1 \rangle_\Gamma$ and $\langle V_2 \rangle_\Gamma$ as the induced signed subgraphs of Γ formed by V_1 and V_2 respectively. The spanning signed subgraph of Γ consisting of the signed edge set $\{(u_i, v_j) \in E(\Gamma) : u_i \in V_1, v_j \in V_2\}$ is denoted by $\langle V_1, V_2 \rangle_\Gamma$. Consider the set of signed edges $E(\widehat{\langle V_1, V_2 \rangle}_\Gamma) = \{(u_j, v_i) : (u_i, v_j) \in E(\langle V_1, V_2 \rangle_\Gamma)\}$. The definition of $E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$ suggests that given any signed edge $(u_i, v_j) \in E(\langle V_1, V_2 \rangle_\Gamma)$ there is a unique signed edge $(u_j, v_i) \in E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$ with the same sign as the sign of edge (u_i, v_j) in $\langle V_1, V_2 \rangle_\Gamma$.

The partial transpose of a signed graph Γ, denoted by Γ^τ, is defined as $\Gamma^\tau = \Gamma - E(\langle V_1, V_2 \rangle_\Gamma) + E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$. Note that, subtracting $E(\langle V_1, V_2 \rangle_\Gamma)$ indicates to remove all the existing signed edges in Γ of the form $(u_i, v_j) \in E(\langle V_1, V_2 \rangle_\Gamma)$. Then we include the signed edges $(u_j, v_i) \in E(\widehat{\langle V_1, V_2 \rangle}_\Gamma)$ to construct Γ^τ. If $i = j$, then the edge (u_i, v_i) will be removed and added again, that is the edge (u_i, v_i) is unaltered under partial transpose. Therefore, partial transpose of a signed graph Γ is an operation on the edge set which replaces the signed edge (u_i, v_j) with the sign $\sigma = \pm 1$, with the corresponding signed edge (u_j, v_i) with the same sign σ.

Example 3.1. Consider the signed graphs Γ_1 and Γ_1^τ as shown in Figure 1. Here, we have $V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1, v_2, v_3\}$ and $E\left(\langle V_1, V_2 \rangle_{\Gamma_1}\right) = \{(u_1, v_1), (u_1, v_3)\}$. Thus, $E\left(\widehat{\langle V_1, V_2 \rangle}_{\Gamma_1}\right) = \{(u_1, v_1), (u_3, v_1)\}$. Here, we replace the signed edge (u_1, v_3) with the signed edge (u_3, v_1).

Remark 3.2. The partial transpose of a signed graph is labelling dependent. Therefore, switching isomorphic signed graphs may have switching non-isomorphic partial transposes, depending on the labellings. The partial transpose keeps $\langle V_1\rangle$ and $\langle V_2\rangle$ unaltered. The total number of vertices and edges remains the same.

A cycle $C_l^{\sigma}(v_1, v_2, \ldots, v_l, v_1)$ in a signed graph $\Gamma = (G, \sigma)$ is a finite sequence of distinct vertices such that $(v_i, v_{i+1}) \in E(\Gamma)$ for all $i = 1, 2, \ldots, l-1$ and $(v_l, v_1) \in E(\Gamma)$. We denote the negative edges in the signed cycle $C_l^{\sigma}(v_1, v_2, \ldots, v_l, v_1)$ by putting the bar over the corresponding adjacent vertices. For example, the cycle $C_4^{\sigma}(v_1, v_2, v_3, v_4, v_1)$ on four vertices such that the only edge $(v_1, v_2) \in E(\Gamma)$ has negative sign will be denoted by $C_4^-(\overline{v_1, v_2}, v_3, v_4, v_1)$. Similarly if only two consecutive edges $(v_1, v_2), (v_2, v_3) \in E(\Gamma)$ have negative signs, then the cycle $C_4^{\sigma}(v_1, v_2, v_3, v_4, v_1)$ will be denoted by $C_4^+(\overline{v_1, v_2, v_3}, v_4, v_1)$. In a signed graph Γ, a signed TU-subgraph H is a signed subgraph whose components are trees or unbalanced unicyclic graphs, namely the unique cycle containing an odd number of negative edges. Thus, if H is a signed TU-subgraph, then $H = T_1 \cup T_2 \cup \cdots \cup T_p \cup U_1 \cup U_2 \cup \cdots \cup U_q$, where $T_i's$ are trees and $U_i's$ are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as $w(H) = 4^q \prod_{i=1}^{p} |T_i|$, where $|T_i|$ is the number of vertices in the tree T_i. Note that we define $\prod_{i=1}^{p} |T_i| = 1$ when $p = 0$. The relation between the coefficients of the Laplacian characteristic polynomial with the TU-subgraphs of a signed graph can be seen in [[3],Theorem 3.9]. If Γ is a signed graph with Laplacian characteristic polynomial $\psi(\Gamma, x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, then its coefficients are given by

$$a_i = (-1)^i \sum_{H \in \mathcal{H}_i(\Gamma)} w(H) \quad (i = 1, 2, \ldots, n), \tag{3.1}$$

where $\mathcal{H}_i(\Gamma)$ denotes the set of signed TU-subgraphs of Γ containing i edges. Two sets of signed TU-subgraphs $\mathcal{H}_i(\Gamma)$ and $\mathcal{H}_i(\Gamma')$ are comparable if

$$\sum_{H \in \mathcal{H}_i(\Gamma)} w(H) = \sum_{H \in \mathcal{H}_i(\Gamma')} w(H).$$

Now, Equation (3.1) suggests that Γ and Γ' are Laplacian cospectral if and only if the sets of their signed TU-subgraphs are comparable for all $i = 1, 2, \ldots m$, where m is the number of edges in the signed graph Γ. We say two signed graphs Γ_1 and Γ_2 are comparable if $\mathcal{H}_i(\Gamma_1)$ and $\mathcal{H}_i(\Gamma_2)$ are comparable for all i. As an example, two signed paths with equal number of vertices are comparable.

Example 3.3. Consider the signed graphs Γ_1 and Γ_1^{τ} as shown in Figure 1. We observe that Γ_1 contains two cycles $C_3^+(u_1, u_2, u_3, u_1)$ and $C_4^-(\overline{u_1, v_1}, v_2, v_3, u_1)$. The partial transpose Γ_1^{τ} of Γ_1 is obtained by replacing the signed edge (u_1, v_3) with (u_3, v_1). Clearly, Γ_1^{τ} contains three cycles $C_3^+(u_1, u_2, u_3, u_1)$, $C_4^-(\overline{u_1, v_1}, u_3, u_2, u_1)$ and $C_3^-(\overline{u_1, v_1}, u_3, u_1)$. The cycle $C_3^+(u_1, u_2, u_3, u_1)$ remains invariant under partial transpose on Γ_1. If the cycle has an odd number of negative edges, then it contributes an unbalanced unicyclic graph in the formation of signed TU-subgraphs. Therefore, the balanced cycle $C_3^+(u_1, u_2, u_3, u_1)$ does not contribute in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^{\tau}}(x)$.

The unbalanced cycle $C_4^-(\overline{u_1, v_1}, v_2, v_3, u_1)$ in Γ_1 is replaced by $C_4^-(\overline{u_1, v_1}, u_3, u_2, u_1)$ in Γ_1^{τ}. Therefore, signed TU-subgraphs whose components contain cycles $C_4^-(\overline{u_1, v_1}, v_2, v_3, u_1)$ and $C_4^-(\overline{u_1, v_1}, u_3, u_2, u_1)$ in Γ_1 and Γ_1^{τ}, respectively, have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^{\tau}}(x)$.

