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Polytopes associated to dihedral groups

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Abstract

In this note we investigate the convex hull of those $n \times n$ permutation matrices that correspond to symmetries of a regular n-gon. We give the complete facet description. As an application, we show that this yields a Gorenstein polytope, and we determine the Ehrhart h^*-vector.

Keywords: Permutation polytopes, dihedral groups, lattice polytopes.

Math. Subj. Class.: 20B35, 52B12; 05E10, 52B05, 52B20

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1 Introduction

To any finite group G of real $n \times n$ permutation matrices we can associate the *permutation polytope* $P(G)$ given by the convex hull of these matrices in the vector space $\mathbb{R}^{n\times n}$. A well-known example of such a polytope is the Birkhoff polytope B_n, which is defined as the convex hull of all $n \times n$ permutation matrices [9, 8]. This polytope appears in various contexts in mathematics from optimization to statistics to enumerative combinatorics. (See, e.g., [24, 20, 21, 2, 1].) It is also a famous example of a Gorenstein polytope (see Section 5). Gorenstein polytopes turn up in connection to mirror symmetry in theoretical physics.

Guralnick and Perkinson [15] studied polytopes associated to general subgroups G of the symmetric group and proved results about their dimension, and about the diameter of their vertex-edge graph. A systematic exposition of general permutation polytopes is given in [5]. There, we studied which groups lead to affinely equivalent polytopes, we considered products of groups and polytopes, classified low-dimensional cases, and we formulated several open conjectures.

In order to get an intuition about what one can expect in general, it is instructive to consider some special classes of permutation groups. A seemingly very difficult case is when G equals the group of even permutation matrices. Just to exhibit exponentially many facets is already a daunting task, for this see [17]. Even for cyclic G we showed in [6] that these polytopes have a surprisingly complex and not yet fully understood facet structure.

In [12] Collins and Perkinson studied polytopes given by Frobenius groups. A special case is the dihedral group D_n for n odd, which was considered in more detail by Steinkamp [22]. Since D_n is the automorphism group of a regular n-gon, the cases where n is even and odd are quite different.

The most recent paper on permutation polytopes [11] focused on determining the volumes of permutation polytopes associated to cyclic groups, dihedral groups, and Frobenius groups. In order to compute the volume of $P(D_n)$, the authors find a Gale dual combinatorial description, which they use to provide an explicit formula for the Ehrhart polynomial of $P(D_n)$.

The dihedral group D_n is the automorphism group $\mathrm{Aut}(C_n)$ of a cycle C_n, and any permutation matrix $M(\sigma)$ of an element $\sigma \in D_n$ commutes with the adjacency matrix A of C_n. So any point in $P(D_n)$ commutes with A, and

$$P(D_n) \subseteq \{M \in \mathbb{R}^{n\times n} \mid M \text{ is doubly stochastic and } MA = AM\}\,.$$

Here, a matrix is doubly stochastic if all entries are non-negative and each row and column sum is 1. Tinhofer [24] asks, more generally, for a classification of those undirected graphs G where the two sets above are equal, i.e. where the commutation condition $MA = AM$ already suffices to characterize the elements of $P(\mathrm{Aut}(G))$ among all doubly stochastic matrices. The Birkhoff-von Neumann theorem is the special case where A is the unit matrix. Tinhofer shows that this also holds for the adjacency matrices of cycles and trees [24, Theorems 2&3].

In this note, we independently investigate $P(D_n)$ in a more direct and elementary way. We give a complete list of its facet inequalities (Theorem 3.3, Theorem 4.1). As an application, we observe that these lattice polytopes are Gorenstein polytopes, and we get a nice description of the generating function of their Ehrhart polynomials (Theorem 5.3, Corollary 5.4).

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2 Notation and preliminary results

Let S_n be the permutation group on $n \geq 3$ elements. Every permutation $\sigma \in S_n$ can be represented by an $n \times n$ matrix M_σ with entries $\delta_{i,(j)\sigma}$. So the entries are in $\{0,1\}$ and there is exactly one 1 in each row and column. Notice that we apply matrices and permutations from the right. We can view such a matrix as a vector in $\mathbb{R}^{n^2}$. For a subgroup G of S_n we define the polytope

$$P_G := \operatorname{conv}(M_\sigma \mid \sigma \in G)\,.$$

This is a $0/1$-polytope, so all matrices are in fact vertices of the polytope.

