Also available at http://amc.imfm.si
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 7 (2014) 40–53

A class of semisymmetric graphs

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Received 20 November 2011, accepted 1 October 2012, published online 14 January 2013

Abstract

A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. Every semisymmetric graph is a bipartite graph with two parts of equal size. Let p be a prime. In this paper, a class of semisymmetric graphs of order $2p^3$ are determined. This work is a partial result for our long term goal to classify all semisymmetric graphs of order $2p^3$.

Keywords: Permutation group, primitive group, vertex-transitive graph, semisymmetric graph.

Math. Subj. Class.: 05C10, 05C25, 20B25

1 Introduction

All graphs considered in this paper are finite, undirected, connected and simple. For a graph X, we use $V(X)$, $E(X)$ and $A := \mathrm{Aut}(X)$ to denote its vertex set, edge set and the full automorphism group, respectively. The graph is said to be vertex-transitive and edge-transitive, if A acts transitively on $V(X)$ and $E(X)$, respectively. If X is bipartite with bipartition $V(X) = W(X) \cup U(X)$, we let A^+ be the subgroup of A preserving both

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$W(X)$ and $U(X)$. Since X is connected, we have that either $|A : A^+| = 2$ or $A = A^+$, depending on whether or not there exists an automorphism which interchanges the two parts. For $G \leq A^+$, the graph X is said to be *G-semitransitive* if G acts transitively on both $W(X)$ and $U(X)$, and *semitransitive* if X is A^+-semitransitive.

We call a graph *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is a semitransitive bipartite graph with two parts of equal size.

The first person who studied semisymmetric graphs was Folkman. In 1967 he constructed several infinite families of such graphs and proposed eight open problems see [13]. Afterwards, Bouwer, Titov, Klin, I.V. Ivanov, A.A. Ivanov and others did much work on semisymmetric graphs see [2, 4, 16, 17, 18, 30]. They gave new constructions of such graphs and nearly solved all of Folkman's open problems. In particular, Iofinova and Ivanov [16] in 1985 classified biprimitive semisymmetric cubic graphs using group-theoretical methods. This was the first classification theorem for such graphs. More recently, following some deep results in group theory which depend on the classification of finite simple groups and some methods from graph coverings, some new results on semisymmetric graphs have appeared. For instance, in [11] Du and Xu classified semisymmetric graphs of order $2pq$ for two different primes p and q. For more papers on semisymmetric graphs see [5, 7, 8, 9, 10, 11, 12, 19, 21, 22, 23, 24, 25, 26, 27, 28, 33].

In [13], Folkman proved that there are no semisymmetric graphs of order $2p$ and $2p^2$ where p is a prime. Then we are interesting in determining semisymmetric graphs of order $2p^3$, where p is prime. Since the smallest semisymmetric graphs have the order 20 [13], we let $p \geq 3$. It is proved in [25] that the Gray graph of order 54 is the only cubic semisymmetric graph of order $2p^3$. To classify all semisymmetric graphs of order $2p^3$ is still one of attractive and difficult problems. These graphs X are naturally divided into two subclasses:

(1) Aut(X) acts unfaithfully on at least one part;

(2) Aut(X) acts faithfully on both parts.

Now we are going to concentrate on Subclass (1). To state our main theorem, we first introduce two concepts.

Let Y be a connected semitransitive and edge-transitive graph with bipartition $V(Y) = W(Y) \cup U(Y)$, where $W(Y) = Z_p^3$ and $U(Y) = Z_p^2$ for an odd prime p. For distinguishing the vertices of $W(Y)$ and $U(Y)$ convenience, the vertices of $W(Y)$ and $U(Y)$ are denoted by $(i, j, k, 0)$ and $(y, z, 1)$, respectively, where $i, j, k, y, z \in Z_p$. Now we define a bipartite graph X with bipartition $W(X) \cup U(X)$, where

$$W(X) = W(Y),\ U(X) = Z_p \times U(Y) = \{(x, y, z, 1) \mid x, y, z \in Z_p\},$$

such that two vertices $(i, j, k, 0) \in W(X)$ and $(x, y, z, 1) \in U(X)$ are adjacent if $\{(i, j, k, 0), (y, z, 1)\} \in E(Y)$. From now on, we shall say that the graph X is the *graph expanded from* Y and that the graph Y is the *graph contracted from* X. Clearly X is edge-transitive and regular. Furthermore, since for any $(y, z, 1) \in U(Y)$, the p vertices $\{(x, y, z, 1) \mid x \in Z_p\}$ in $U(X)$ have the same neighborhood, X is semisymmetric, provided there exist no two vertices in $W(X)$ which have the same neighborhood. Clearly, Aut(X) acts unfaithfully on $W(X)$ and Aut$(X)/S_p^p \cong$ Aut(Y).

Note that the semisymmetric graphs where two vertices have the same neighbourhood have been studied in several papers see [11, 20, 29, 33], with different definitions, for instance, *X is a derived graph from Y*, *X is a unworthy graph*, *X is contracted to from Y* and so on.

Let Y be a connected graph and $\mathcal{B}$ an imprimitive system of $\mathrm{Aut}(Y)$. Define a graph Z with the vertex set $\mathcal{B}$ such that two blocks are adjacent in Z if there exists at least one edge in Y between two blocks. This graph Z is called the *block graph of Y*. Moreover, if $\mathcal{B}$ is the set of orbits of some nontrival normal subgroup N of $\mathrm{Aut}(Y)$, then we call Z the *block graph induced by N*.