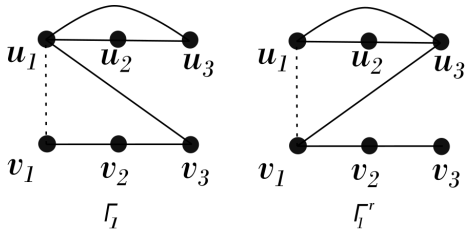

Figure 1: Signed graph Γ_1 and its partial transpose Γ_1^τ.

The signed edges $(u_1, v_1), (u_1, v_3)$ and (u_1, u_3) induce star signed graph $K_{1,3}$ in Γ_1. It is replaced by an unbalanced unicyclic TU-subgraph $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ in Γ_1^τ. The signed graphs $K_{1,3}$ and $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ have the same weights equal to 4 as a components to the signed TU-subgraphs. Therefore, the signed TU-subgraphs formed by $K_{1,3}$ and $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ in Γ_1 and Γ_1^τ, respectively, have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$. Moreover, the role of signed TU-subgraphs which contain the signed edges $(u_1, v_1), (u_1, v_3)$ and (u_1, u_3) in Γ_1 are replaced by the signed TU-subgraphs which contain the signed cycle $C_3^-(\overline{u_1, v_1}, u_3, u_1)$ in Γ_1^τ. Thus, all the signed TU-subgraphs of Γ_1 and Γ_1^τ are comparable. Hence they have same Laplacian characteristic polynomials, which can be calculated by Equation (3.1) and is given as

$$\psi_{\Gamma_1}(x) = \psi_{\Gamma_1^\tau}(x) = x^6 - 14x^5 + 73x^4 - 176x^3 + 196x^2 - 88x + 12.$$

In a signed graph $\Gamma = (G, \sigma)$ with $w_i, w_j \in V(\Gamma)$, if the signed edge (w_i, w_j) is added, then the resultant signed graph is denoted by $\Gamma' = \Gamma + \{(w_i, w_j)\}$. Similarly, $\Gamma' = \Gamma - \{(w_i, w_j)\}$ denotes the signed graph obtained by removing an edge (w_i, w_j). Whether the added/removed edge (w_i, w_j) is positive or negative, we denote a negative edge by $\overline{(w_i, w_j)}$, and a positive edge without a bar over the edge (w_i, w_j).

Theorem 3.4. *Let the signed subgraphs $\langle V_1 \rangle_\Gamma$ and $\langle V_2 \rangle_\Gamma$ of the signed graph Γ be two paths on n vertices with each edge being positive. If $\langle V_1, V_2 \rangle_\Gamma$ is an empty signed graph, then*

(i) *the signed graph $\Gamma_1 = \Gamma + \left\{ (u_1, u_n), \overline{(u_1, v_1)}, (u_1, v_n) \right\}$ is switching non-isomorphic and Laplacian cospectral to its partial transpose Γ_1^τ.*

(ii) *the signed graphs $\Gamma_2 = \Gamma_1 - \left\{ (u_{n-1}, u_n), \overline{(u_1, v_1)} \right\} + \left\{ \overline{(u_{n-1}, u_n)}, (u_1, v_1) \right\}$ and $\Gamma_3 = \Gamma_1^\tau - \{(u_{n-1}, u_n)\} + \left\{ \overline{(u_{n-1}, u_n)} \right\}$ are switching non-isomorphic and Laplacian cospectral.*

Proof. (i) The cycles in Γ_1 generated by additional three edges and their incidence with existing edges of Γ are $C_n^+(u_1, u_2, u_3, \ldots, u_n, u_1)$ and $C_{n+1}^-(\overline{u_1, v_1}, v_2, \ldots, v_n, u_1)$. The signed spanning subgraph $\langle V_1, V_2 \rangle_\Gamma$ contains only two signed edges which are $\overline{(u_1, v_1)}$ and (u_1, v_n). Partial transpose replaces (u_1, v_n) with (u_n, v_1). Clearly, Γ_1^τ contains three cycles $C_n^+(u_1, u_2, u_3, \ldots, u_n, u_1)$, $C_{n+1}^-(\overline{u_1, v_1}, u_n, \ldots, u_2, u_1)$ and $C_3^-(\overline{u_1, v_1}, u_n, u_1)$.

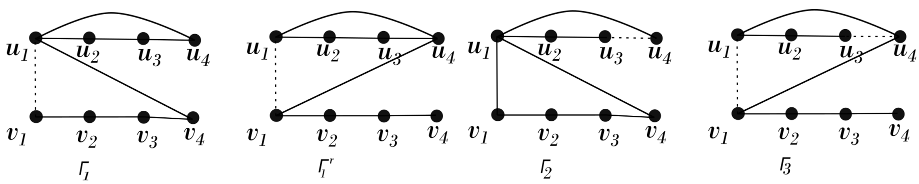

Figure 2: Signed graphs Γ_1, Γ_1^τ, Γ_2 and Γ_3.

The cycle $C_n^+\,(u_1, u_2, u_3, \ldots, u_n, u_1)$ remains invariant under partial transpose on Γ_1. Removing an edge from a cycle generates a tree. If the cycle has an odd number of negative edges, then it contributes an unbalanced unicyclic graph in the formation of signed TU-subgraphs. Therefore, the balanced cycle $C_n^+\,(u_1, u_2, u_3, \ldots, u_n, u_1)$ does not contribute in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

The unbalanced cycle $C_{n+1}^-\,(\overline{u_1, v_1}, v_2, \ldots, v_n, u_1)$ in Γ_1 is replaced by $C_{n+1}^-(\overline{u_1, v_1}, u_n, \ldots, u_2, u_1)$ in Γ_1^τ. Therefore, signed TU-subgraphs whose components contain cycles $C_{n+1}^-\,(\overline{u_1, v_1}, v_2, \ldots, v_n, u_1)$ and $C_{n+1}^-\,(\overline{u_1, v_1}, u_n, \ldots, u_2, u_1)$ in Γ_1 and Γ_1^τ, respectively, have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

The signed edges $\overline{(u_1, v_1)}, (u_1, v_n)$ and (u_1, u_n) induce star signed graph $K_{1,3}$ in Γ_1. It is replaced by an unbalanced unicyclic TU-subgraph $C_3^-\,(\overline{u_1, v_1}, u_n, u_1)$ in Γ_1^τ. They have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$ which is seen in Example 3.3. Moreover, the role of signed TU-subgraphs which contain the signed edges $\overline{(u_1, v_1)}, (u_1, v_n)$ and (u_1, u_n) in Γ_1 are replaced by the signed TU-subgraphs which contain the signed cycle $C_3^-\,(\overline{u_1, v_1}, u_n, u_1)$ in Γ_1^τ. Therefore, all the signed TU-subgraphs of Γ_1 and Γ_1^τ are comparable. Thus, they have the same Laplacian characteristic polynomials, which proves the result in this case.

(ii) If a signed graph is switching equivalent to a signed graph whose each edge is negative, then its Laplacian matrix coincides with the signless Laplacian matrix of its underlying graph. If n is odd, then the result follows by Corollary 2 of [8]. For even n, we observe that Γ_2 contains two cycles $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$ and $C_{n+1}^+\,(u_1, v_1, \ldots, v_n, u_1)$. The underlying graph of Γ_3 is the partial transpose of the underlying graph of Γ_2. The signed graph Γ_3 contains three cycles $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$, $C_{n+1}^+(\overline{u_1, v_1}, \overline{u_n, u_{n-1}}, u_{n-2}, \ldots, u_2, u_1)$ and $C_3^-(\overline{u_1, v_1}, u_n, u_1)$. The cycle $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$ is common in Γ_2 and Γ_3. Therefore, the signed $TU-$subgraphs formed by the unbalanced cycle $C_n^-\,(u_1, u_2, \ldots, \overline{u_{n-1}, u_n}, u_1)$ in Γ_2 and Γ_3 contribute same in $\psi_{\Gamma_2}(x)$ and $\psi_{\Gamma_3}(x)$. The balanced cycle $C_{n+1}^+\,(u_1, v_1, \ldots, v_n, u_1)$ in Γ_2 is replaced by a balanced cycle $C_{n+1}^+(\overline{u_1, v_1}, \overline{u_n, u_{n-1}}, u_{n-2}, \ldots, u_2, u_1)$ in Γ_3. Therefore, they do not contribute in $\psi_{\Gamma_2}(x)$ and $\psi_{\Gamma_3}(x)$.

