

The edge-transitive polytopes that are not vertex-transitive*

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Abstract

In 3-dimensional Euclidean space there exist two exceptional polyhedra, the *rhombic dodecahedron* and the *rhombic triacontahedron*, the only known polytopes (besides polygons) that are edge-transitive without being vertex-transitive. We show that these polyhedra do not have higher-dimensional analogues, that is, that in dimension $d \geq 4$, edge-transitivity of convex polytopes implies vertex-transitivity.

More generally, we give a classification of all convex polytopes which at the same time have all edges of the same length, an edge in-sphere and a bipartite edge-graph. We show that any such polytope in dimension $d \geq 4$ is vertex-transitive.

Keywords: Convex polytopes, symmetry of polytopes, vertex-transitive, edge-transitive.

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1 Introduction

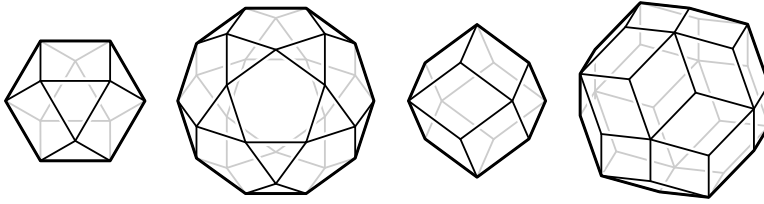
A d -dimensional (convex) polytope $P \subset \mathbb{R}^d$ is the convex hull of finitely many points. P is said to be *vertex-transitive* resp. *edge-transitive* if its (orthogonal) symmetry group $\text{Aut}(P) \subset O(\mathbb{R}^d)$ acts transitively on its vertices resp. edges. For a general overview over the state of the art regarding symmetries in convex and abstract polytopes we refer to [9].

It has long been known that there are exactly *nine* edge-transitive polyhedra in \mathbb{R}^3 (see e.g. [6]). These are the five Platonic solids (tetrahedron, cube, octahedron, icosahedron and dodecahedron) together with the cuboctahedron, the icosidodecahedron, and their duals, the *rhombic dodecahedron* and the *rhombic triacontahedron* (depicted in this order):

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Little is known about analogous questions in higher dimensions. Branko Grünbaum writes in “Convex Polytopes” [5, page 413]

No serious consideration seems to have been given to polytopes in dimension $d \geq 4$ about which transitivity of the symmetry group is assumed only for faces of suitably low dimensions, [...].

Even though families of higher-dimensional edge-transitive polytopes have been studied, to the best of our knowledge, no classification of these has been achieved so far. Equally striking, all the known examples of such polytopes in dimension at least four are simultaneously *vertex-transitive*. In dimension up to three, certain polygons (see Figure 1), as well as the rhombic dodecahedron and rhombic triacontahedron are edge- but *not* vertex-

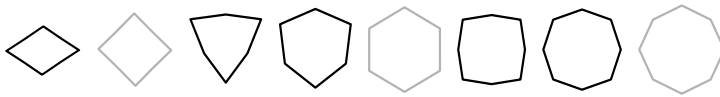


Figure 1: Some examples of edge-transitive $2n$ -gons with $2n \in \{4, 6, 8\}$ (the same works for all n). The polygons depicted with black boundary are not vertex-transitive.

transitive. No higher dimensional example of this kind has been found. In this paper we prove that this is not for lack of trying:

Theorem 1.1. *In dimension $d \geq 4$, edge-transitivity of convex polytopes implies vertex-transitivity.*

As immediate consequence, we obtain the classification of all polytopes that are edge- but not vertex-transitive. The list is quite short:

Corollary 1.2. *If $P \subset \mathbb{R}^d, d \geq 2$ is edge- but not vertex-transitive, then P is one of the following:*

- (i) *a non-regular $2k$ -gon (see Figure 1),*
- (ii) *the rhombic dodecahedron, or*
- (iii) *the rhombic triacontahedron.*

Theorem 1.1 is proven by embedding the class of edge- but not vertex-transitive polytopes in a larger class of polytopes, defined by geometric regularities instead of symmetry. In Theorem 2.4 we show that a polytope $P \subset \mathbb{R}^d$ which is edge- but not vertex-transitive must have all of the following properties:

- (i) all edges are of the same length,

- (ii) it has a bipartite edge-graph $G_P = (V_1 \cup V_2, E)$, and
- (iii) there are radii $r_1 \leq r_2$, so that $\|v\| = r_i$ for all $v \in V_i$.

We compile this into a definition: a polytope that has these three properties shall be called *bipartite* (cf. Definition 2.1). The edge- but not vertex-transitive polytopes then form a subclass of the bipartite polytopes, but the class of bipartite polytopes is much better behaved. For example, faces of bipartite polytopes are bipartite (Proposition 2.5), something which is not true for edge/vertex-transitive polytopes¹. Our quest is then to classify all bipartite polytopes. The surprising result: already being bipartite is very restrictive:

Theorem 1.3. *If $P \subset \mathbb{R}^d$, $d \geq 2$ is bipartite, then P is one of the following:*

- (i) an edge-transitive $2k$ -gon (see Figure 1),
- (ii) the rhombic dodecahedron,
- (iii) the rhombic triacontahedron, or
- (iv) a Γ -permutahedron for some finite reflection group $\Gamma \subset O(\mathbb{R}^d)$ (see Definition 2.10; some 3-dimensional examples are shown in Figure 2).

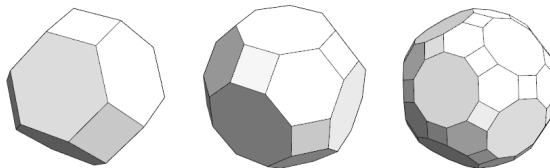


Figure 2: From left to right: the A_3 -, B_3 and H_3 -permutahedron.