The following proposition gives a characterization for Subclass (1) given in [31]:

Proposition 1.1. *Suppose X is a semisymmetric graph of order $2p^3$, where p is an odd prime, such that $\mathrm{Aut}(X)$ acts unfaithfully on at least one part. Then $\mathrm{Aut}(X)$ must act unfaithfully on one part and faithfully on the other part, and X is the graph expanded from the graph Y with bipartition $V(Y) = W(Y) \cup U(Y)$, where $W(Y) = Z_p^3$ and $U(Y) = Z_p^2$. Moreover, we have that either*

(1) *$p = 3$, $\mathrm{Aut}(Y) \cong S_3 \wr S_3$ which acts primitively on $W(Y)$; or*

(2) *$\mathrm{Aut}(Y)$ has blocks of length p^2 on $W(Y)$ and of length p on $U(Y)$. Let $\overline{Y}$ be the block graph of Y. Then either*

   (2.1) *the block graph $\overline{Y}$ is of valency at least 3, and $\mathrm{Aut}(Y)$ is solvable and contains a normal regular subgroup on $W(Y)$; or*

   (2.2) *the block graph $\overline{Y}$ is of valency 2, where $\mathrm{Aut}(Y)$ may be solvable or insolvable.*

Following Proposition 1.1, in this paper we shall determine the graphs in Case (2.2), while Cases (1) and (2.1) will be determined in our another paper. Before giving the main theorem of this paper, we first define six families of graphs Y.

Definition 1.2. We shall define six families of bipartite graphs X with bipartition $V(X) = W(X) \cup U(X)$, where

$$W(X) = \{(i, j, k, 0) \mid i, j, k \in Z_p\}, \quad U(X) = \{(x, y, z, 1) \mid x, y, z \in Z_p\},$$

and edge set

$$\begin{aligned} E(X) \;=\; & \{\{(i, j, k, 0), (x, i + b, k + \tfrac{p-1}{2}, 1)\} \mid i, j, k, x \in Z_p, b \in \Sigma\} \cup \\ & \{\{(i, j, k, 0), (x, j + sb, k + \tfrac{p+1}{2}, 1)\} \mid i, j, k, x \in Z_p, b \in \Sigma\}, \end{aligned}$$

where $s = \theta^{\frac{p-1}{2r}}$, $Z_p^* = \langle\theta\rangle$ for the family of graphs $X_2(p, r)$, $s = 1$ for other five families of graphs $X_i(p, r)$, and Σ is given by

(1) **Graphs $X_1(p, r)$:** Let $p \geq 3$ and let Σ be a subgroup of Z_p^* of order r, where $(p, r) \neq (7, 3), (11{,}5)$. Moreover, the valency of $X_1(p, r)$ is $2pr$ and the smallest examples are $X_1(3, 1)$ and $X_1(3, 2)$.

(2) **Graphs $X_2(p, r)$:** Let $p \geq 5$ and let Σ be a subgroup of Z_p^* of order $r \geq 2$, where $(p, r) \neq (7, 3), (11, 5)$ and $2r \mid (p - 1)$. Moreover, the valency of $X_2(p, r)$ is $2pr$ and the smallest example is $X_2(5, 2)$.

(3) **Graphs $X_3(11, 5)$:** Let $p = 11$ and let $\Sigma = \{0, 2, 3, 4, 8\} \subset Z_{11}$. Moreover, the valency of $X_3(11, 5)$ is 110.

(4) **Graphs** $X_4(11,6)$**:** Let $p = 11$ and $\Sigma = \{1,5,6,7,9,10\} \subset Z_{11}$. Moreover, the valency of $X_4(11,6)$ is 132.

(5) **Graphs** $X_5(p,r)$**:** Choose a point $\langle v\rangle$ and a hyperplane $\mathcal{L}$ in the project space $\mathrm{PG}(n-1,q)$, where $\frac{q^n-1}{q-1} = p \geq 7$, and let $G = \langle t\rangle$ be a Singer subgroup of $\mathrm{PGL}(n,q)$. Let $\Sigma = \{l \in Z_p \mid \langle v\rangle \in \mathcal{L}^{t^l}\}$, where $r = |\Sigma| = \frac{q^{n-1}-1}{q-1}$. Moreover, the valency of $X_5(p,r)$ is $2p\frac{q^{n-1}-1}{q-1}$ and the smallest example is $X_5(7,3)$.

(6) **Graphs** $X_6(p,r)$: Adopting the same notation as in (5), set $\Sigma = \{l \in Z_p \mid \langle v\rangle \notin \mathcal{L}^{t^l}\}$, where $r = q^{n-1}$. Moreover, the valency of $X_6(p,r)$ is $2pq^{n-1}$ and the smallest example is $X_6(7,4)$.

Remark 1.3. For $1 \leq i \leq 6$, let $X_i(p,r)$ be as in Definition 1.2. Then

(1) For any given $y,z \in Z_p$, the p vertices $\{(x,y,z,1) \mid x \in Z_p\}$ have the same neighborhood. Let $Y_i(p,r)$ be the contracted graph from $X_i(p,r)$, obtained by contracting each such p vertices into one vertex while preserving the adjacent relation, that is,

$$W(Y_i(p,r)) = W(X_i(p,r)),\ U(Y_i(p,r)) = \{(y,z,1) \mid y,z \in Z_p\}.$$

Then we shall see from the proof of Theorem 1.1 that $\mathrm{Aut}(Y_i(p,r)) = K \rtimes D_{2p}$, where the subgroup K is the following

(i) $Y_1(p,r)$ and $Y_2(p,r)$: $K = S_p^p$ if $r \in \{1,p-1\}$; $K = (Z_p \rtimes Z_r)^p$ if $r \notin \{1,p-1\}$;

(ii) $Y_3(11,5)$ and $Y_4(11,6)$: $K = (\mathrm{PSL}(2,11))^p$;

(iii) $Y_5(p,r)$ and $Y_6(p,r)$: $K = (\mathrm{P\Gamma L}(n,q))^p$.

(2) For any $k \in Z_p$, let

$$W_k(Y) = \{(i,j,k,0) \in W(Y) \mid i,j \in Z_p\}, \quad U_z(Y) = \{(y,z,1) \mid y \in Z_p\}.$$

Then we shall see from the proof of Theorem 1.1 that $\{W_k(Y) \mid k \in Z_p\}$ and $\{U_z(Y) \mid z \in Z_p\}$ are orbits of the group K on $W(Y)$ and $U(Y)$, respectively. Let $\overline{Y}$ be the block graph induced by K. Then $\overline{Y}$ is a cycle of length $2p$.