The signed edges $(u_1, v_1), (u_1, v_n)$ and (u_1, u_n) induce star signed graph $K_{1,3}$ in Γ_2. It is replaced by an unbalanced unicyclic TU-subgraph $C_3^-\,(\overline{u_1, v_1}, u_n, u_1)$ in Γ_3 and proceeding similarly as in (i), we get the result. □

Example 3.5. Consider the signed graphs Γ_1, Γ_1^τ, Γ_2 and Γ_3 as given in Figure 2. They are constructed by using Theorem 3.4. Their Laplacian characteristic polynomials are respectively given as.

$$\psi_{\Gamma_1}(x) = \psi_{\Gamma_1^\tau}(x) = x^8 - 18x^7 + 131x^6 - 498x^5 + 1061x^4 - 1256x^3 + 764x^2 - 200x + 16$$

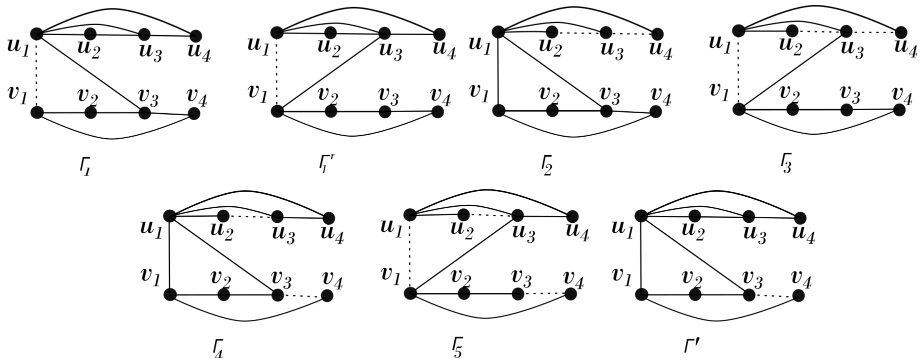

Figure 3: Signed graphs Γ_1 , Γ_1^τ, Γ_2, Γ_3, Γ_4, Γ_5 and Γ'.

and

$$\psi_{\Gamma_2}(x) = \psi_{\Gamma_3}(x) = x^8 - 18x^7 + 131x^6 - 498x^5 + 1065x^4 - 1284x^3 + 824x^2 - 240x + 20.$$

Clearly, the signed graphs Γ_1 and Γ_1^τ are switching non-isomorphic and Laplacian cospectral. Also, Γ_2 and Γ_3 are switching non-isomorphic and Laplacian cospectral signed graphs.

The proof of the following result is similar to Theorem 1 of [8] but for the self containment of the paper, we include it here.

Theorem 3.6. *Let the signed subgraphs $\langle V_1\rangle_\Gamma$ and $\langle V_2\rangle_\Gamma$ of Γ be two cycles on n vertices with each edge being positive. Let $\langle V_1, V_2\rangle_\Gamma$ be an empty signed graph. Given two non-adjacent vertices u_i and u_j with $i < j$, construct a new signed graph $\Gamma_1 = \Gamma + \left\{(u_i, u_j), \overline{(u_i, v_i)}, (u_i, v_j)\right\}$. Then*

(i) *the signed graph Γ_1 is switching non-isomorphic and Laplacian cospectral to its partial transpose Γ_1^τ,*

(ii) *the signed graphs $\Gamma_2 = \Gamma_1 - \left\{(u_{n-1}, u_n), \overline{(u_i, v_i)}, (u_{j-1}, u_j)\right\} + \left\{\overline{(u_{n-1}, u_n)}, (u_1, v_1), \overline{(u_{j-1}, u_j)}\right\}$ and $\Gamma_3 = \Gamma_1^\tau - \{(u_{n-1}, u_n), (u_{j-1}, u_j)\} + \left\{\overline{(u_{n-1}, u_n)}, \overline{(u_{j-1}, u_j)}\right\}$ are switching non-isomorphic and Laplacian cospectral,*

(iii) *the signed graphs $\Gamma_4 = \Gamma_1 - \left\{(v_{n-1}, v_n), \overline{(u_i, v_i)}, (u_{j-1}, u_j)\right\} + \left\{\overline{(v_{n-1}, v_n)}, (u_1, v_1), \overline{(u_{j-1}, u_j)}\right\}$ and $\Gamma_5 = \Gamma_1^\tau - \{(v_{n-1}, v_n), (u_{j-1}, u_j)\} + \left\{\overline{(v_{n-1}, v_n)}, \overline{(u_{j-1}, u_j)}\right\}$ are switching non-isomorphic and Laplacian cospectral.*

Proof. (i) The set of all cycles in Γ_1 includes two cycles of Γ. They are denoted by γ_1 and γ_2. The new cycles formed by additional three signed edges and their incidence with existing edges in Γ are $\gamma_3 = C^+_{j-i+1}(u_i, u_{i+1}, u_{i+2}, \ldots, u_j, u_i)$, $\gamma_4 = C^+_{n-(j-i)+1}(u_1, u_2, \ldots, u_i, u_j, u_{j+1}, \ldots, u_n, u_1)$, $\gamma_5 = C^-_{j-i+2}(\overline{u_i, v_i}, v_{i+1}, v_{i+2}, \ldots v_j, u_i)$, and $\gamma_6 =$

$C^-_{n-(j-i)+2}(v_1, v_2, \ldots, \overline{v_i, u_i}, v_j, v_{j+1}, \ldots, v_n, v_1)$. Note that, $\langle V_1, V_2\rangle_{\Gamma_1}$ contains only two edges which are (u_i, v_i) and (u_i, v_j). Partial transpose replaces (u_i, v_j) with (u_j, v_i). The cycles $\gamma_1, \gamma_2, \gamma_3$ and γ_4 remain invariant under partial transpose on Γ_1. If the cycle has an odd number of negative edges, then it contributes an unbalanced unicyclic graph in the formation of signed TU-subgraphs. Therefore, signed TU-subgraphs formed by the cycles $\gamma_1, \gamma_2, \gamma_3$ and γ_4 in Γ_1 and Γ_1^τ have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$.

Now, the signed cycle γ_5 in Γ_1 is replaced by $\gamma_5' = C^-_{j-i+2}(\overline{u_i, v_i}, u_j, u_{j-1}, \ldots, u_{i+1}, u_i)$ in Γ_1^τ. They have equal length and equal contribution in the characteristic coefficients. The cycle γ_6 in Γ_1 and its counterpart $\gamma_6' = C^-_{n-(j-i)+2}(u_1, u_2, \ldots, u_i, v_i, u_j, u_{j+1}, \ldots, u_q, u_1)$ in Γ_1^τ have equal lengths. If $(v_k, v_{k+1}) \in \gamma_6 \cap \langle V_2\rangle_{\Gamma_1}$ in Γ_1, then $(u_k, u_{k+1}) \in \gamma_6' \cap \langle V_1\rangle_{\Gamma_1^\tau}$ in Γ_1^τ. A signed TU-subgraphs containing more than $n-(j-i)+2$ edges contains edges from γ_1 in Γ_1. The role of γ_1 in Γ_1 is replaced by the edges of γ_2 in Γ_1^τ. We have assumed that γ_1 and γ_2 have equal length. Therefore, replacement of γ_6 in Γ_1^τ does not make any difference in the characteristic coefficients.