The Γ -permutahedra are vertex-transitive, and all the other entries in the list are of dimension $d \leq 3$. This immediately implies Theorem 1.1.

Remarkably, despite the definition of bipartite polytope being purely geometric, all bipartite polytopes are highly symmetric, that is, at least vertex- or facet-transitive, and sometimes even edge-transitive.

Overview

In Section 2 we introduce the central notion of *bipartite polytope* and prove its most relevant properties: that being bipartite generalizes being edge- but not vertex-transitive, and that all faces of bipartite polytopes are again bipartite. We then investigate certain subclasses of bipartite polytopes: bipartite polygons and inscribed bipartite polytopes. We prove that the latter coincide with the Γ -permutahedra, a class of vertex-transitive polytopes. It therefore remains to classify the non-inscribed cases, the so-called *strictly bipartite polytopes*. We

¹For example, consider a vertex-transitive but not uniform antiprism. Its faces are non-regular triangles, which are thus not vertex-transitive. Alternatively, consider the (n, n) -duoprism, $n \neq 4$, that is, the cartesian product of a regular n -gon with itself. This polytope is edge-transitive, but its facets are n -gonal prisms (the cartesian product of a regular n -gon with an edge), which are not edge-transitive.

show that the classification of these reduces to the classification of bipartite *polyhedra*, i.e., the case $d = 3$.

From Section 3 on the investigation is focused on the class of strictly bipartite polyhedra. We successively determine restrictions on the structure of such, e.g. the degrees of their vertices and the shapes of their faces. This quite elaborate process uses many classical geometric results and techniques, including spherical polyhedra, the classification of rhombic isohedra and the realization of graphs as edge-graphs of polyhedra. As a result, we can exclude all but two cases, namely, the rhombic dodecahedron, and the rhombic triacontahedron. Additionally, we shall find a remarkable near-miss, that is, a polyhedron which fails to be bipartite only by a tiny (but quantifiable) amount.

2 Bipartite polytopes

From this section on let $P \subset \mathbb{R}^d, d \geq 2$ denote a d -dimensional polytope of full dimension (i.e., P is not contained in a proper affine subspace). By $\mathcal{F}(P)$ we denote the face lattice of P , and by $\mathcal{F}_\delta(P) \subset \mathcal{F}(P)$ the subset of δ -dimensional faces.

Definition 2.1. P is called *bipartite*, if

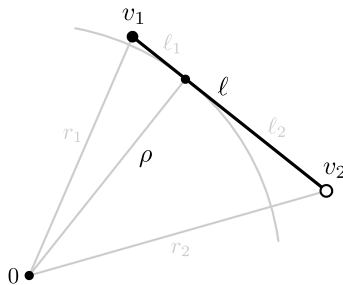
- (i) all its edges are of the same length ℓ ,
- (ii) its edge-graph is bipartite, which we write as $G_P = (V_1 \sqcup V_2, E)$, and
- (iii) there are radii $r_1 \leq r_2$ so that $\|v\| = r_i$ for all $v \in V_i$.

If $r_1 < r_2$, then P is called *strictly bipartite*. A vertex $v \in V_i$ is called an *i-vertex*. The numbers r_1, r_2 and ℓ are called the *parameters* of a bipartite polytope.

Remark 2.2. Since P is full-dimensional by convention, Definition 2.1 only defines *full-dimensional* bipartite polytopes.

To extend this notion to not necessarily full-dimensional polytopes, we shall call a polytope *bipartite* even if it is just bipartite as a subset of its affine hull where we made an appropriate choice of origin in the affine hull (note that whether a polytope is bipartite depends on its placement relative to the origin and that there is at most one such placement if the polytope is full-dimensional). This comes in handy when we discuss faces of bipartite polytopes.

Remark 2.3. An alternative definition of bipartite polytope would replace (iii) by the condition that P has an *edge in-sphere*, that is, a sphere that touches each edge of P in a single point (this definition was used in the abstract). The configuration depicted below (an edge of P connecting two vertices $v_1 \in V_1$ and $v_2 \in V_2$) shows how any one of the four quantities r_1, r_2, ℓ and ρ (the radius of the edge in-sphere) is determined from the other three by solving the given set of equations:



$$\begin{aligned} \rho^2 + \ell_1^2 &= r_1^2 \\ \rho^2 + \ell_2^2 &= r_2^2 \\ \ell_1 + \ell_2 &= \ell \end{aligned}$$

There is a subtlety: for the edge in-sphere to actually touch the edge (rather than only its affine hull outside of the edge) it is necessary that the perpendicular projection of the origin onto the edge ends up inside the edge (equivalently, that the triangle $\text{conv}\{0, v_1, v_2\}$ is acute at v_1 and v_2). One might regard this as intuitively clear since we are working with convex polytopes, but this will also follow formally as part of our proof of Proposition 3.7 (as we shall mention there in a footnote).

This alternative characterization of bipartite polytopes via edge in-spheres will become relevant towards the end of the classification (in Section 3.9). Still, for the larger part of our investigation, Definition 2.1(iii) is the more convenient version to work with.

2.1 General observations

Proposition 2.4. *If P is edge- but not vertex-transitive, then P is bipartite.*

This is a geometric analogue to the well known fact that every edge- but not vertex-transitive graph is bipartite. A proof of the graph version can be found in [4]. The following proof can be seen as a geometric analogue:

Proof of Proposition 2.4. Clearly, all edges of P are of the same length.

Fix some edge $e \in \mathcal{F}_1(P)$ with end vertices $v_1, v_2 \in \mathcal{F}_0(P)$. Let V_i be the orbit of v_i under $\text{Aut}(P)$. We prove that $V_1 \cup V_2 = \mathcal{F}_0(P)$, $V_1 \cap V_2 = \emptyset$ and that the edge graph G_P is bipartite with partition $V_1 \cup V_2$.