Now we give the main theorem of this paper.

Theorem 1.1. For an odd prime p, suppose that X is a semisymmetric graph of order $2p^3$ expanded from a graph Y such that $\mathrm{Aut}(Y)$ has the blocks of length p^2 on $W(Y)$ and of length p on $U(Y)$ while the block graph $\overline{Y}$ is a cycle of length $2p$. Then X is isomorphic to one of graphs $X_i(p,r)$ where $1 \leq i \leq 6$, defined in Definition 1.2.

After this introductory section, some preliminary results will be given in Section 2, and the main theorem will be proved in Sections 3. For group-theoretic concepts and notation not defined here the reader is refereed to [6, 15].

2 Preliminaries

First we introduce some notation. By H char G, we mean that H is a characteristic subgroup of G. Given a group G and a subgroup H of G, by $\mathrm{Cos}(G, H)$ we denote the set of right cosets of H in G. The action of G on $\mathrm{Cos}(G, H)$ is always assumed to be the right multiplication action. For two subgroups $N \lhd G$ and $H \leq G$, by $N \rtimes H$ we denote the semi-direct product of N by H, where N is normal. For a group G, by Exp (G) we denote the least common multiple of orders of all the elements of G. By $H \wr K$, we denote the wreath product of H and K.

A group-theoretic construction of semitransitive and semisymmetric graphs were given in [11]. Here we quote one definition and two results.

Definition 2.1. Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \bigcup_i Rd_iL$. Define a bipartite graph $X = (G, L, R; D)$ with bipartition $V(X) = \mathrm{Cos}(G, L) \cup \mathrm{Cos}(G, R)$ and edge set $E(X) = \{(Lg, Rdg) \bigm| g \in G, d \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D.

Proposition 2.1. *[11] The graph $X = \mathbf{B}(G, L, R; D)$ is a well-defined bipartite graph. Under the right multiplication action of G on $V(X)$, the graph X is G-semitransitive. The kernel of the action of G on $V(X)$ is $\mathrm{Core}_G(L) \cap \mathrm{Core}_G(R)$, the intersection of the cores of the subgroups L and R in G. Furthermore, we have*

(i) *X is G-edge-transitive if and only if $D = RdL$ for some $d \in G$;*

(ii) *the degree of any vertex in $\mathrm{Cos}(G, L)$ (resp. $\mathrm{Cos}(G, R)$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so X is regular if and only if $|L| = |R|$;*

(iii) *X is connected if and only if G is generated by elements of $D^{-1}D$;*

(vi) *$X \cong \mathbf{B}(G, L^a, R^b; D')$ where $D' = \bigcup_i R^b\ (b^{-1}d_ia)L^a$, for any $a, b \in G$;*

(v) *$X \cong \mathbf{B}(\hat{G}, L^\sigma, R^\sigma; D^\sigma)$ where σ is an isomorphism from G to $\hat{G}$ (it does not appear in [11] but it is easy to prove.)*

Proposition 2.2. *[11] Suppose Y is a G-semitransitive graph with bipartition $V(Y) = U(Y) \cup W(Y)$. Take $u \in U(Y)$ and $w \in W(Y)$. Set $D = \{g \in G \bigm| w^g \in Y_1(u)\}$. Then D is a union of double cosets of G_w and G_u in G, and $Y \cong \mathbf{B}(G, G_u, G_w; D)$.*

Proposition 2.3. *[32, 11.6, 11.7] Every permutation group of prime degree p is either insolvable and 2-transitive, or isomorphic to $Z_p \rtimes Z_s$ for some s dividing $p - 1$.*

Proposition 2.4. *[14] The insolvable permutation groups of prime degree p are given as follows, where T denotes be the socle of the group and H denotes a point stabilizer of T:*

(i) *$T = A_p$ and $H = A_{p-1}$;*

(ii) *$T = \mathrm{PSL}(n, q)$ and H is the stabilizer of a projective point or a hyperplane in $\mathrm{PG}(n - 1, q)$, and $|T : H| = (q^n - 1)/(q - 1) = p$;*

(iii) *$T = \mathrm{PSL}(2, 11)$ and $H = A_5$, and T has two conjugacy classes of subgroups isomorphic to A_5;*

(iv) *$T = M_{11}$ and $H = M_{10}$;*

(v) $T = M_{23}$ *and* $H = M_{22}$.

Lemma 2.5. *[31] Let G be an imprimitive transitive group of degree p^2 with $p \geq 3$ and $p^3 \mid |G|$. Suppose that G has an imprimitive system $\mathcal{B}$ with p-blocks and the kernel K. Let P be a Sylow p-subgroup of G and $N = P \cap K$. Then*

(1) $\mathrm{Exp}\,(P) \leq p^2$, $|Z(P)| = p$ *and* $P = N\langle t\rangle$, *where* $t^p \in Z(P)$;

(2) K *is solvable,* N char K *and so* $N \lhd G$, *provided either* $p = 3$; *or* $p \geq 5$ *and* $|N| \leq p^{p-1}$.

3 Proof of the main theorem

To prove Theorem 1.1, we assume that p is an odd prime and that X is a semisymmetric graph of order $2p^3$ expanded from the graph Y, where $\mathrm{Aut}(Y)$ acts edge transitively on Y and has blocks of length p^2 on $W(Y)$ and of length p on $U(Y)$, and the block graph $\overline{Y}$ is a cycle C_{2p} of length $2p$.