The new edges $(u_i, u_j), (u_i, v_i)$ and (u_i, v_j) form a star $K_{1,3}$ in Γ_1. It is replaced by a unicyclic signed TU-subgraph $C_3^-\,(u_i, u_j, \overline{v_i, u_i})$ in Γ_1^τ. They have equal contribution in $\psi_{\Gamma_1}(x)$ and $\psi_{\Gamma_1^\tau}(x)$ which is seen in Example 3.3. Therefore, all the signed $TU-$subgraphs of Γ_1 and Γ_1^τ are comparable as well as they form equal characteristic polynomials. This proves (i) The proof of (ii) and (iii) is similar to (i). Hence, the result follows. □

Example 3.7. Consider the signed graphs $\Gamma_1, \Gamma_1^\tau, \Gamma_2, \Gamma_3, \Gamma_4$ and Γ_5 as shown in Figure 3. Here $n = 4, i = 1$ and $j = 3$. The signed graph Γ_1, which is constructed by Theorem 3.6 is switching non-isomorphic and Laplacian cospectral to its partial transpose Γ_1^τ. We obtain the signed graph Γ_2 from Γ_1 by replacing the positive edges (u_2, u_3) and (u_3, u_4) with negative edges $\overline{(u_2, u_3)}$ and $\overline{(u_3, u_4)}$ and negative edge $\overline{(u_1, v_1)}$ with the positive edge (u_1, v_1). Also, the signed graph Γ_3 is obtained from Γ_1^τ by replacing the positive edges (u_2, u_3) and (u_3, u_4) with negative edges $\overline{(u_2, u_3)}$ and $\overline{(u_3, u_4)}$. The signed graphs Γ_2 and Γ_3 are switching non-isomorphic and Laplacian cospectral. Similarly the switching non-isomorphic and Laplacian cospectral signed graphs Γ_4 and Γ_5 are obtained from Γ_1 and Γ_1^τ, respectively, as in Theorem 3.6.

Remark 3.8. In Example 3.7, we have seen that Γ_1 and Γ_1^τ are Laplacian cospectral signed graphs. Also, we have mentioned that Γ_1^τ is the partial transpose of Γ_1. But, not all signed graphs are Laplacian cospectral to their partial transpose, for instance, consider the signed graphs Γ and Γ^τ as given in Figure 4. It is easy to calculate that the Laplacian characteristic polynomials of Γ and Γ^τ are $\psi_\Gamma(x) = x^6 - 12x^5 + 51x^4 - 96x^3 + 81x^2 - 30x + 4$ and $\psi_{\Gamma^\tau}(x) = x^6 - 12x^5 + 51x^4 - 94x^3 + 72x^2 - 18x$.

Let G be a graph and $\Gamma = (G, \sigma)$ be a signed graph on G. Hou et al. [15] raised the following two problems.

Problem 1. Let G be a graph, $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs on G, and $\det(L(\Gamma_1)) = \det(L(\Gamma_2))$. Are $L(\Gamma_1)$ and $L(\Gamma_2)$ cospectral?

Problem 2. Do there exist pairs $\Gamma_1 = (G_1, \sigma_1)$ and $\Gamma_2 = (G_2, \sigma_2)$ of signed graphs that have either of the following properties (i) and (ii)?

(i) Γ_1 and Γ_2 are not balanced but Laplacian cospectral such that G_1 and G_2 are nonisomorphic.

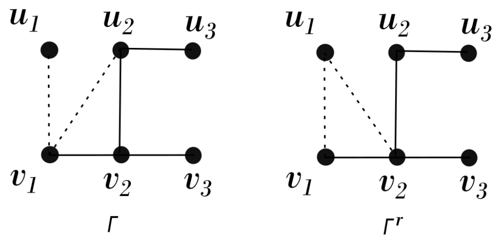

Figure 4: Signed graphs Γ and Γ^τ.

(ii) Γ_1 and Γ_2 are not balanced but Laplacian cospectral such that G_1 and G_2 are not cospectral.

The statement of Problem 1 is not always true. To see this, let Γ_1 be a signed graph as shown in Figure 3. Let Γ' (shown in Figure 3) be the signed graph obtained from Γ_1 by replacing the negative edge $\overline{(u_1, v_1)}$ with positive edge (u_1, v_1) and positive edge (v_3, v_4) with negative edge $\overline{(v_3, v_4)}$. The Laplacian characteristic polynomials of Γ_1 and Γ' are respectively given by

$$\psi_{\Gamma_1}(x) = x^8 - 22x^7 + 197x^6 - 928x^5 + 2476x^4 - 3736x^3 + 2976x^2 - 1056x + 128$$

and

$$\psi_{\Gamma'}(x) = x^8 - 22x^7 + 197x^6 - 928x^5 + 2476x^4 - 3748x^3 + 3048x^2 - 1152x + 128.$$

The underlying graphs of Γ_1 and Γ' are isomorphic and $\det(L(\Gamma_1)) = \det(L(\Gamma'))$. It is clear that the signed graphs Γ_1 and Γ' are not Laplacian cospectral and this answers Problem 1.

For Problem 2, consider the signed graph Γ_1 and its partial transpose Γ_1^τ as given in Figure 3. Clearly, the underlying graphs of Γ_1 and Γ_1^τ are non-isomorphic. The unbalanced signed graphs Γ_1 and Γ_1^τ are Laplacian cospectral. Also, it is easy to see that the underlying graph of Γ_1 and Γ_1^τ are not cospectral and this answers Problem 2.

4 Constructing switching non-isomorphic cospectral signed graphs, integral signed graphs and equienergetic signed graphs

The novel non-isomorphic cospectral graph constructions have implications for the complexity of the graph isomorphism problem. This necessitates the creation of methods for detecting and/or creating non-isomorphic cospectral graphs. Seidel switching, Godsil–McKay (GM) switching, and others are well-known approaches for constructing cospectral graphs. In 2019, Belardo et al. [1] used the Godsil-Mckay-type procedures developed for graphs to construct the pairs of switching non-isomorphic cospectral signed graphs. In this section, we will introduce a new operation on signed graphs. This operation establishes the relationship of the adjacency spectrum of one signed graph with the Laplacian spectrum of another signed graph. Furthermore, this operation will be utilized to construct the pairs of switching non-isomorphic cospectral signed graphs and integral signed graphs. Before that, we need the following motivation which can also be seen in [3].

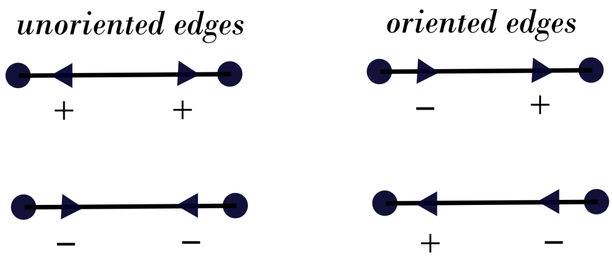

Figure 5: Bidirected edges in signed graphs.

The usual orientation of edges in digraphs differs from the orientation of signed graphs. In fact in signed graphs, instead of one arrow, we can use two arrows assigned to edges, which results in bidirected graphs. An orientated signed graph, more exactly, is an ordered pair $\Gamma_\vartheta = (\Gamma, \vartheta)$, where

$$\vartheta \colon V(G) \times E(G) \to \{0, 1, -1\} \tag{4.1}$$

satisfying the following three conditions.