Let $v \in \mathcal{F}_0(P)$ be some vertex and $\tilde{e} \in \mathcal{F}_1(P)$ an incident edge. By edge-transitivity, there is a symmetry $T \in \text{Aut}(P)$ that maps \tilde{e} onto e , and therefore maps v onto v_i for some $i \in \{1, 2\}$. Thus, v is in the orbit V_i . This holds for all vertices of P , and therefore $V_1 \cup V_2 = \mathcal{F}_0(P)$.

The orbits of v_1 and v_2 must either be identical or disjoint. Since $V_1 \cup V_2 = \mathcal{F}_0(P)$, from $V_1 = V_2$ it would follow $V_1 = \mathcal{F}_0(P)$, stating that P has a single orbit of vertices. But since P is *not* vertex-transitive, this cannot be. Thus, $V_1 \cap V_2 = \emptyset$, and therefore $V_1 \cup V_2 = \mathcal{F}_0(P)$.

Let $\tilde{e} \in \mathcal{F}_1(P)$ be an edge with end vertices \tilde{v}_1 and \tilde{v}_2 . By edge-transitivity, \tilde{e} can be mapped onto e by some symmetry $T \in \text{Aut}(P)$. Equivalently $\{T\tilde{v}_1, T\tilde{v}_2\} = \{v_1, v_2\}$. Since v_1 and v_2 belong to different orbits under $\text{Aut}(P)$, so do \tilde{v}_1 and \tilde{v}_2 . Hence \tilde{e} has one end vertex in V_1 and one end vertex in V_2 . This holds for all edges, and thus, G_P is bipartite with partition $V_1 \cup V_2$.

It remains to determine the radii $r_1 \leq r_2$. Set $r_i := \|v_i\|$ (assuming w.l.o.g. that $\|v_1\| \leq \|v_2\|$). Then for every $v \in V_i$ there is a symmetry $T \in \text{Aut}(P) \subset O(\mathbb{R}^d)$ so that $Tv_i = v$, and thus

$$\|v\| = \|Tv_i\| = \|v_i\| = r_i. \quad \square$$

Bipartite polytopes are more comfortable to work with than edge- but not vertex-transitive polytopes because their faces are again bipartite polytopes (in the sense as explained in Remark 2.2). Later, this will enable us to reduce the problem to an investigation in lower dimensions.

Proposition 2.5. *Let $\sigma \in \mathcal{F}(P)$ be a face of P . Then it holds*

- (i) *if P is bipartite, so is σ .*

(ii) if P is strictly bipartite, then so is σ , and $v \in \mathcal{F}_0(\sigma) \subseteq \mathcal{F}_0(P)$ is an i -vertex in P if and only if it is an i -vertex in σ .

(iii) if $r_1 \leq r_2$ are the radii of P and $\rho_1 \leq \rho_2$ are the radii of σ , then there holds

$$h^2 + \rho_i^2 = r_i^2,$$

where h is the height of σ , that is, the distance of $\text{aff}(\sigma)$ from the origin.

Proof. Properties clearly inherited by σ are that all edges are of the same length and that the edge graph is bipartite. It remains to show the existence of the radii $\rho_1 \leq \rho_2$ compatible with the bipartition of the edge-graph of σ .

Let $c \in \text{aff}(\sigma)$ be the orthogonal projection of 0 onto $\text{aff}(\sigma)$. Then $\|c\| = h$, the height of σ as defined in (iii). For any vertex $v \in \mathcal{F}_0(\sigma)$ which is an i -vertex in P , the triangle $\Delta := \text{conv}\{0, c, v\}$ has a right angle at c . Set $\rho_i := \|v - c\|$ and observe

$$\rho_i^2 := \|v - c\|^2 = \|v\|^2 - \|c\|^2 = r_i^2 - h^2. \tag{*}$$

In particular, the value ρ_i does only depend on i . In other words, σ is a bipartite polytope when considered as a subset of its affine hull, where the origin is chosen to be c (cf. Remark 2.2). This proves (i), and (*) is equivalent to the equation in (iii). From (*) also follows $r_1 < r_2 \Leftrightarrow \rho_1 < \rho_2$, which proves (ii). \square

The following observation will be of use later on.

Observation 2.6. Given two adjacent vertices $v_1, v_2 \in \mathcal{F}_0(P)$ with $v_i \in V_i$, and if P has parameters r_1, r_2 and ℓ , then

$$\ell^2 = \|v_1 - v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 - 2\langle v_1, v_2 \rangle = r_1^2 + r_2^2 - 2r_1r_2 \cos \angle(v_1, v_2),$$

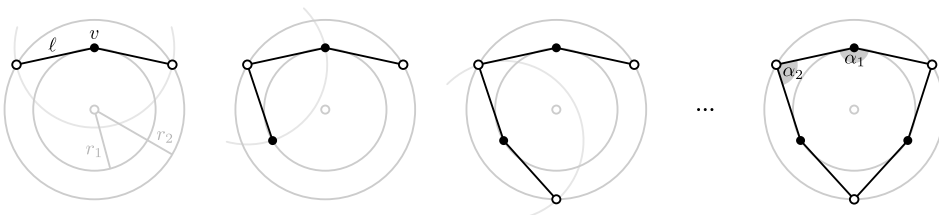
This can be rearranged for $\cos \angle(v_1, v_2)$. While the exact value of this expression is not of relevance to us, this shows that this angle is determined by the parameters and does not depend on the choice of the adjacent vertices v_1 and v_2 .

2.2 Bipartite polygons

The easiest to describe (and to explicitly construct) are the bipartite *polygons*.

Foremost, the edge-graph is bipartite, and thus, a bipartite polygon must be a $2k$ -gon for some $k \geq 2$. One can show that the bipartite polygons are exactly the edge-transitive $2k$ -gons (cf. Figure 1), and that such one is *strictly* bipartite if and only if it is *not* vertex-transitive (or equivalently, not regular). We will not make use of these symmetry properties of bipartite polygons.