Let $F = \mathrm{Aut}(Y)$ and let

$$\mathcal{B} = \{B_0, B_1, \cdots, B_{p-1}\} \qquad \text{and} \qquad \mathcal{B}' = \{B'_0, B'_1, \cdots, B'_{p-1}\}$$

be blocks system of F on $U(Y)$ and $W(Y)$, respectively. Label

$$E(\overline{Y}) = \{(B_0, B'_{\frac{p+1}{2}}), (B'_{\frac{p+1}{2}}, B_1), \cdots, (B_{\frac{p-1}{2}}, B'_0), (B'_0, B_{\frac{p+1}{2}}), \cdots, (B_{p-1}, B'_{\frac{p-1}{2}}), (B'_{\frac{p-1}{2}}, B_0)\},$$

so that $\overline{Y} \cong C_{2p}$. Set

$$\sigma = (0, 1, \cdots, p-1) \quad \text{and} \quad \tau = (0)(1, -1)\cdots(\frac{p-1}{2}, \frac{p+1}{2}) \in S_p.$$

Then $\mathrm{Aut}(\overline{Y}) \cong \langle\sigma, \tau\rangle \cong D_{2p}$, by defining $(B_i)^\gamma = B_{i^\gamma}$ and $(B'_j)^\gamma = B'_{j^\gamma}$ for any $\gamma \in \langle\sigma, \tau\rangle$.

Label the vertices in B_i by a_{ji} for $j \in Z_p$. By considering the imprimitive action of F on $U(Y)$, we know that

$$F \leq S_p \wr \langle\sigma, \tau\rangle = S_p^p \rtimes \langle\sigma, \tau\rangle,$$

where, for any

$$e = (e^{(0)}, e^{(1)}, \cdots, e^{(p-1)}) \in S_p^p \quad \text{and} \quad \gamma \in \langle\sigma, \tau\rangle,$$

we have

$$a_{ji}{}^{(e;\gamma)} = a_{j^{e^{(i)}}i^\gamma}.$$

In particular, by identifying $(1, \gamma)$ with γ so that $a_{ji}^\gamma = a_{ji^\gamma}$, we have that $\langle\sigma, \tau\rangle$ can be viewed as a subgroup of F.

From now on, for any $t \in T \leq S_p$ and $i \in Z_p$, we set

$$t_i = (\overbrace{1, 1, \cdots,}^{i+1} t, 1, \cdots, 1) \quad \text{and} \quad T_i = \langle t_i \mid t \in T\rangle,$$

where T_i acts transitively on B_i and fixes B_j pointwise for all $j \neq i$. Moreover, we have

$$t_i^{(e;\gamma)} = t_{i^\gamma}^{e^{(i)}},\ T_0^{\sigma^i} = T_{0^{\sigma^i}} = T_i,\ B_0^{\sigma^i} = B_i.$$

Since K^{B_i} is a transitive group of degree p, following Propositions 2.3 and 2.4 we need to consider the following four cases separately in four subsections:

(i) $p \geq 5$ and K^{B_i} is insolvable;

(ii) $p \geq 5$ and $K^{B_i} \cong Z_p \rtimes Z_r$ for $r \neq 1$;

(iii) $p \geq 5$ and $K^{B_i} \cong Z_p$.

(iv) $p = 3$.

3.1 K^{B_i} is insolvable for $p \geq 5$

Lemma 3.1. *Suppose that $p \geq 5$ and K^{B_i} is insolvable. Then Y is isomorphic to one of the following graphs:*

(i) $Y_1(p,r)$, *and* $\mathrm{Aut}(Y) = S_p \wr D_{2p}$, *where $r = 1$ or $p-1$;*

(ii) $Y_3(11,5)$ *and* $Y_4(11,6)$, *and* $\mathrm{Aut}(Y) = \mathrm{PSL}(2,11) \wr D_{22}$;

(iii) $Y_5(p, \frac{q^{n-1}-1}{q-1})$ *and* $Y_6(p, q^{n-1})$, *and* $\mathrm{Aut}(Y) = \mathrm{P\Gamma L}(n,q) \wr D_{2p}$.

Proof. Suppose that $p \geq 5$ and K^{B_i} is insolvable. Then by Lemma 2.5 we know that $K = T_0 \times T_1 \times \cdots \times T_{p-1}$, where T is an insolvable group of degree p and T_i is defined as before. In particular, a Sylow p-subgroup of F is of order p^{p+1}, and so we may assume that F contains σ defined as above.

Let $u \in B_0$ and take an element $g_0 \in F_u \setminus K$. Since g_0 fixes B_0 setwise and exchanges $B'_{\frac{p-1}{2}}$ and $B'_{\frac{p+1}{2}}$, there exists a $d = (d^{(0)}, d^{(1)}, \cdots, d^{(p-1)}) \in S_p^p$ such that $g_0 = d\tau$, where τ is defined as before. Since $F/K \cong D_{2p}$, by considering the order of F we get $F = KR$ where $R = \langle \sigma, d\tau \rangle$.

Let $H_0 = (T_0)_u$. Then

$$K_u = H_0 \times T_1 \times \cdots \times T_{p-1} \quad \text{and} \quad F_u = K_u \rtimes \langle d\tau \rangle.$$

By $K_u^{d\tau} = K_u$, we know that $d^{(0)} \in N_{(S_p)_0}(H_0)$ and $d^{(i)} \in N_{(S_p)_i}(T_i)$ for $i \neq 0$.

Now $d\tau$ fixes the block B'_0 setwise and exchanges $B_{\frac{p-1}{2}}$ and $B_{\frac{p+1}{2}}$. Take $w \in B'_0$. Since $T_{\frac{p-1}{2}} \times T_{\frac{p+1}{2}}$ fixes u and acts transitively on B'_0, there exists a $k \in T_{\frac{p-1}{2}} \times T_{\frac{p+1}{2}} \leq K_u$ such that $kd\tau$ fixes both u and w, where without loss of generality, we denote kd by d again so that $d\tau$ fixes both u and w. Then

$$K_w = T_0 \times \cdots \times T_{\frac{p-3}{2}} \times L_{\frac{p-1}{2}} \times N_{\frac{p+1}{2}} \times T_{\frac{p+3}{2}} \times \cdots \times T_{p-1} \quad \text{and} \quad F_w = K_w \langle d\tau \rangle.$$

By $K_w^{d\tau} = K_w$, we know that $L = N$ and $d^{(i)} \in N_{(S_p)_i}(L_i)$ for $i \in \{\frac{p-1}{2}, \frac{p+1}{2}\}$.

Now the corresponding groups H and L are two maximal subgroups of T of index p. Following Proposition 2.4 we need to consider three cases separately.