(a) $\vartheta(u, vw) = 0$ whenever $u \neq v, w; u, v, w \in V(G)$ and $vw \in E(G)$,

(b) $\vartheta(v, vw) = 1$ (or -1) if an arrow at v is going into (rep. out of) v. For illustration, see Figure 5,

(c) $\vartheta(v, vw)\, \vartheta(w, vw) = -\sigma(vw)$.

As a result, positive edges are oriented edges, whereas negative edges are unoriented (see Figure 5). Therefore, every bidirected graph is also a signed graph. The converse is likewise true, however, one arrow (at any end) can be taken at random, whereas the other arrow (in light of (c) above) cannot. For an oriented signed graph Γ_ϑ, its incidence matrix $B_\vartheta = (b_{ij})$ is a matrix, whose rows correspond to vertices and columns to edges of G, with $b_{ij} = \vartheta(v_i, e_j)$ (here $v_i \in V(G)$, $e_j \in E(G)$). Usually, when only Γ is given, then we use an arbitrary orientation. So each row of the incidence matrix corresponding to vertex v_i contains d_{v_i} non-zero entries, each equal to $+1$ or -1. On the other hand, each column of the incidence matrix corresponding to edge e_j contains two non-zero entries, each equal to $+1$ or -1. Therefore, even in the case that multiple edges exist, we easily obtain

$$B_\vartheta B_\vartheta^T = D(G) - A(\Gamma_\vartheta) = L(\Gamma_\vartheta), \tag{4.2}$$

where $D(G)$ is the diagonal matrix of vertex degrees of G. It is easy to observe that $L(\Gamma_\vartheta)$ is positive-semidefinite.

The subdivision signed graph $S(\Gamma_\vartheta)$ is the signed graph whose underlying graph is $S(G)$ with vertex set $V(G) \cup E(G)$. It preserves the orientation ϑ and its adjacency matrix can be represented in the block form as follows

$$A(S(\Gamma_\vartheta)) = \begin{pmatrix} O_n & B_\vartheta \\ B_\vartheta^T & O_m \end{pmatrix},$$

where $O_r \in M_r(\mathbb{R})$. It is easy to see that the signature σ of the subdivision signed graph is defined by $\sigma(v_i e_j) = \vartheta_{ij}$. An example of a subdivision signed graph of a signed graph is shown in Figure 6.

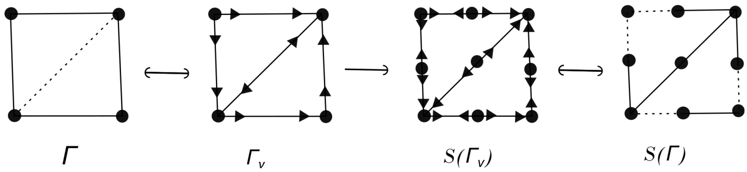

Figure 6: A signed graph and the corresponding subdivision signed graph.

Remark 4.1. Any orientation (random) ϑ to the edges of Γ gives rise to the same matrices $A(\Gamma_\vartheta) = A(\Gamma)$ and $L(\Gamma_\vartheta) = L(\Gamma)$, while the matrix $A(S(\Gamma_\vartheta))$ does depend on ϑ. Let S be a ± 1 diagonal matrix such that $B_{\vartheta'} = B_\vartheta S$. Clearly, $A(S(\Gamma_{\vartheta'})) = [I_n \dot{+} S] A(S(\Gamma_\vartheta))[I_n \dot{+} S]$, where $\dot{+}$ denotes the direct sum of two matrices. From now on, the subscript ϑ in B_ϑ will be not specified anymore.

Lemma 4.2 ([3]). *If B is the incidence matrix of a connected signed graph $\Gamma = (G, \sigma)$ having n vertices. Then*

$$rank(B) = \begin{cases} n-1 & \text{if } \Gamma \text{ is balanced,} \\ n & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Operation. Let Γ be a signed graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Let $S(\Gamma)$ be a subdivision signed graph of a signed graph Γ with vertex set $V(G) \cup E(G)$. In $S(\Gamma)$, replace each vertex v_i, $i = 1, 2, \ldots, n$, by k vertices and join every vertex to the neighbours of v_i with the same sign as that of the signed edge joining v_i with corresponding neighbours in $S(\Gamma)$. Then in the resulting signed graph, replace each vertex e_j, $j = 1, 2, \ldots, m$, by p vertices and join every vertex to the neighbours of e_j with the same sign as that of the signed edge joining e_j with corresponding neighbours in $S(\Gamma)$. The resulting signed graph is denoted by $S_{k,p}(\Gamma)$. That is, for a given signed graph Γ with a compatible orientation ϑ, the signed graph $S_{k,p}(\Gamma)$ has vertex set $V(\Gamma) \times \{1, 2, \ldots, k\} \cup E(\Gamma) \times \{1, 2, \ldots, p\}$($k$ copies of $V(\Gamma)$ and p copies of $E(\Gamma)$) and the edges of $S_{k,p}(\Gamma)$ are all between pairs of vertices (v, i) and (e, j), where $e \in E(\Gamma)$ is incident to $v \in V(\Gamma)$ in Γ, and with sign given by $\vartheta(v, e)$.

If $k = p = 1$, then $S_{k,p}(\Gamma)$ coincides with the subdivision signed graph $S(\Gamma)$. For convenience, if $k = 1$, then $S_{k,p}(\Gamma)$ will be denoted by $S_p(\Gamma)$. To illustrate the above operation, $S_2(\Gamma)$ is shown in Figure 7 and $S_{2,2}(\Gamma)$ for $k = p = 2$ is shown in Figure 8.

Theorem 4.3. *Let Γ be a signed graph with n vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ be the Laplacian eigenvalues of the signed graph Γ. Then the adjacency spectrum of $S_p(\Gamma)$ is $Spec(S_p(\Gamma)) =$*

$$\begin{cases} \{0^{(pm-n+2)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \ldots, \pm\sqrt{p\mu_{n-1}}^{(1)}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(pm-n)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \ldots, \pm\sqrt{p\mu_{n-1}}^{(1)}, \pm\sqrt{p\mu_n}^{(1)}\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Proof. We first label the vertices of $S_p(\Gamma)$ as follows. Let $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and let $\{u_1^j, u_2^j, \ldots, u_p^j\}$, $j = 1, 2, \ldots, m$, denote the vertex set replaced corresponding to the vertex e_j in $S(\Gamma)$. Denote by

$$V_i = \{u_i^1, u_i^2, \ldots, u_i^m\}, \quad i = 1, 2, \ldots, p.$$

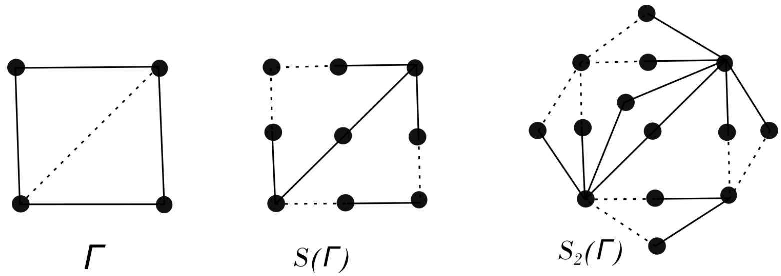

Figure 7: Signed graphs Γ, $S(\Gamma)$ and $S_2(\Gamma)$.