The parameters r_1, r_2 and ℓ uniquely determine a bipartite polygon, as can be seen by explicit construction:



One starts with an arbitrary 1-vertex $v \in \mathbb{R}^2$ placed on the circle $\mathbf{S}_{r_1}(0)$. Its neighboring vertices are then uniquely determined as the intersections $\mathbf{S}_{r_2}(0) \cap \mathbf{S}_\ell(v)$. The procedure is repeated with the new vertices until the edge cycle closes (which only happens if the parameters are chosen appropriately).

The procedure also makes clear that the interior angle $\alpha_i \in (0, \pi)$ at an i -vertex only depends on i , but not on the chosen vertex $v \in V_i$.

Corollary 2.7. *A bipartite polygon $P \subset \mathbb{R}^2$ is a $2k$ -gon with alternating interior angles $\alpha_1, \alpha_2 \in (0, \pi)$ (α_i being the interior angle at an i -vertex), and its shape is uniquely determined by its parameters (up to congruence).*

The exact values for the interior angles are not of relevance. Instead, we only need the following properties:

Proposition 2.8. *The interior angles $\alpha_1, \alpha_2 \in (0, \pi)$ satisfy*

$$\alpha_1 + \alpha_2 = 2\alpha_{\text{reg}}^k \quad \text{and} \quad \alpha_2 \leq \alpha_{\text{reg}}^k \leq \alpha_1, \tag{2.1}$$

where $\alpha_{\text{reg}}^k := (1 - 1/k)\pi$ is the interior angle of a regular $2k$ -gon, and the inequalities are satisfied with equality if and only $r_1 = r_2$.

Proof. The sum of interior angles of a $2k$ -gon is $2(k-1)\pi$, and thus $k\alpha_1 + k\alpha_2 = 2(k-1)\pi$, which, after division by k , yields the first part of (2.1).

For two adjacent vertices $v_1, v_2 \in \mathcal{F}_0(P)$ (where $v_i \in V_i$), consider the triangle $\Delta := \text{conv}\{0, v_1, v_2\}$ whose edge lengths are r_1, r_2 and ℓ , and whose interior angles at v_1 resp. v_2 are $\alpha_1/2$ resp. $\alpha_2/2$. From $r_1 \leq r_2$ (resp. $r_1 < r_2$) and the law of sine follows $\alpha_1 \geq \alpha_2$ (resp. $\alpha_1 > \alpha_2$). With $\alpha_1 + \alpha_2 = 2\alpha_{\text{reg}}^k$ this yields the second part of (2.1). \square

Observation 2.9. For later use (in Corollary 3.18), consider Proposition 2.8 with $2k = 4$. In this case we find,

$$\alpha_2 \leq \frac{\pi}{2} \leq \alpha_1,$$

that is, α_1 is never acute, and α_2 is never obtuse.

2.3 The case $r_1 = r_2$

We classify the inscribed bipartite polytopes, that is, those with coinciding radii $r_1 = r_2$. This case is made especially easy by a classification result from [10]. We need the following definition:

Definition 2.10. Let $\Gamma \subset \text{O}(\mathbb{R}^d)$ be a finite reflection group and $v \in \mathbb{R}^d$ a generic point w.r.t. Γ (i.e., v is not fixed by a non-identity element of Γ). The orbit polytope

$$\text{Orb}(\Gamma, v) := \text{conv}\{Tv \mid T \in \Gamma\} \subset \mathbb{R}^d$$

is called a Γ -permutahedron.

The relevant result then reads

Theorem 2.11 (Corollary 4.6. in [10]). *If P has only centrally symmetric 2-dimensional faces (that is, it is a zonotope), has all vertices on a common sphere and all edges of the same length, then P is a Γ -permutahedron.*

This provides a classification of bipartite polytopes with $r_1 = r_2$.

Theorem 2.12. *If $P \subset \mathbb{R}^d$ is bipartite with $r_1 = r_2$, then it is a Γ -permutahedron.*

Proof. If $r_1 = r_2$, then all vertices are on a common sphere (that is, P is inscribed). By definition, all edges are of the same length. Both statements then also hold for the faces of P , in particular, the 2-dimensional faces. An inscribed polygon with a unique edge length is necessarily regular. With Corollary 2.7 the 2-faces are then regular $2k$ -gons, therefore centrally symmetric.

Summarizing, P is inscribed, has all edges of the same length, and all 2-dimensional faces of P are centrally symmetric. By Theorem 2.11, P is a Γ -permutahedron. \square

Γ -permutahedra are vertex-transitive by definition, hence do not provide examples of edge- but not vertex-transitive polytopes.

2.4 Strictly bipartite polytopes

It remains to classify the *strictly* bipartite polytopes. This problem is divided into two independent cases: dimension $d = 3$, and dimension $d \geq 4$. The detailed study of the case $d = 3$ (which turns out to be the actual hard work) is postponed until Section 3, the result of which is the following theorem:

Theorem 2.13. *If $P \subset \mathbb{R}^3$ is strictly bipartite, then P is the rhombic dodecahedron or the rhombic triacontahedron.*

Presupposing Theorem 2.13, the case $d \geq 4$ is done quickly.

Theorem 2.14. *There are no strictly bipartite polytopes in dimension $d \geq 4$.*

Proof. It suffices to show that there are no strictly bipartite polytopes in dimension $d = 4$, as any higher-dimensional example has a strictly bipartite 4-face (by Proposition 2.5).