(1) H and L are conjugate in T.

Without loss of generality, let $H = L$. For any almost simple group T in S_p, its point stabilizers have two orbits in each block B_i with the respective length 1 and $p-1$. We may therefore let $T = S_p$ so that $H = S_{p-1}$ and $F = S_p^p R = S_p^p \langle \sigma, d\tau \rangle = S_p^p \langle \sigma, \tau \rangle$. Thus, we may set $d = 1$. For later use, we set $t = (0, 1, \cdots, p-1)$, $\Sigma_1 = \{0\}$ and $\Sigma_2 = Z_p^*$.

(2) $\mathrm{soc}(T) = \mathrm{PSL}(2,11)$, and H and L are not conjugate in T.

In this case $T = \mathrm{PSL}(2, 11)$, and T has two nonequivalent representations on the set of right cosets of A_5 of cardinality 11. Now $F = T^p\langle\sigma, d\tau\rangle$. Since $d^{(i)} \in N_{S_{11}}(T_i) = T_i$, we have $d \in T^p$ and so $F = T^p\langle\sigma, \tau\rangle$. Therefore, we set $d = 1$.

Moreover, T may be considered as the automorphism group of a (11, 5, 2)-design $\mathcal{D}$. Let $V = Z_{11}$ be the point set and let $t = (0, 1, \cdots, 10)$ be an element of order 11 in T. Then $M = \{0, 2, 3, 4, 8\} \subset V$ is a block (see [1, p.55]) of $\mathcal{D}$. Without loss of generality, we choose L and H to be the stabilizes of the block M and point 0, respectively. Again, for later use, set $\Sigma_3 = M$ and $\Sigma_4 = Z_{11} \setminus M$.

(3) $\mathrm{soc}(T) = \mathrm{PSL}(n, q)$, and H and L are not conjugate in T.

In this case, $\mathrm{PSL}(n, q) \leq T \leq N_{S_p}(\mathrm{PSL}(n, q))$=$\mathrm{P\Gamma L}(n, q)$. With the same reason as (1), we let $T = N_{S_p}(\mathrm{PSL}(n, q))$ and $d = 1$. Let S_1 and S_2 be the set of points and hyperplanes of $\mathrm{PG}(n, q-1)$, respectively, where $|S_1| = |S_2| = \frac{q^n-1}{q-1} = p$. Without loss of generality, we choose L and H to be the stabilizers of a given point $\langle v\rangle$ and a hyperplane $\mathcal{L}$, respectively. Let $G = \langle t\rangle \cong Z_p$ be a singer subgroup of $\mathrm{PGL}(n, q)$. Let

$$\Sigma_5 = \{l \in Z_p \mid \langle v\rangle \in \mathcal{L}^{t^l}\},\ \Sigma_6 = Z_p \setminus \Sigma_1 = \{l \in Z_p \mid \langle v\rangle \notin \mathcal{L}^{t^l}\},$$

where $|\Sigma_5| = \frac{q^{n-1}-1}{q-1}$ and $|\Sigma_6| = q^{n-1}$.

Now for the above three cases (1)-(3), we have

$$\mathrm{Cos}(F, F_w) = \{F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k \mid i, j, k \in Z_p\},\ \mathrm{Cos}(F, F_u) = \{F_u t_0^y \sigma^z \mid y, z \in Z_p\}.$$

Clearly, F_w has two orbits on $B_{\frac{p-1}{2}} \cup B_{\frac{p+1}{2}}$, that is,

$$D_l = \{F_u t_0^b \sigma^{\frac{p-1}{2}}, F_u t_0^b \sigma^{\frac{p+1}{2}} \mid b \in \Sigma_l\},$$

where $l = 1, 3, 4, 5, 6$. For any point $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k$ in $W(Y)$, since

$$\begin{aligned} F_u t_0^b \sigma^{\frac{p-1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k &= F_u t_0^b (t^{\sigma^{\frac{p+1}{2}}}_{\frac{p-1}{2}})^i (t^{\sigma^{\frac{p+1}{2}}}_{\frac{p+1}{2}})^j \sigma^{\frac{p-1}{2}+k} = F_u t_0^b t_0^i t_1^j \sigma^{\frac{p-1}{2}+k} \\ &= F_u t_0^{i+b} \sigma^{k+\frac{p-1}{2}}, \end{aligned}$$

and similarly,

$$F_u t_0^b \sigma^{\frac{p+1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k = F_u t_0^{j+b} \sigma^{k+\frac{p+1}{2}},$$

its neighbor is

$$D_l t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k = \{F_u t_0^{i+b} \sigma^{k+\frac{p-1}{2}}, F_u t_0^{j+b} \sigma^{k+\frac{p+1}{2}} \mid b \in \Sigma_l\}.$$

By labeling $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k$ by $(i, j, k, 0)$ and $F_u t_0^y \sigma^z$ by $(y, z, 1)$, we get the respective edge set of two graphs $Y(l)$

$$\begin{aligned} E_l \;=\; \{((i, j, k, 0), (y, z, 1)) \mid\ & y = i + b, z = k + \tfrac{p-1}{2}; \text{and} \\ & y = j + b, z = k + \tfrac{p+1}{2}, i, j, k, y, z \in Z_p, b \in \Sigma_l\}. \end{aligned}$$

In cases (2) and (3), we get the graphs $Y_3(11,5), Y_4(11,6)$ with the automorphism group $\mathrm{PSL}(2,11) \wr D_{2p}$, and $Y_5(p,r), Y_6(p,r)$ with the automorphism group $\mathrm{P\Gamma L}(n,q) \wr D_{2p}$.