We denote by D_n the subgroup of S_n corresponding to the symmetry group of the regular n-gon, the *dihedral group of order* $2n$. This group is generated by two elements. If n is odd, then these may taken to be the rotation ρ of the n-gon by $360/n$ degrees, and the reflection τ along a line through one vertex and the midpoint of the opposite edge. If n is even, then the second generator τ is instead the reflection along a line through two opposite vertices. Thus ρ is the permutation $(1,2,\ldots,n)$ and τ the reflection $(2,n)(3,n-1)\cdots((n+1)/2,(n+3)/2)$ if n is odd and $(2,n)(3,n-1)\cdots(n/2,(n/2)+2)$ if n is even.

The associated permutation polytope is the convex hull of the corresponding matrices,

$$\mathrm{DP}_n := \operatorname{conv}(M_\sigma \mid \sigma \in D_n)\,.$$

The dihedral group D_n has $2n$ elements

$$\rho^0, \rho^1, \rho^2, \ldots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \tau\rho^3, \ldots, \tau\rho^{n-1}.$$

We label the vertices of DP_n by $v_0, \ldots, v_{n-1}, w_0, \ldots, w_{n-1}$ in this order. Let us give a more convenient way to write these matrices.

Let I be the n-dimensional identity matrix and R be the $n \times n$ matrix that has 0's everywhere except at the n entries (i,j), where $0 \leq i,j \leq n-1$ and $j \equiv i+1 \mod n$:

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Reading the matrices M_σ row by row, we can identify M_σ with a (row) vector in $\mathbb{R}^{n^2}$. For instance, the 2×2 identity matrix would be identified with $(1\ 0\ 0\ 1)$. Under this identification the vertices of DP_n are (in the order given above) the rows of the $2n \times n^2$ matrix

$$\begin{bmatrix} R^0 & R^1 & R^2 & \cdots & R^{n-1} \\ R^0 & R^{-1} & R^{-2} & \cdots & R^{-(n-1)} \end{bmatrix}.$$

Permuting the coordinates (corresponding to a linear automorphism of $\mathbb{R}^{n^2}$) we may write the vertices in the form

$$V = \begin{bmatrix} I & I & I & \cdots & I \\ I & R^{-2} & R^{-4} & \cdots & R^{-2(n-1)} \end{bmatrix}. \tag{2.1}$$

Clearly, the first $2n$ coordinates of the vertices linearly determine the remaining coordinates. So we can project onto $\mathbb{R}^{2n}$ without changing the combinatorics of the polytope. Hence, we observe that the dimension of DP_n is at most $2n$.

3 The situation for odd n

In this section we completely describe DP_n for n odd. As it will turn out, it is useful to introduce a new polytope that will serve as a basic building block for both situations of even n and odd n.

Definition 3.1. Let Q_n be the polytope defined as the convex hull of the rows of the $2n \times n^2$ matrix

$$W := \begin{bmatrix} I & I & I & \dots & I \\ I & R^1 & R^2 & \dots & R^{n-1} \end{bmatrix}. \tag{3.1}$$

While Q_n differs from DP_n for even n, for odd n the R^{2k} for $0 \leq k \leq n-1$ are a permutation of the R^k for $0 \leq k \leq n-1$. So we deduce from (2.1) that, for n odd, Q_n is up to a permutation of coordinates just the polytope DP_n.

Proposition 3.2. *For odd n, the polytopes DP_n and Q_n are affinely isomorphic.* □

The following theorem examines the structure of Q_n for arbitrary n. For n odd, this result is a special case of Theorem 4.4 in [12].

Let us fix some convenient notation. We denote by Δ_r the r-dimensional simplex. We also use for any two integers s, k, the term $[s]_k \in \{0, \dots, k-1\}$ to denote the remainder of s upon division by k. The *free sum* of two polytopes P and P' of dimensions d and d' is the polytope

$$P \oplus P' := \mathrm{conv}(\{(p, 0) \in \mathbb{R}^{d+d'} \mid p \in P\} \cup \{(0, p') \in \mathbb{R}^{d+d'} \mid p' \in P'\}).$$

Theorem 3.3 (Collins&Perkinson [12])**.** *Let n be odd or even. The polytope Q_n has dimension $2n-2$ and is a free sum of two copies of Δ_{n-1}. Taking coordinates $x_0, \dots, x_{n^2-1}$ for $\mathbb{R}^{n\times n}$, its affine hull is given by the equations*

$$1 = \sum_{i=ln}^{(l+1)n-1} x_i \tag{aff}$$

$$0 = x_{kn+[j]_n} - x_{(k+1)n+[j]_n} - x_{(k+1)n+[j+1]_n} + x_{(k+2)n+[j+1]_n} \tag{$A_{j,k}$}$$

for $0 \leq l \leq n-1$, $0 \leq j \leq n-2$, $0 \leq k \leq n-3$.