Let $P \subset \mathbb{R}^4$ be a strictly bipartite 4-polytope. Let $e \in \mathcal{F}_1(P)$ be an edge of P . Then there are $s \geq 3$ cells (aka. 3-faces) $\sigma_1, \dots, \sigma_s \in \mathcal{F}_3(P)$ incident to e , each of which is again strictly bipartite (by Proposition 2.5). By Theorem 2.13 each σ_i is a rhombic dodecahedron or rhombic triacontahedron.

The dihedral angle of the rhombic dodecahedron resp. triacontahedron is 120° resp. 144° at every edge [3]. However, the dihedral angles meeting at e must sum up to less than 2π . With the given dihedral angles this is impossible. \square

3 Strictly bipartite polyhedra

In this section we derive the classification of strictly bipartite polyhedra. The main goal is to show that there are only two: the rhombic dodecahedron and the rhombic triacontahedron.

From this section on, let $P \subset \mathbb{R}^3$ denote a fixed *strictly bipartite polyhedron* with radii $r_1 < r_2$ and edge length ℓ . The 2-faces of P will be shortly referred to as just *faces* of P . Since they are bipartite, they are necessarily $2k$ -gons.

Definition 3.1. We use the following terminology:

- (i) a face of P is of *type $2k$* (or called a *$2k$ -face*) if it is a $2k$ -gonal polygon.

- (ii) an edge of P is of type $(2k_1, 2k_2)$ (or called a $(2k_1, 2k_2)$ -edge) if the two incident faces are of type $2k_1$ and $2k_2$ respectively.
- (iii) a vertex of P is of type $(2k_1, \dots, 2k_s)$ (or called a $(2k_1, \dots, 2k_s)$ -vertex) if its incident faces can be enumerated as $\sigma_1, \dots, \sigma_s$ so that σ_i is a $2k_i$ -face (note, the order of the numbers does not matter).

We write $\tau(v)$ for the type of a vertex $v \in \mathcal{F}_0(P)$.

3.1 General observations

In a given bipartite polyhedron, the type of a vertex, edge or face already determines much of its metric properties. We prove this for faces:

Proposition 3.2. *For some face $\sigma \in \mathcal{F}_2(P)$, any of the following properties of σ determines the other two:*

- (i) its type $2k$,
- (ii) its interior angles $\alpha_1 > \alpha_2$.
- (iii) its height h (that is, the distance of $\text{aff}(\sigma)$ from the origin).

Corollary 3.3. *Any two faces of P of the same height, or the same type, or the same interior angles, are congruent.*

Proof of Proposition 3.2. Fix a face $\sigma \in \mathcal{F}_2(P)$.

Suppose that the height h of σ is known. By Proposition 2.5, a face of P of height h is bipartite with radii $\rho_i^2 := r_i^2 - h^2$ and edge length ℓ . By Corollary 2.7, these parameters then uniquely determine the shape of σ , which includes its type and its interior angles. This shows (iii) \implies (i), (ii).

Suppose now that we know the interior angles $\alpha_1 > \alpha_2$ of σ (it actually suffices to know one of these, say α_1). Fix a 1-vertex $v \in V_1$ of σ and let $w_1, w_2 \in V_2$ be its two adjacent 2-vertices in σ . Consider the simplex $S := \text{conv}\{0, v, w_1, w_2\}$. The length of each edge of S is already determined, either by the parameters alone, or by additionally using the known interior angles via

$$\begin{aligned} \|w_1 - w_2\|^2 &= \|w_1 - v\|^2 + \|w_2 - v\|^2 - 2\langle w_1 - v, w_2 - v \rangle \\ &= 2\ell^2(1 - \underbrace{\cos \angle(w_1 - v, w_2 - v)}_{\alpha_1}). \end{aligned}$$

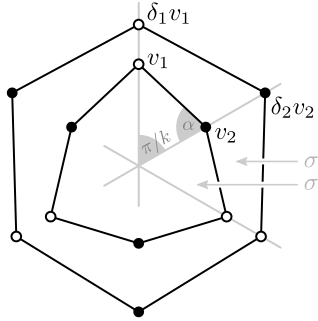
Thus, the shape of S is determined. In particular, this determines the height of the face $\text{conv}\{v, w_1, w_2\} \subset S$ over the vertex $0 \in S$. Since $\text{aff}\{v, w_1, w_2\} = \text{aff}(\sigma)$, this determines the height of σ in P . This proves (ii) \implies (iii).

Finally, suppose that the type $2k$ is known. We then want to show that the height h is uniquely determined.² For the sake of contradiction, suppose that the type $2k$ does *not*

²The reader motivated to prove this himself should know the following: it is indeed possible to write down a polynomial in h of degree four whose coefficients involve only r_1, r_2, ℓ and $\cos(\pi/k)$, and whose zeroes include all possible heights of any $2k$ -face of P . However, it turns out to be quite tricky to work out which zeroes correspond to feasible solutions. For certain values of the coefficients there are multiple positive solutions for h , some of which correspond to *non-convex* $2k$ -faces. There seems to be no easy way to tell them apart.

uniquely determine the height of the face. Then there is another $2k$ -face $\sigma' \in \mathcal{F}_2(P)$ of some height $h' \neq h$. W.l.o.g. assume $h > h'$.

Visualize both faces embedded in \mathbb{R}^2 , on top of each other and centered at the origin as shown in the figure below:



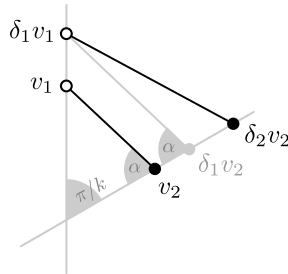
The vertices in both polygons are equally spaced by an angle of π/k (cf. Observation 2.6) and we can therefore assume that the vertex v_i of σ (resp. v'_i of σ') is a positive multiple of $(\sin(i\pi/k), \cos(i\pi/k)) \in \mathbb{R}^2$ for $i \in \{1, \dots, 2k\}$. There are then factors $\delta_i \in \mathbb{R}_+$ with $v'_i = \delta v_i$.