For case (1), the graph with the edge set E_2 is exactly $Y_1(p,p-1)$ for $p \geq 5$. As for the graph with the edge set E_1, let ϕ be a map on $W(Y) \cup U(Y)$ which fixes $W(Y)$ pointwise and sends $(y,z,1)$ to $(y+1,z,1)$. Then ϕ is an isomorphism between the present graph and $Y_1(p,1)$. From the proof we know that both $Y_1(p,1)$ and $Y_1(p,p-1)$ have the automorphism group $S_p \wr D_{2p}$. □

3.2 $K^{B_i} \cong Z_p \rtimes Z_r$ for $p \geq 5$ and $r \neq 1$

Lemma 3.2. *Suppose $K^{B_i} \cong Z_p \rtimes Z_r$ for $p \geq 5$ and $r \neq 1$. Then $Y \cong Y_1(p,r)$ or $Y_2(p,r)$ where $p \geq 5$, $r \neq 1, p-1$ and $(p,r) \neq (7,3),(11,5)$, where* $\mathrm{Aut}(Y) = (Z_p \rtimes Z_r) \wr D_{2p}$.

Proof. Step 1: Determination of the structure of F.

Proof. Suppose $K^{B_i} \cong Z_p \rtimes Z_r$ for $r \neq 1$. Let $S = \langle t \rangle \rtimes \langle c \rangle \cong Z_p \rtimes Z_{p-1} \leq S_p$. Then we may set $T = \langle t \rangle \rtimes \langle h \rangle$, where $h = c^{\frac{p-1}{r}}$. Let P be a Sylow p-subgroup of F and take $d_0\sigma \in P$ where

$$d_0 \in \langle t_0 \rangle \times \langle t_1 \rangle \times \cdots \times \langle t_{p-1} \rangle \cong Z_p^p.$$

Then $K \leq T^p$ and $F = K\langle d_0\sigma, d\tau \rangle$ for some $d \in S_p^p$. Moreover,

$$F \leq \hat{F} = T^p\langle d_0\sigma, d\tau \rangle = T^p\langle \sigma, d\tau \rangle \quad \text{and} \quad \langle \sigma, d\tau \rangle/(T^p \cap \langle \sigma, d\tau \rangle) \cong D_{2p}.$$

Let $w \in B_0'$ and $(B_0', B_{\frac{p-1}{2}}), (B_0', B_{\frac{p+1}{2}}) \in E(\overline{Y})$. Let $(w,u_1) \in E(Y)$ for $u_1 \in B_{\frac{p-1}{2}}$. Then $E = (w,u_1)^F \leq (w,u_1)^{\hat{F}}$. Since the orbits of F_w and $\hat{F}_w$ on the block $B_{\frac{p-1}{2}}$ in $U(Y)$ are completely the same, we have $|(w,u_1)^{\hat{F}}| = 2rp^3 = |E|$, which implies $E = (w,u_1)^{\hat{F}}$. Therefore, we may just consider the case $F = \hat{F} = T^p\langle \sigma, d\tau \rangle$.

As in the last Lemma, we choose two vertices $u \in B_0$ and $w \in B_0'$ which are fixed by $d\tau$. Without loss of generality, let $H = \langle h \rangle$ so that

$$F_u = (H_0 \times T_1 \times \cdots \times T_{p-1})\langle d\tau \rangle,\ F_w = (T_0 \times T_1 \times \cdots \times H_{\frac{p-1}{2}} \times H_{\frac{p+1}{2}} \times \cdots \times T_{p-1})\langle d\tau \rangle.$$

We then need to determine the element d.

Let $d = (d^{(0)}, d^{(1)}, \cdots, d^{(p-1)}) \in S_p^p$. Since $d\tau$ normalizes K, K_u and K_w, it follows that $d^{(i)} \in N_{S_p}(H_i) = \langle c \rangle$ for $i \in \{0, \frac{p\pm1}{2}\}$, and $d^{(i)} \in N_{S_p}(T_i) = S = \langle t \rangle\langle c \rangle$ for $i \notin \{0, \frac{p\pm1}{2}\}$. Suppose that $i \in \{0, \frac{p\pm1}{2}\}$ and write $d^{(i)} = t^m c^n$. Since $T_i \leq F_u$ and F_w we may re-choose $d^{(i)} = c^n$. Therefore, for any $i \in Z_p$, we get

$$d^{(i)} \in \langle c \rangle. \tag{1}$$

Since $(d\tau)^2 \in K$, we have

$$d\tau d\tau = ((d^{(0)})^2, d^{(1)}d^{(p-1)}, \cdots, d^{(p-1)}d^{(1)}) \in K,$$

and by taking into account (1) we get

$$(d^{(0)})^2, d^{(1)}d^{(p-1)}, \cdots, d^{(\frac{p-1}{2})}d^{(\frac{p+1}{2})} \in H. \tag{2}$$

Since K_w fixes only one point $u^{\sigma^{\frac{p-1}{2}}}$ in $B_{\frac{p-1}{2}}$ and $u^{\sigma^{\frac{p+1}{2}}}$ in $B_{\frac{p+1}{2}}$ and since $d\tau$ normalizes K_w and exchanges $B_{\frac{p-1}{2}}$ and $B_{\frac{p+1}{2}}$, it follows that $d\tau$ must exchange these two points. Therefore,

$$\begin{aligned} F_u\sigma^{\frac{p-1}{2}}(d\tau) &= F_u(d^{(\frac{p-1}{2})}, d^{(\frac{p+1}{2})}, \cdots, d^{(p-1)}, d^{(0)}, d^{(1)}, \cdots, d^{(\frac{p-3}{2})})\tau\sigma^{\frac{p+1}{2}} \\ &= F_u d(d^{(\frac{p-1}{2})}, d^{(\frac{p-3}{2})}, \cdots, d^{(1)}, d^{(0)}, d^{(p-1)}, \cdots, d^{(\frac{p+1}{2})})\sigma^{\frac{p+1}{2}} \\ &= F_u(d^{(0)}d^{(\frac{p-1}{2})}, d^{(1)}d^{(\frac{p-3}{2})}, \cdots, d^{(\frac{p-1}{2})}d^{(0)}, d^{(\frac{p+1}{2})}d^{(p-1)}, \\ &\quad \cdots, d^{(p-1)}d^{(\frac{p+1}{2})})\sigma^{\frac{p+1}{2}} \\ &= F_u\sigma^{\frac{p+1}{2}}. \end{aligned}$$