An irredundant system of inequalities defining the polytope inside its affine hull is given by the inequalities

$$x_i \geq 0$$

for $0 \leq i \leq n^2 - 1$.

Proof. All the given equations are satisfied by the vertices of Q_n. There are n equations of type (aff) and $n^2 - 3n + 2$ equations of type ($A_{j,k}$). They are easily seen to be linearly independent, so the dimension of Q_n is at most $2n - 2$. On the other hand, deleting any row of W leaves us with a linearly (and hence affinely) independent set of row vectors. (Observe that deleting a row leaves us with a column that contains exactly one 1.) Hence, $\dim(Q_n) = 2n - 2$ and the given equations define the affine hull of Q_n in $\mathbb{R}^{n^2}$.

Further, we see that every $2n - 1$ of the $2n$ rows of W span the affine hull of Q_n. So any facet of Q_n has $2n - 2$ vertices. Since the inequalities $x_j \geq 0$ are 0 on exactly $2n - 2$ of the rows, they all define facets.

In order to prove that Q_n is a free sum of simplices we observe that the first n and the last n vertices define $(n - 1)$-dimensional simplices sitting in transversal subspaces (intersecting in the matrix corresponding to the row vector $(1/n, \ldots, 1/n)$). Therefore, the combinatorial dual of Q_n corresponds to the product of Δ_{n-1} with itself. In particular, Q_n has precisely n^2 facets, so the facet description given above is complete. □

4 The situation for even n

Recall that the *join* $P \star Q$ of two polytopes P and Q is the convex hull of $P \cup Q$ after embedding P and Q in skew affine subspaces. The dimension of $P \star Q$ equals $\dim(P) + \dim(Q) + 1$. For instance, the join of two intervals is a tetrahedron.

Theorem 4.1. *Let n be even. The polytope DP_n is a join of two copies of $Q_{n/2}$. In particular, its dimension is $2n - 3$.*

Combined with Theorem 3.3, this result gives a complete description of the facet inequalities and the affine hull equations of DP_n for n even.

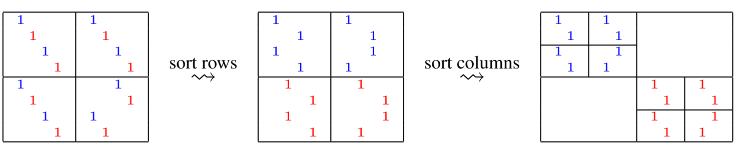
Proof. Permuting the coordinates, we can transform V (see (2.1)) into

$$\begin{bmatrix} I & I & I & \cdots & I & & I & I & I & \cdots & I \\ R^0 & R^2 & R^4 & \cdots & R^{n-2} & & R^0 & R^2 & R^4 & \cdots & R^{n-2} \end{bmatrix}.$$

Clearly, projecting onto the first $\frac{n^2}{2}$ coordinates yields an affine isomorphism of DP_n onto the convex hull of the rows of the $2n \times \frac{n^2}{2}$ matrix

$$\begin{bmatrix} I & I & I & \ldots & I \\ R^0 & R^2 & R^4 & \ldots & R^{n-2} \end{bmatrix}.$$

In the representation given by this matrix let us partition the set of $2n$ vertices (labelled from 0 to $2n - 1$) into two sets: consisting of the n rows with even index and the n rows with odd index.



Then we permute the $\frac{n^2}{2}$ coordinates in such a way that in the first set of rows (corresponding to even vertices) all nonzero entries are in the first half (i.e. in the first $\frac{n^2}{4}$ columns).