Then $V(\Gamma)\cup V_1\cup V_2\cup\cdots\cup V_p$ is a partition of $V(S_p(\Gamma))$. With this partition, the adjacency matrix of $S_p(\Gamma)$ can be written as

$$A(S_p(\Gamma))=\begin{pmatrix} O & B & B & \dots & B\\ B^T & O & O & \dots & O\\ B^T & O & O & \dots & O\\ \vdots & \vdots & \ddots & \vdots & \\ B^T & O & O & \dots & O\end{pmatrix}.$$

Now, we have

$$\begin{aligned} A(S_p(\Gamma))^2 &= \begin{pmatrix} O & B & B & \dots & B\\ B^T & O & O & \dots & O\\ B^T & O & O & \dots & O\\ \vdots & \vdots & \ddots & \vdots & \\ B^T & O & O & \dots & O\end{pmatrix}\begin{pmatrix} O & B & B & \dots & B\\ B^T & O & O & \dots & O\\ B^T & O & O & \dots & O\\ \vdots & \vdots & \ddots & \vdots & \\ B^T & O & O & \dots & O\end{pmatrix}\\ &= \begin{pmatrix} pBB^T & O & O & \dots & O\\ O & B^TB & B^TB & \dots & B^TB\\ O & B^TB & B^TB & \dots & B^TB\\ \vdots & \vdots & \ddots & \vdots & \\ O & B^TB & B^TB & \dots & B^TB\end{pmatrix}\\ &= \begin{pmatrix} pBB^T & O_{1\times p}\\ O_{p\times 1} & J_{p\times p}\otimes B^TB\end{pmatrix}, \end{aligned}$$

where $J_{p\times p}$ is a square matrix whose all entries are equal to 1. Therefore

$$Spec(A(S_p(\Gamma))^2)=Spec(pBB^T)\cup Spec(J_{p\times p}\otimes B^TB).$$

As B^TB is a real symmetric matrix of order m, so all its eigenvalues are real. Let $x_1\geq x_2\geq\cdots\geq x_m$ be the eigenvalues of the matrix B^TB. Note that $rank(BB^T)=rank(B^TB)=rank(B)$. Therefore, by Lemma 4.2, we have

$$Spec(B^TB)=\begin{cases}\{0^{(m-n+1)},x_1,x_2,\dots,x_{n-1}\} & \text{if } \Gamma \text{ is balanced,}\\ \{0^{(m-n)},x_1,x_2,\cdots,x_{n-1},x_n\} & \text{if } \Gamma \text{ is unbalanced,}\end{cases}$$

and

$$Spec(BB^T) = Spec(L(\Gamma)) = \begin{cases} \{0, \mu_1, \mu_2, \dots, \mu_{n-1}\} & \text{if } \Gamma \text{ is balanced,} \\ \{\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n\} & \text{if } \Gamma \text{ is unbalanced,} \end{cases}$$

where $x_n \neq 0$ and $\mu_n \neq 0$. As Spec($J_{p\times p}$) is $\{0^{p-1}, p\}$, then by Lemma 2.2, we have

$$Spec(J_{p\times p} \otimes B^T B) = \begin{cases} \{0^{(pm-n+1)}, px_1, px_2, \cdots, px_{n-1}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(pm-n)}, px_1, px_2, \dots, px_{n-1}, px_n\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

We know that the underlying graph of a subdivision signed graph is always bipartite. Similarly, the underlying graph of $S_p(\Gamma)$ is always bipartite. Note that the eigenvalues of $B^T B$ are given by the eigenvalues of BB^T, together with 0 of multiplicity $m-n$. Therefore, by Lemmas 2.5 and 2.6, we have $Spec(S_p(\Gamma)) =$

$$\begin{cases} \{0^{(pm-n+2)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \dots, \pm\sqrt{p\mu_{n-1}}^{(1)}\} & \text{if } \Gamma \text{ is balanced,} \\ \{0^{(pm-n)}, \pm\sqrt{p\mu_1}^{(1)}, \pm\sqrt{p\mu_2}^{(1)}, \dots, \pm\sqrt{p\mu_{n-1}}^{(1)}, \pm\sqrt{p\mu_n}^{(1)}\} & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Hence, the result follows. □

The following result can also be seen in Theorem 2.2 of [3].

Corollary 4.4. *Let Γ be a signed graph with n vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ be the Laplacian eigenvalues of the signed graph Γ. Then the adjacency spectrum of $S(\Gamma)$ is $Spec(S(\Gamma)) =$*

$$\begin{cases} \{0^{(m-n+2)}, \pm\sqrt{\mu_1}^{(1)}, \pm\sqrt{\mu_2}^{(1)}, \dots, \pm\sqrt{\mu_{n-1}}^{(1)}\} & \textit{if } \Gamma \textit{ is balanced,} \\ \{0^{(m-n)}, \pm\sqrt{\mu_1}^{(1)}, \pm\sqrt{\mu_2}^{(1)}, \dots, \pm\sqrt{\mu_{n-1}}^{(1)}, \pm\sqrt{\mu_n}^{(1)}\} & \textit{if } \Gamma \textit{ is unbalanced.} \end{cases}$$

Theorem 4.5. *Let Γ be a signed graph with n vertices and m edges. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n \geq 0$ be the Laplacian eigenvalues of the signed graph Γ. Then the adjacency spectrum of $S_{k,p}(\Gamma)$, where $p \in \{k, k-1\}$, is $Spec(S_{k,p}(\Gamma)) =$*

$$\begin{cases} \{0^{((k-2)n+pm+2)}, \pm\sqrt{pk\mu_1}^{(1)}, \dots, \pm\sqrt{pk\mu_{n-1}}^{(1)}\} & \textit{if } \Gamma \textit{ is balanced,} \\ \{0^{((k-2)n+pm)}, \pm\sqrt{pk\mu_1}^{(1)}, \dots, \pm\sqrt{pk\mu_{n-1}}^{(1)}, \pm\sqrt{pk\mu_n}^{(1)}\} & \textit{if } \Gamma \textit{ is unbalanced.} \end{cases}$$

Proof. We first label the vertices of $S_{k,p}(\Gamma)$ as follows. Let $\left\{v_1^j, v_2^j, \dots, v_k^j\right\}$, $j = 1, 2, \dots, n$, denote the vertex set replaced corresponding to the vertex v_j and $\{u_1^j, u_2^j, \dots, u_p^j\}$, $j = 1, 2, \dots, m$, denote the vertex set replaced corresponding to the vertex e_j in $S(\Gamma)$. Denote by

$$V^i = \left\{v_i^1, v_i^2, \dots, v_i^n\right\}, \quad i = 1, 2, \dots, k,$$

and

$$V_i = \left\{u_i^1, u_i^2, \dots, u_i^m\right\}, \quad i = 1, 2, \dots, p.$$

Then $V^1 \cup V_1 \cup V^2 \cup V_2 \cup \cdots \cup V^k \cup V_p$ is a partition of $V(S_{k,p}(\Gamma))$ when $k = p$. With this partition, the adjacency matrix of $S_{k,p}(\Gamma)$ can be written as

$$A(S_{k,p}(\Gamma)) = \begin{pmatrix} O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \\ O & B & O & \dots & O & B \\ \vdots & \vdots & \ddots & \vdots & & \\ O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \end{pmatrix}.$$

If $p = k-1$, then $V^1 \cup V_1 \cup V^2 \cup V_2 \cup \cdots \cup V_{k-1} \cup V^k$ is a partition of $V(S_{k,p}(\Gamma))$. With this partition, the adjacency matrix of $S_{k,p}(\Gamma)$ is given by

$$A(S_{k,p}(\Gamma)) = \begin{pmatrix} O & B & O & \dots & B & O \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \\ \vdots & \vdots & \ddots & \vdots & & \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \end{pmatrix}.$$

To prove the result, the following two cases arise.