Hence,

$$d^{(0)}d^{(\frac{p-1}{2})}, d^{(1)}d^{(\frac{p-3}{2})}, \cdots, d^{(\frac{p-1}{2})}d^{(0)}, d^{(\frac{p+1}{2})}d^{(p-1)}, \cdots, d^{(p-1)}d^{(\frac{p+1}{2})} \in H. \qquad (3)$$

From (2) and (3) we get

$$d^{(0)}, d^{(1)} \cdots, d^{(p-1)} \in H, \text{ or } d^{(0)}, d^{(1)} \cdots, d^{(p-1)} \in \langle c^{\frac{p-1}{2r}}\rangle \setminus H \text{ if } 2r \bigm| (p-1). \quad (4)$$

Therefore, if $2r \nmid (p-1)$ then we set $d = 1$; if $2r \bigm| (p-1)$, we set $d = (c'^m, c'^m, \cdots c'^m)$ where $c' = c^{\frac{p-1}{2r}}$ and $m = 0, 1$. To unify these two cases, in the first case we still write $d = (c'^m, c'^m, \cdots c'^m)$ for $m = 0$.

Suppose that $2r \bigm| (p-1)$. Let $F_1 = K \rtimes \langle\sigma, \tau\rangle$ and $F_2 = K \rtimes \langle\sigma, d\tau\rangle$, where $d = (c', \cdots, c')$ with $c' = c^{\frac{p-1}{2r}}$, noting that $c' \notin T$. we may then state the following fact

Fact: $F_1 \ncong F_2$

Proof: Assume the contrary. Suppose that γ is an isomorphism from F_1 to F_2. Since $\langle(t, t, \cdots, t)\rangle$ is characteristic in F_1 and F_2, we get

$$\gamma((t, t \cdots t)) = (t^k, t^k, t^k \cdots, t^k)$$

for some $k \in F_p^*$. Assume that $\gamma(\tau) = ed\tau$, where $e = (e^{(0)}, e^{(1)} \cdots, e^{(p-1)}) \in K$. Since

$$\tau^{-1}(t, t, \cdots, t)\tau = (t, t, \cdots, t).$$

we have

$$\gamma(\tau^{-1})\gamma((t, t, \cdots, t))\gamma(\tau) = \gamma(t, t, \cdots, t),$$

that is

$$(ed\tau)^{-1}(t^k, t^k, \cdots, t^k)(ed\tau) = (t^k, t^k, \cdots, t^k),$$

which implies

$$(t^k)^{e^{(0)}c'} = t^k.$$

Therefore, $e^{(0)}c' \in \langle t\rangle$ and so $c' \in T$, a contradiction.

Step 2: Determination of the bicoset graphs.

Set $D(l) = F_u t^l \sigma^{\frac{p-1}{2}} F_w$ and by $Z = Z(p, r, d, l)$ we denote the corresponding bicoset graph. We consider two cases separately.

(1) $l = 0$.

Since $F_u\sigma^{\frac{p-1}{2}}K_w = F_u\sigma^{\frac{p-1}{2}}$ and $F_u\sigma^{\frac{p-1}{2}}K_w\langle d\tau\rangle = F_u\sigma^{\frac{p+1}{2}}$, we have

$$D(0) = F_u\sigma^{\frac{p-1}{2}}F_w = \{F_u\sigma^{\frac{p-1}{2}}, F_u\sigma^{\frac{p+1}{2}}\}.$$

For any point $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k$ in $W(Z)$, since

$$F_u\sigma^{\frac{p-1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u(t^{\sigma^{\frac{p+1}{2}}}_{\frac{p-1}{2}})^i(t^{\sigma^{\frac{p+1}{2}}}_{\frac{p+1}{2}})^j\sigma^{k+\frac{p-1}{2}} = F_u t^i_0 t^j_1\sigma^{k+\frac{p-1}{2}} = F_u t^i_0\sigma^{k+\frac{p-1}{2}},$$

and similarly

$$F_u\sigma^{\frac{p+1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u t^j_0\sigma^{k+\frac{p+1}{2}},$$

its neighbor is $N = \{F_u t^i_0\sigma^{k+\frac{p-1}{2}}, F_u t^j_0\sigma^{k+\frac{p+1}{2}}\}$. In this case, $d(w) = 2$. Let

$$\rho : F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k \to F_w t^{i+1}_{\frac{p-1}{2}} t^{j+1}_{\frac{p+1}{2}}\sigma^k, \quad F_u t^y_0\sigma^z \to F_u t^y_0\sigma^z$$

be the mapping of $V(Z(p, r, d, 0))$ to $V(Y_1(p, 1))$. Then one may check that ρ is an isomorphism from $Z(p, r, d, 0)$ to $Y_1(p, 1)$. Therefore, $\mathrm{Aut}(Z(p, r, d, 0)) \cong S_p \wr D_{2p}$, contrary to our hypothesis $K^{B_i} \cong Z_p \rtimes Z_r$.

(2) $l \neq 0$.

In $S_p \wr D_{2p}$, there exists some $c^{l'}$ such that the inner automorphism $I(c^{l'})$ fixes F_u and F_w and maps $D(1)$ to $D(l)$. Therefore, up to graph isomorphism, we only consider $Z(p, r, d, 1)$.