Then all nonzero entries in the second set of rows (corresponding to the odd vertices) will be in the second half (i.e. in the last $\frac{n^2}{4}$ columns). By a permutation of the coordinates within the first half we get that the rows of even vertices yield precisely the vertex set of $Q_{n/2} \times \{0\}$ (for $0 \in \mathbb{R}^{\frac{n^2}{4}}$). In the same way, the coordinates in the second half can be permuted so that the rows of odd vertices equal the vertices of $\{0\} \times Q_{n/2}$ (for $0 \in \mathbb{R}^{\frac{n^2}{4}}$). Since 0 is not in the affine hull of $Q_{n/2}$, we deduce that DP_n is a join of two copies of $Q_{n/2}$. Hence, its dimension equals $2\dim(Q_{n/2}) + 1 = 2(n-2) + 1 = 2n - 3$ by Theorem 3.3. □

5 Lattice properties

DP_n and Q_n are *lattice polytopes*, i.e. their vertices lie in the lattice $\mathbb{Z}^{n^2}$ of integral vectors. It is readily checked that all above affine isomorphisms respect lattice points. In this section, we will show that these lattice polytopes have especially nice properties which allow us to completely describe their Ehrhart h^*-vectors.

A d-dimensional lattice polytope P containing 0 in its interior is *reflexive*, if its polar (or dual) polytope

$$P^* := \{x \in \mathbb{R}^d \mid \langle x, v \rangle \geq -1 \; \forall \, v \in P\}$$

is again a lattice polytope (in the dual lattice). This notion was introduced by Batyrev in [3]. A generalization of this is the class of Gorenstein polytopes. A lattice polytope is a *Gorenstein polytope of codegree* k, if there is a positive integer k and an interior lattice point m in kP such that $kP - m$ is a reflexive polytope. Such polytopes play an important role in the classification of Calabi-Yau manifolds for string theory. See [4] for basic properties. The next proposition tells us that the polytopes Q_n belong to this class. The *normalized volume of* $\mathbb{R}^n$ is the volume form which assigns to the standard simplex the volume 1.

Proposition 5.1. *Let n be odd or even. The polytope Q_n is Gorenstein of codegree n and normalized volume n.*

Proof. By Theorem 3.3, the point $\frac{1}{n}(1, 1, \ldots, 1)$ is an interior point of Q_n with equal integral distance $1/n$ to all facets, and $m := (1, 1, \ldots, 1)$ is the unique interior lattice point in nQ_n. Hence $nQ_n - m$ is a reflexive polytope.

By Theorem 3.3, all facets of Q_n are simplices of facet width 1, hence they are all unimodular. As we have seen, multiplying by n gives (up to translation) a reflexive polytope with the unique interior lattice point $m = (1, 1, \ldots, 1)$. The normalized volume of nQ_n is the sum of the volumes of n^2 pyramids over facets with apex m. But in nQ_n each facet has normalized volume n^{2n-3}, and the apex has lattice distance 1 from the facet, so each pyramid has normalized volume n^{2n-3}. There are n^2 of these pyramids, so the normalized volume of nQ_n equals n^{2n-1}. Dividing by n^{2n-2} to get from nQ_n back to Q_n gives the normalized volume n of Q_n. □

A polytope P is *compressed* if every so-called pulling triangulation is regular and unimodular. Equivalently, P is compressed if for any supporting inequality $a^t x \leq b$ with a primitive integral normal a, i.e. with a normal vector whose entries are integers and which is not an integral multiple of some other integer vector, the polytope is contained in the set $\{x \mid b - 1 \leq a^t x \leq b\}$. For a more detailed explanation of these terms we refer to [13].

This property has strong implications on the associated toric ideal, see e.g. [23]. The next proposition follows immediately from Theorem 1.1 of [19] and Theorem 3.3.

Proposition 5.2. *Let n be odd or even. The polytope Q_n is compressed.* □

The *Ehrhart polynomial* $L_P(k) := |kP \cap \mathbb{Z}^d|$ of a d-dimensional lattice polytope counts the number of integral points in integral dilates of P. It is well known that the generating function of L_P is given by

$$\sum_{m\geq 0} L_P(m)t^m = \frac{h^*(t)}{(1-t)^{d+1}}$$

for some polynomial h^* of degree at most d with integral non-negative coefficients, see [7]. Hence, determining the Ehrhart polynomial is equivalent to finding the h^*-vector (also called the δ-vector) of coefficients of $h^*(t)$. As is well-known, P is Gorenstein if and only if the h^*-vector is symmetric. The following theorem shows that in our case this vector has a particularly nice form.