Case 1. Let Γ be a balanced signed graph with n vertices and m edges. Let $Z = \begin{pmatrix} X \\ Y \end{pmatrix} \in M_{(n+m)\times 1}(\mathbb{R})$, where $X \in M_{n\times 1}(\mathbb{R})$ and $Y \in M_{m\times 1}(\mathbb{R})$, be an eigenvector corresponding to the non-zero eigenvalue λ_i, $1 \leq i \leq 2n-2$, of $S(\Gamma)$. Then $A(S(\Gamma))Z = \lambda_i Z$ implies that $BY = \lambda_i X$ and $B^T X = \lambda_i Y$. To find the eigenvalues of $S_{k,p}(\Gamma)$, consider the following two subcases.

Subcase 1.1. If $k = p$, then let $U = \begin{pmatrix} X \\ Y \\ \vdots \\ X \\ Y \end{pmatrix}$. Clearly, $U \in M_{(kn+pm)\times 1}(\mathbb{R})$ is a non-zero column vector. We have

$$A(S_{k,p}(\Gamma))U = \begin{pmatrix} O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \\ O & B & O & \dots & O & B \\ \vdots & \vdots & \ddots & \vdots & & \\ O & B & O & \dots & O & B \\ B^T & O & B^T & \dots & B^T & O \end{pmatrix} \begin{pmatrix} X \\ Y \\ \vdots \\ X \\ Y \end{pmatrix} = \begin{pmatrix} p\lambda_i X \\ p\lambda_i Y \\ \vdots \\ p\lambda_i X \\ p\lambda_i Y \end{pmatrix}$$
$$= p\lambda_i U.$$

Therefore $p\lambda_i$ is an eigenvalue of $S_{k,p}(\Gamma)$ corresponding to an eigenvector U. As $k = p$, thus $p\lambda_i$ can be written as $\sqrt{kp}\lambda_i$. Hence the result follows by Corollary 4.4.

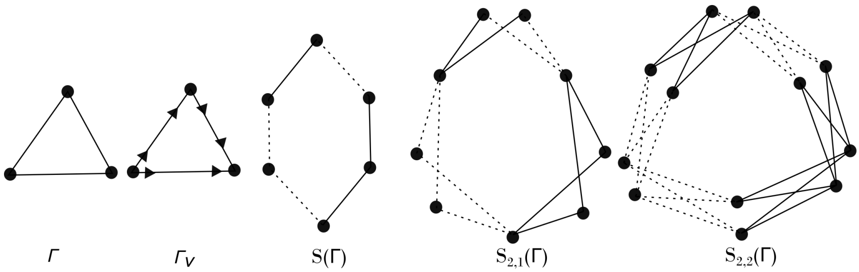

Figure 8: Signed graphs Γ, Γ_ϑ, $S(\Gamma)$, $S_{2,1}(\Gamma)$ and $S_{2,2}(\Gamma)$.

Subcase 1.2. If $p = k-1$, then let $U = \begin{pmatrix} \sqrt{p}X \\ \sqrt{k}Y \\ \vdots \\ \sqrt{p}X \\ \sqrt{k}Y \\ \sqrt{p}X \end{pmatrix}$. Clearly, $U \in M_{(kn+pm)\times 1}(\mathbb{R})$ is a non-zero column vector. We have

$$A(S_{k,p}(\Gamma))U = \begin{pmatrix} O & B & O & \dots & B & O \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \\ \vdots & \vdots & \ddots & \vdots & & \\ B^T & O & B^T & \dots & O & B^T \\ O & B & O & \dots & B & O \end{pmatrix} \begin{pmatrix} \sqrt{p}X \\ \sqrt{k}Y \\ \vdots \\ \sqrt{p}X \\ \sqrt{k}Y \\ \sqrt{p}X \end{pmatrix} = \begin{pmatrix} p\lambda_i\sqrt{k}X \\ k\lambda_i\sqrt{p}Y \\ \vdots \\ p\lambda_i\sqrt{k}X \\ k\lambda_i\sqrt{p}Y \\ p\lambda_i\sqrt{k}X \end{pmatrix}$$
$$= \sqrt{kp}\lambda_i U.$$

Therefore $\sqrt{kp}\lambda_i$ is an eigenvalue of $S_{k,p}(\Gamma)$ corresponding to an eigenvector U. Hence the result follows by Corollary 4.4.

Case 2. When Γ is an unbalanced signed graph with n vertices and m edges, the proof is similar to that of Case 1. □

Various constructions for non-isomorphic cospectral regular graphs, non-isomorphic Laplacian cospectral graphs and non-isomorphic signless Laplacian cospectral graphs can be seen in [5, 7, 8, 10, 11, 13]. The following results show that these constructions including the constructions obtained in the last section can be utilized to obtain infinite families of switching non-isomorphic cospectral signed graphs.

Corollary 4.6. *Let Γ_1 and Γ_2 be two switching non-isomorphic signed graphs which are Laplacian cospectral. Then*

(i) *the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are switching non-isomorphic and cospectral,*

(ii) *for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$ are switching non-isomorphic and cospectral.*

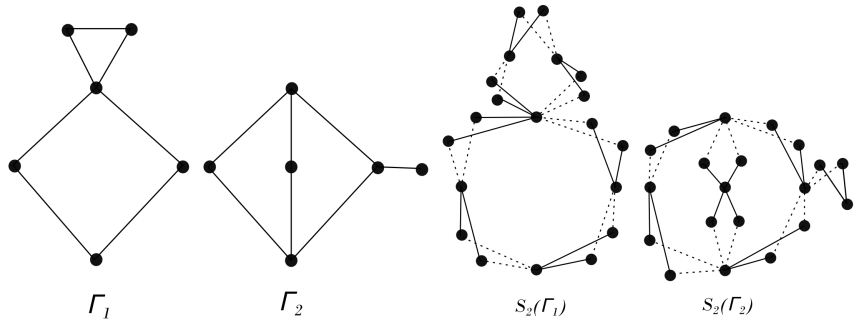

Figure 9: Cospectral signed graphs $S_2(\Gamma_1)$ and $S_2(\Gamma_2)$.

Proof. Let Γ_1 and Γ_2 be two switching non-isomorphic signed graphs. Then, clearly $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are switching non-isomorphic signed graphs and $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$ are switching non-isomorphic signed graphs. Hence the result follows by Theorems 4.3 and 4.5. □

Example 4.7. Consider the two non-isomorphic signed graphs Γ_1 and Γ_2 as shown in Figure 9. Their Laplacian spectrum is respectively given by $Spec_L(\Gamma_1) = \{0, 2, 3^{(2)}, 3 + \sqrt{5}, 3 - \sqrt{5}\}$ and $Spec_L(\Gamma_2) = \{0, 2, 3^{(2)}, 3 + \sqrt{5}, 3 - \sqrt{5}\}$. So Γ_1 and Γ_2 are Laplacian cospectral. It is easy to see that $S_2(\Gamma_1)$ and $S_2(\Gamma_2)$ are non-isomorphic signed graphs which are cospectral as their adjacency spectrum are, respectively, given by $Spec(S_2(\Gamma_1))$ $= \{0^{(10)}, \pm 2, \pm\sqrt{6}^{(2)}, \pm(\sqrt{6 + \sqrt{20}}), \pm(\sqrt{6 - \sqrt{20}})\}$ and $Spec(S_2(\Gamma_2)) = \{0^{(10)}, \pm 2,$ $\pm\sqrt{6}^{(2)}, \pm(\sqrt{6 + \sqrt{20}}), \pm(\sqrt{6 - \sqrt{20}})\}$.