Since

$$F_u t_0\sigma^{\frac{p-1}{2}}K_w = F_u t_0\sigma^{\frac{p-1}{2}}(T_0 \times T_1 \times \cdots \times H_{\frac{p-1}{2}} \times H_{\frac{p+1}{2}} \times \cdots T_{p-1}) = F_u(t^H)_0\sigma^{\frac{p-1}{2}},$$

$$\begin{aligned} F_u t_0\sigma^{\frac{p-1}{2}}K_w d\tau &= F_u(t^H)_0\sigma^{\frac{p-1}{2}}d\tau = F_u(t^H)_0(c'^m, c'^m, \cdots, c'^m)\tau\sigma^{\frac{p+1}{2}} \\ &= F_u(t^{Hc'^m})_0\sigma^{\frac{p+1}{2}}, \end{aligned}$$

it follows that

$$D(1) = \{F_u(t^{h'})_0\sigma^{\frac{p-1}{2}}, F_u(t^{h'c'^m})_0\sigma^{\frac{p+1}{2}} \mid h' \in H\}.$$

For any $i, j, k \in Z_p$, since

$$F_u(t^{h'})_0\sigma^{\frac{p-1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u(t^{h'})_0 t^i_0\sigma^{k+\frac{p-1}{2}} = F_u(t^{h'+i})_0\sigma^{k+\frac{p-1}{2}}$$

and

$$F_u(t^{h'c'^m})_0\sigma^{\frac{p+1}{2}} t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k = F_u(t^{h'c'^m})_0 t^j_0\sigma^{k+\frac{p+1}{2}} = F_u(t^{h'c'^m+j})_0\sigma^{k+\frac{p+1}{2}},$$

the neighbor of $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}}\sigma^k$ is

$$\{F_u(t^{h'+i})_0\sigma^{k+\frac{p-1}{2}}, F_u(t^{h'c'^m+j})_0\sigma^{k+\frac{p+1}{2}} \mid h' \in H\}.$$

Suppose $d(w) = 2(p-1)$. Then $K^{B_i} \cong Z_p \rtimes Z_{p-1}$ and $F = K\langle\sigma,\tau\rangle$. In this case, for any $F_w t^i_{\frac{p-1}{2}} t^j_{\frac{p+1}{2}} \sigma^k$ in $W(Y)$, its neighbor is

$$N = \{F_u(t^{h'+i})_0\sigma^{k+\frac{p-1}{2}}, F_u(t^{h'+j})_0\sigma^{k+\frac{p+1}{2}} \mid h' \in H\}$$

where $H \cong Z_{p-1}$. Clearly, the corresponding graph is isomorphic to $Y_1(p,p-1)$, with $K^{B_i} \cong S_p$, a contradiction. Therefore, $2 < d(w) < 2(p-1)$.

Let $t^{h'} = t^b$ and $b \in \Sigma$ where Σ is a subgroup of Z_p^* of order r. Define a mapping ϕ from $V(Z)$ to $V(Y_2(p,r))$, for $r \neq 1, p-1$, by

$$F_w t^i_{p-1} t^j_1 \sigma^k \to F_w t^i_{p-1} t^j_1 \sigma^k, \quad \text{and} \quad F_u t^y_0 \sigma^z \to F_u t^y_0 \sigma^z.$$

Then ϕ is clearly an isomorphism between the two graphs.

Step 3: Determination of isomorphic classes and automorphism groups.

Let $\tilde{A} = \mathrm{Aut}(Z)$ and K_Z be the kernel of $\tilde{A}$ on $\overline{Z}$, where $\overline{Z}$ is a cycle of length $2p$. Clearly, $\tilde{A}/K \cong D_{2p}$.

If $(p,r) = (7,3)$ and $(11,5)$, then Z is $Y_2(7,3)$ with $K^{B_i} \cong P\Gamma L(3,2)$ and $Y_1(11,5)$ with $\mathrm{PSL}(2,11)$, respectively, contradicting our condition.

Suppose that $(p,r) \neq (7,3), (11,5)$. Since $r \mid (p-1)$ and $r \neq 1, p-1$, K^{B_i} can not be insolvable and hence an affine group. Therefore, $K \leq K_Z = (Z_p \rtimes Z_r)^p$ and then $K_Z = K$. Therefore, $\tilde{A} = F$.

In the case of $2r \mid (p-1)$, let $F_1 = K \rtimes \langle\sigma,\tau\rangle$ and $F_2 = K \rtimes \langle\sigma, d\tau\rangle$ be defined as in Step 1. Let Z_1 and Z_2 be the corresponding graphs. Suppose that ρ is an isomorphism from Z_1 to Z_2. Then $F_2 \leq \langle\rho^{-1}F_1\rho, F_2\rangle \leq \mathrm{Aut}(Z_2) = F_2$. Therefore, $F_2 = \rho^{-1}F_1\rho \cong F_1$, a contradiction. Therefore, the two graphs are not isomorphic.

□

3.3 $K^{B_i} \cong Z_p$ for $p \geq 5$

Lemma 3.3. *The case $K^B \cong Z_p$ cannot occur.*

Proof. Suppose that $K^B \cong Z_p$. Then $|E(Y)| = 2p^3$. As above, let $w \in B'_0$ and $(B'_0, B_{\frac{p-1}{2}}), (B'_0, B_{\frac{p+1}{2}}) \in E(\overline{Y})$. Let $(w,u_1) \in E(Y)$ for $u_1 \in B_{\frac{p-1}{2}}$. Then $E = (w,u_1)^F$. We may consider the group $\hat{F} = S_p^p \rtimes \langle\sigma,\tau\rangle \geq F$. From the proof of Lemma 3.1, we may construct two representations of $\hat{F}$ with respective degree p^3 and p^2 such that both K_w and $(S_p^p)_w$ fix u_1. Then $(w,u_1)^F \subset (w,u_1)^{\hat{F}}$. Since $|(w,u_1)^F| = 2p^3 = |(w,u_1)^{\hat{F}}|$, we have $(w,u_1)^{\hat{F}} = (w,u_1)^F = E(Y)$ and so $\mathrm{Aut}(Y) \cong \hat{F}$, contrary to our hypothesis $K^{B_i} \cong Z_p$. Therefore, this case cannot occur. □

3.4 $p = 3$

Lemma 3.4. *If $p = 3$, then $Y \cong Y_1(3,r)$ for $r = 1, 2$.*

Proof. In this case, take $F = S_3 \wr D_6$ and $H = L = Z_2$. Checking the proof of Lemma 3.1(1), one may find that the arguments in there still hold for $p = 3$. Therefore, $Y \cong Y_1(3,r)$ for $r = 1, 2$. □

Acknowledgments: The authors thank the referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China (10971144 and 11271267) and Natural Science Foundation of Beijing (1092010).

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