The norms of vectors $v_1, v_2, \delta_1 v_1$ and $\delta_2 v_2$ are the radii of the bipartite polygons σ and σ' . With Proposition 2.5(iii) from $h > h'$ follows $\|v_1\| < \|\delta_1 v_1\|$ and $\|v_2\| < \|\delta_2 v_2\|$, and thus, (*) $\delta_1, \delta_2 > 1$. W.l.o.g. assume $\delta_1 \leq \delta_2$.

Since both faces have edge length ℓ , we have $\|v_1 - v_2\| = \|\delta_1 v_1 - \delta_2 v_2\| = \ell$. Our goal is to derive the following contradiction:

$$\ell = \|v_1 - v_2\| \stackrel{(*)}{<} \delta_1 \|v_1 - v_2\| = \|\delta_1 v_1 - \delta_1 v_2\| \stackrel{(**)}{<} \|\delta_1 v_1 - \delta_2 v_2\| = \ell,$$

To prove (**), consider the triangle Δ with vertices $\delta_1 v_1, \delta_2 v_2$ and $\delta_1 v_2$:



Since σ is convex, the angle α is smaller than 90° . It follows that the interior angle of Δ at $\delta_1 v_2$ is obtuse (here we are using $\delta_1 \leq \delta_2$). Hence, by the sine law, the edge of Δ opposite to $\delta_1 v_2$ is the longest, which translates to (**). \square

As a consequence of Proposition 3.2, the interior angles of a face of P do only depend on the type of the face (and the parameters), and so we can introduce the notion of the interior angle $\alpha_i^k \in (0, \pi)$ of a $2k$ -face at an i -vertex. Furthermore, set $\epsilon_k := (\alpha_1^k - \alpha_2^k)/2\pi$. By Proposition 2.8 we have $\epsilon_k > 0$ and

$$\alpha_1^k = \left(1 - \frac{1}{k} + \epsilon_k\right)\pi, \quad \alpha_2^k = \left(1 - \frac{1}{k} - \epsilon_k\right)\pi.$$

Definition 3.4. If $\tau = (2k_1, \dots, 2k_s)$ is the type of a vertex, then define

$$K(\tau) := \sum_{i=1}^s \frac{1}{k_i}, \quad E(\tau) := \sum_{i=1}^s \epsilon_{k_i}.$$

Both quantities are strictly positive.

Proposition 3.5. Let $v \in \mathcal{F}_0(P)$ be a vertex of type $\tau = (2k_1, \dots, 2k_s)$.

- (i) If $v \in V_1$, then $E(\tau) < K(\tau) - 1$ and $s = 3$.
- (ii) If $v \in V_2$, then $E(\tau) > s - 2 - K(\tau)$.

Proof. Let $\sigma_1, \dots, \sigma_s \in \mathcal{F}_2(P)$ be the faces incident to v , so that σ_j is a $2k_j$ -face. The interior angle of σ_j at v is $\alpha_i^{k_j}$, and the sum of these must be smaller than 2π . In formulas

$$2\pi > \sum_{j=1}^s \alpha_i^{k_j} = \sum_{j=1}^s \left(1 - \frac{1}{k_j} \pm \epsilon_{k_j}\right)\pi = (s - K(\tau) \pm E(\tau))\pi,$$

where \pm is the plus sign for $i = 1$, and the minus sign for $i = 2$. Rearranging for $E(v)$ yields $(*) \mp E(\tau) > s - 2 - K(\tau)$. If $i = 2$, this proves (ii). If $i = 1$, note that from the implication $k_j \geq 2 \implies K(\tau) \leq s/2$ follows

$$s < -E(\tau) + K(\tau) + 2 \leq 0 + \frac{s}{2} + 2 \implies s < 4.$$

The minimum degree of a vertex in a polyhedron is at least three, hence $s = 3$, and $(*)$ becomes (i). \square

This allows us to exclude all but a manageable list of types for 1-vertices. Note that a vertex $v \in V_1$ has a type of some form $(2k_1, 2k_2, 2k_3)$.

Corollary 3.6. For a 1-vertex $v \in V_1$ of type τ holds $K(\tau) > 1 + E(\tau) > 1$. One checks that this leaves exactly the options in Table 1.

τ	$K(\tau)$	Γ
(4, 4, 4)	3/2	$I_1 \oplus I_1 \oplus I_1$
(4, 4, 6)	4/3	$I_1 \oplus I_2(3)$
(4, 4, 8)	5/4	$I_1 \oplus I_2(4)$
\vdots	\vdots	\vdots
(4, 4, 2k)	$1 + 1/k$	$I_1 \oplus I_2(k)$
(4, 6, 6)	7/6	$A_3 = D_3$
(4, 6, 8)	13/12	B_3
(4, 6, 10)	31/30	H_3

Table 1: Possible types of 1-vertices, their K -values and the Γ of the Γ -permutahedron in which all vertices have this type.

The types in Table 1 are called the *possible types* of 1-vertices. Each of the possible types is realizable in the sense that there exists a bipartite polyhedron in which all 1-vertices have this type. Examples are provided by the Γ -permutahedra (the Γ of that Γ -permutahedron is listed in the right column of Table 1). These are not *strictly* bipartite though.

The convenient thing about Γ -permutahedra is that all their vertices are of the same type. We cannot assume this for general strictly bipartite polyhedra, not even for all 1-vertices.

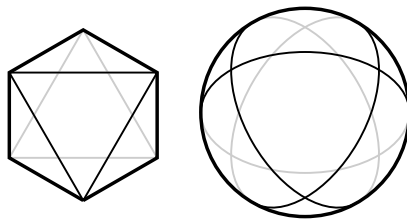
3.2 Spherical polyhedra

The purpose of this section is to define a second notion of interior angle for each face. These angles can be defined in several equivalent ways, one of which is via spherical polyhedra.