Corollary 4.8. *Let Γ_1 and Γ_2 be two switching non-isomorphic cospectral $r-$regular signed graphs. Then*

(i) *the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are switching non-isomorphic and cospectral,*

(ii) *for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$ are switching non-isomorphic and cospectral.*

Proof. If Γ_1 and Γ_2 are two switching non-isomorphic cospectral regular signed graphs, then $L(\Gamma_1) = D(\Gamma_1) - A(\Gamma_1)$ and $L(\Gamma_2) = D(\Gamma_2) - A(\Gamma_2)$ are cospectral. Hence the result follows by Corollary 4.6. □

Corollary 4.9. *Let Γ be a signed graph whose all Laplacian eigenvalues are perfect squares. Then*

(i) *the signed graph $S_p(\Gamma)$ is integral, if p is a perfect square,*

(ii) *for $p \in \{k, k-1\}$, the signed graph $S_{k,p}(\Gamma)$ is integral, if kp is a perfect square.*

Example 4.10. Let K_n be a balanced complete signed graph on n vertices, where $n = t^2$, $t \geq 2$ is a positive integer. Then

(i) the signed graph $S_p(K_n)$ is integral, if p is a perfect square,

(ii) the signed graph $S_{k,p}(K_n)$ is integral, if kp is a perfect square.

The following result is the graceful implication of Lemma 2.7 and Corollaries 4.6 and 4.8.

Theorem 4.11. *For infinitely many n, there exists a family of 2^k pairwise switching nonisomorphic cospectral signed graphs on n vertices, where $k > \frac{n}{(2log_2(n))}$.*

The next result directly follows from Theorems 4.3 and 4.5.

Theorem 4.12. *Let Γ be a signed graph with n vertices and m edges. Then*

(i) $\mathcal{E}(S_p(\Gamma)) = \sqrt{p}\mathcal{E}(S(\Gamma))$,

(ii) $\mathcal{E}(S_{k,p}(\Gamma)) = \sqrt{pk}\mathcal{E}(S(\Gamma))$, *where* $p \in \{k, k-1\}$.

Theorem 4.13. *Let Γ be an unbalanced unicyclic signed graph with at least one edge and having Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0$. Then $S(\Gamma) \times K_2$ and $S(\Gamma) \otimes K_2$, where K_2 is a complete signed graph on 2 vertices, are noncospectral and equienergetic if and only if $\mu_n \geq 1$.*

Proof. Let Γ be an unbalanced unicyclic signed graph. Then, by Theorem 4.3, we have

$$Spec(S(\Gamma)) = \{\pm\sqrt{\mu_1}^{(1)}, \pm\sqrt{\mu_2}^{(1)}, \ldots, \pm\sqrt{\mu_{n-1}}^{(1)}, \pm\sqrt{\mu_n}^{(1)}\}.$$

First, assume that $\mu_n \geq 1$. This implies that $|\sqrt{\mu_j}| \geq 1$, for all $j = 1, 2, \ldots, n$. Also,

$$\mathcal{E}(S(\Gamma) \times K_2) = 2\sum_{j=1}^{n}(|\sqrt{\mu_j} + 1| + |\sqrt{\mu_j} - 1|).$$

As $|\sqrt{\mu_j}| \geq 1$, for all $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}\mathcal{E}(S(\Gamma) \times K_2) &= 2\sum_{j=1}^{n}(|\sqrt{\mu_j}| + 1 + |\sqrt{\mu_j}| - 1)\\ &= 2\mathcal{E}(S(\Gamma))\\ &= \mathcal{E}(S(\Gamma))\mathcal{E}(K_2) = \mathcal{E}(S(\Gamma) \otimes K_2).\end{aligned}$$

Note that $\sqrt{\mu_1} + 1 \in Spec(S(\Gamma) \times K_2)$ but $\sqrt{\mu_1} + 1 \notin Spec(S(\Gamma) \otimes K_2)$. Therefore $S(\Gamma) \times K_2$ and $S(\Gamma) \otimes K_2$ are noncospectral. The converse is similar to that of the converse in Lemma 2.4. □

Example 4.14. Let $C_3^- = (C_3, -)$ be an unbalanced unicyclic signed graph on 3 vertices. Its Laplacian spectrum is given by $Spec_L(C_3^-) = \{4, 1, 1\}$. Therefore C_3^- meets the requirement of Theorem 4.13. Hence $S(C_3^-) \times K_2$ and $S(C_3^-) \otimes K_2$ are noncospectral and equienergetic.

The following corollary directly follows from Theorem 4.12.

Corollary 4.15. *Let Γ_1 and Γ_2 be two signed graphs whose subdivision signed graphs are noncospectral and equienergetic. Then*

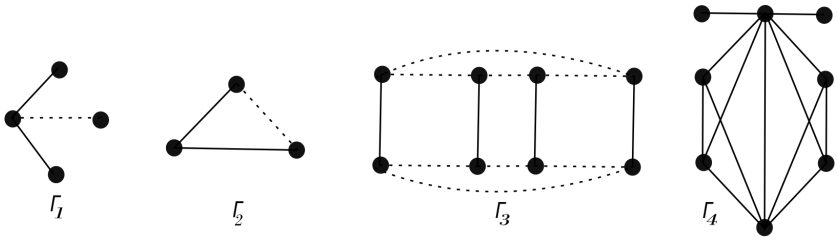

Figure 10: Signed graphs Γ_1, Γ_2, Γ_3 and Γ_4.

(i) *the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$are noncospectral and equienergetic,*

(ii) *for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_1)$ and $S_{k,p}(\Gamma_2)$are noncospectral and equienergetic.*

Example 4.16. Consider the signed graphs Γ_1, Γ_2, Γ_3 and Γ_4 as shown in Figure 10. The adjacency spectrum of their subdivision signed graphs is respectively given by $Spec(S(\Gamma_1)) = \{\pm 2, \pm 1^{(2)}, 0\}$, $Spec(S(\Gamma_2)) = \{\pm 2, \pm 1^{(2)}\}$, $Spec(S(\Gamma_3)) = \{\pm 2^{(3)}, \pm \sqrt{2}^{(3)}, \pm\sqrt{6}, 0^{(6)}\}$ and $Spec(S(\Gamma_4)) = \{\pm 1^{(2)}, \pm 2^{(2)}, \pm\sqrt{2}, \pm 2\sqrt{2}, \pm\sqrt{6}, 0^{(7)}\}$. Clearly, the signed graphs $S(\Gamma_1)$ and $S(\Gamma_2)$ are noncospectral and equienergetic. Similarly, the signed graphs $S(\Gamma_3)$ and $S(\Gamma_4)$ are noncospectral and equienergetic. Thus, by Corollary 4.15, we have

(i) the signed graphs $S_p(\Gamma_1)$ and $S_p(\Gamma_2)$ are noncospectral and equienergetic,

(ii) the signed graphs $S_p(\Gamma_3)$ and $S_p(\Gamma_4)$ are noncospectral and equienergetic,

(iii) for $p \in \{k, k-1\}$, the signed graphs $S_{k,p}(\Gamma_3)$ and $S_{k,p}(\Gamma_4)$ are noncospectral and equienergetic.

Conclusion. In this paper, we generalized the construction of the subdivision graph $S(\Gamma)$ to $S_{k,p}(\Gamma)$ of a signed graph Γ. The adjacency spectrum of $S_{1,p}(\Gamma)(S_p(\Gamma))$, $S_{p,p-1}(\Gamma)$ and $S_{p,p}(\Gamma)$ is completely determined by the Laplacian spectrum of Γ. Now, it remains a problem to investigate the adjacency spectrum of $S_{k,p}(\Gamma)$ for other values of k and p.

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