Theorem 5.3. *Let n be odd or even. The h^*-vector of Q_n satisfies $h^*_i = 1$ for $0 \leq i \leq n-1$ and $h^*_i = 0$ otherwise.*

Proof. Since the codegree of Q_n is n and its dimension is $2n-2$ by Theorem 3.3, the maximal non-zero entry of the h^*-vector has to be h^*_{n-1}, see [7]. By a theorem of Bruns and Römer [10] we know that the h^*-vector of a Gorenstein polytope that has a regular unimodular triangulation is symmetric and unimodal. In particular, $h^*_i \geq 1$ for $i = 0, \dots, n-1$. Since by Proposition 5.1 the sum of the entries of the h^*-vector equals n, the statement follows. □

In particular, if n is odd, the previous result describes the h^*-vector of DP_n. Finally, let us deal with the even case.

Corollary 5.4. *Let n be even. The h^*-vector of DP_n equals*

$$(1, 2, 3, \dots, \frac{n}{2}-1, \frac{n}{2}, \frac{n}{2}-1, \dots, 2, 1).$$

In particular, the polytope DP_n is Gorenstein of codegree n and normalized volume $n^2/4$.

Proof. By the proof of Theorem 4.1, DP_n is given up to coordinate permutation as the convex hull of the rows of

$$\begin{bmatrix} \tilde{W} & 0 \\ 0 & \tilde{W} \end{bmatrix},$$

where $\tilde{W}$ is the $n \times (\frac{n}{2})^2$ matrix whose rows are the vertices of $Q_{\frac{n}{2}}$ as given in (3.1). The integral linear functional which sums the first $\frac{n}{2}$ coordinates evaluates to 1 on the first $\frac{n}{2}$ rows, and to 0 on the second half. Hence, the two copies of $Q_{\frac{n}{2}}$ (say, $P_1 \times \{0\}$ and $\{0\} \times P_2$) have lattice distance 1 in the lattice $\mathbb{Z}^{\frac{n^2}{2}} \cap \operatorname{aff} \mathrm{DP}_n$. In other words, there is an affine isomorphism respecting lattice points which maps DP_n onto the convex hull of $P_1 \times \{0\} \times \{1\}$ and $\{0\} \times P_2 \times \{0\}$ in $\mathbb{R}^{\frac{n^2}{2}+1}$. Therefore, the statement follows from the well-known fact [7, Example 3.32] that in this case the h^*-polynomial equals the product of the h^*-polynomials of P_1 and P_2. □

6 Substructures

In [5] the authors discussed which subgroups of a permutation group yield faces of $P(G)$. An obvious class of such subgroups are stabilizers:

Take a partition $[n] := \{1, \ldots, n\} = \bigsqcup I_i$. Then the polytope of the stabilizer of the subsets I_i

$$\mathrm{stab}(G; (I_i)_i) := \{\sigma \in G \mid \sigma(I_i) = I_i \text{ for all } i\} \leq G$$

is a face of $P(G)$. The authors conjecture that there are no other examples.

Conjecture 5.8 [5] Let $G \leq S_n$. Suppose $H \leq G$ is a subgroup such that $P(H) \preceq P(G)$ is a face. Then $H = \mathrm{stab}(G; (I_i)_i)$ for a partition $[n] = \bigsqcup I_i$.

We have verified the conjecture for $G = S_n$ as well as for cyclic subgroups $G \leq S_n$, see Proposition 5.9 of [5]. Meanwhile Jessica Nowack and Daniel Heinrich studied this question for the dihedral groups in their Diploma theses.

Proposition 6.1. *(Heinrich, Nowack [16, 18]) Conjecture 5.8 holds for $G = D_n \leq S_n$ for every n.*

Sketch of the proof. For n odd Heinrich first shows that, if H is the subgroup of all rotations of G, then P_H is not a face of P_G. The remaining subgroups are precisely the stabilizers of their orbits, see Theorem 7.1.1 of [16].

For n even the main work is to show that the subgroup of all rotations, the subgroup of the squares of the rotations and finally the subgroup generated by the squares of the rotations and by the reflections through two edges are precisely those subgroups H of G for which P_H is not a face of P_G. Nowack shows that the remaining subgroups are precisely the stabilizers of their orbits, see Section 4.2 of [18]. □

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