A *spherical polyhedron* is an embedding of a planar graph into the unit sphere, so that all edges are embedded as great circle arcs, and all regions are convex³. If $0 \in \text{int}(P)$, we can associate to P a spherical polyhedron P^S by applying central projection

$$\mathbb{R}^3 \setminus \{0\} \rightarrow \mathbf{S}_1(0), \quad x \mapsto \frac{x}{\|x\|}$$

to all its vertices and edges (this process is visualized below).



The vertices, edges and faces of P have spherical counterparts in P^S obtained as projections onto the unit sphere. Those will be denoted with a superscript “ S ”. For example, if $e \in \mathcal{F}_1(P)$ is an edge of P , then e^S denotes the corresponding “spherical edge”, which is a great circle arc obtained as the projection of e onto the sphere.

We still need to justify that the spherical polyhedron of P is well-defined, by proving that P contains the origin:

Proposition 3.7. $0 \in \text{int}(P)$.

Proof. The proof proceeds in several steps.

Step 1: Fix a 1-vertex $v \in V_1$ with neighbors $w_1, w_2, w_3 \in V_2$, and let $u_i := w_i - v$ be the direction of the edge $\text{conv}\{v, w_i\}$ emanating from v . Let $\sigma_{ij} \in \mathcal{F}_2(P)$ denote the $2k$ -face containing v, w_i and w_j . The interior angle of σ_{ij} at v is then $\angle(u_i, u_j)$, which by Proposition 2.8 and $k \geq 2$ satisfies

$$\angle(u_i, u_j) > \left(1 - \frac{1}{k}\right)\pi \geq \frac{\pi}{2} \implies \langle u_i, u_j \rangle < 0.$$

Step 2: Besides v , the polyhedron P contains another 1-vertex $v' \in V_1$. It then holds $v' \in v + \text{cone}\{u_1, u_2, u_3\}$, which means that there are non-negative coefficients

³Convexity on the sphere means that the shortest great circle arc connecting any two points in the region is also contained in the region.

$a_1, a_2, a_3 \geq 0$, at least one positive, so that $v + a_1u_1 + a_2u_2 + a_3u_3 = v'$. Rearranging and applying $\langle v, \cdot \rangle$ yields

$$\begin{aligned} a_1\langle v, u_1 \rangle + a_2\langle v, u_2 \rangle + a_3\langle v, u_3 \rangle &= \langle v, v' \rangle - \langle v, v \rangle \\ &= r_1^2 \cos \angle(v, v') - r_1^2 < 0. \end{aligned} \quad (*)$$

The value $\langle v, u_i \rangle$ is independent of i (see Observation 2.6). Since there is at least one positive coefficient a_i , from (*) follows $\langle v, u_i \rangle < 0$.⁴

Step 3: By the previous steps, $\{v, u_1, u_2, u_3\}$ is a set of four vectors with pair-wise negative inner product. The convex hull of such an arrangement in 3-dimensional Euclidean space does necessarily contain the origin in its interior, or equivalently, there are positive coefficients $a_0, \dots, a_3 > 0$ with $a_0v + a_1u_1 + a_2u_2 + a_3u_3 = 0$ (for a proof, see Proposition A.1). In other words: $0 \in v + \text{int}(\text{cone}\{u_1, u_2, u_3\})$.

Step 4: If $H(\sigma)$ denotes the half-space associated with the face $\sigma \in \mathcal{F}_2(P)$, then

$$0 \in v + \text{int}(\text{cone}\{u_1, u_2, u_3\}) = \bigcap_{\sigma \sim v} \text{int}(H(\sigma)).$$

Thus, $0 \in \text{int}(H(\sigma))$ for all faces σ incident to v . But since every face is incident to a 1-vertex, we obtain $0 \in \text{int}(H(\sigma))$ for all $\sigma \in \mathcal{F}_2(P)$, and thus $0 \in \text{int}(P)$ as well. \square

The main reason for introducing spherical polyhedra is that we can talk about the *spherical interior angles* of their faces.

Let $\sigma \in \mathcal{F}_2(P)$ be a face, and $v \in \mathcal{F}_0(\sigma)$ one of its vertices. Let $\alpha(\sigma, v)$ denote the interior angle of σ at v , and $\beta(\sigma, v)$ the spherical interior angle of σ^S at v^S . It only needs a straight-forward computation (involving some spherical geometry) to establish a direct relation between these angles: e.g. if v is a 1-vertex, then

$$\sin^2(\ell^S) \cdot (1 - \cos \beta(\sigma, v)) = \left(\frac{\ell}{r_2}\right)^2 \cdot (1 - \cos \alpha(\sigma, v)),$$

where ℓ^S denotes the arc-length of an edge of P^S (indeed, all edges are of the same length). An equivalent formula exists for 2-vertices. The details of the computation are not of relevance, but can be found in Appendix A.2.

The core message is that the value of $\alpha(\sigma, v)$ uniquely determines the value of $\beta(\sigma, v)$ and vice versa. In particular, since the value of $\alpha(\sigma, v) = \alpha_i^{k_i}$ does only depend on the type of the face and the partition class of the vertex, so does $\beta(\sigma, v)$, and it makes sense to introduce the notion $\beta_i^{k_i}$ for the spherical interior angle of a $2k$ -gonal spherical face of P^S at (the projection of) an i -vertex. Thus, we have

$$\beta_i^{k_1} = \beta_i^{k_2} \iff \alpha_i^{k_1} = \alpha_i^{k_2} \stackrel{3.2}{\iff} k_1 = k_2, \quad (3.1)$$

where we use Proposition 3.2 for the last equivalence.

Observation 3.8. The spherical interior angles $\beta_i^{k_i}$ have the following properties:

- (i) The spherical interior angles surrounding a vertex add up to exactly 2π . That is, for an i -vertex $v \in \mathcal{F}_0(P)$ of type $(2k_1, \dots, 2k_s)$ holds

$$\beta_i^{k_1} + \dots + \beta_i^{k_s} = 2\pi.$$

⁴Note that this provides the formal proof mentioned in Remark 2.3, namely, that the triangle $\text{conv}\{0, v_1, v_2\}$ is acute at v_1 and v_2 .

(ii) The sum of interior angles of a spherical polygon always exceed the interior angle sum of a respective flat polygon. That is, it holds

$$k\beta_1^k + k\beta_2^k > 2(k - 1)\pi \implies \beta_1^k + \beta_2^k > 2\left(1 - \frac{1}{k}\right)\pi.$$

This has some consequences for the strictly bipartite polyhedron P :

Corollary 3.9. P contains at most two different types of 1-vertices, and if there are two, then one is of the form $(4, 4, 2k)$, and the other one is of the form $(4, 6, 2k')$ for distinct $k \neq k'$ and $2k' \in \{6, 8, 10\}$.

Proof. Each possible type listed in Table 1 is either of the form $(4, 4, 2k)$ or of the form $(4, 6, 2k')$ for some $2k \geq 4$ or $2k' \in \{6, 8, 10\}$.

If P contains simultaneously 1-vertices of type $(4, 4, 2k_1)$ and $(4, 4, 2k_2)$, apply Observation 3.8(i) to see

$$\beta_1^2 + \beta_1^2 + \beta_1^{k_1} \stackrel{(i)}{=} \beta_1^2 + \beta_1^2 + \beta_1^{k_2} \implies \beta_1^{k_1} = \beta_1^{k_2} \xrightarrow{(3.1)} k_1 = k_2.$$

If P contains simultaneously 1-vertices of type $(4, 6, 2k'_1)$ and $(4, 6, 2k'_2)$, then

$$\beta_1^2 + \beta_1^3 + \beta_1^{k'_1} \stackrel{(i)}{=} \beta_1^2 + \beta_1^3 + \beta_1^{k'_2} \implies \beta_1^{k'_1} = \beta_1^{k'_2} \xrightarrow{(3.1)} k'_1 = k'_2.$$

Finally, if P contains simultaneously 1-vertices of type $(4, 4, 2k)$ and $(4, 6, 2k')$, then

$$\beta_1^2 + \beta_1^2 + \beta_1^k \stackrel{(i)}{=} \beta_1^2 + \beta_1^3 + \beta_1^{k'} \implies \beta_1^k - \beta_1^{k'} = \underbrace{\beta_1^3 - \beta_1^2}_{\neq 0 \text{ by (3.1)}} \xrightarrow{(3.1)} k \neq k'. \quad \square$$

Since each edge of P is incident to a 1-vertex, we obtain

Observation 3.10. If P has only 1-vertices of types $(4, 4, 2k)$ and $(4, 6, 2k')$, then each edge of P is of one of the types

$$\underbrace{(4, 4), (4, 2k)}_{\text{from a } (4, 4, 2k)\text{-vertex}}, \underbrace{(4, 6), (4, 2k') \text{ or } (6, 2k')}_{\text{from a } (4, 6, 2k')\text{-vertex}}.$$

Corollary 3.11. The dihedral angle of an edge $e \in \mathcal{F}_1(P)$ of P only depends on its type.

Proof. Suppose that e is a $(2k_1, 2k_2)$ -edge. Then e is incident to a 1-vertex $v \in V_1$ of type $(2k_1, 2k_2, 2k_3)$. By Observation 3.8(i) holds $\beta_1^{k_3} = 2\pi - \beta_1^{k_1} - \beta_1^{k_2}$, which further determines k_3 . By Proposition 3.2 we have uniquely determined interior angles $\alpha_1^{k_1}, \alpha_1^{k_2}$ and $\alpha_1^{k_3}$.

It is known that for a simple vertex (that is, a vertex of degree three) the interior angles of the incident faces already determine the dihedral angles at the incident edges (for a proof, see the Appendix, Proposition A.2). Consequently, the dihedral angle at e is already determined. \square

The next result shows that Γ -permutahedra are the only bipartite polytopes in which a 1-vertex and a 2-vertex can have the same type.

Corollary 3.12. *P cannot contain a 1-vertex and a 2-vertex of the same type.*

Proof. Let $v \in \mathcal{F}_0(P)$ be a vertex of type $(2k_1, 2k_2, 2k_3)$. The incident edges are of type $(2k_1, 2k_2)$, $(2k_2, 2k_3)$ and $(2k_3, 2k_1)$ respectively. By Corollary 3.11 the dihedral angles of these edges are uniquely determined, and since v is simple (that is, has degree three), the interior angles of the incident faces are also uniquely determined (cf. Appendix, Proposition A.2). In particular, we obtain the same angles independent of whether v is a 1-vertex or a 2-vertex.

A 1-vertex is always simple, and thus, a 1-vertex and a 2-vertex of the same type would have the same interior angles at all incident faces, that is, $\alpha_1^k = \alpha_2^k$ for each incident $2k$ -face. But this is not possible if P is strictly bipartite (by Proposition 2.5(ii) and Proposition 2.8). \square

3.3 Adjacent pairs

Given a 1-vertex $v \in V_1$ of type $\tau_1 = (2k_1, 2k_2, 2k_3)$, for any two distinct $i, j \in \{1, 2, 3\}$, v has a neighbor $w \in V_2$ of type $\tau_2 = (2k_i, 2k_j, *, \dots, *)$, where $*$ are placeholders for unknown entries. The pair of types

$$(\tau_1, \tau_2) = ((2k_1, 2k_2, 2k_3), (2k_i, 2k_j, *, \dots, *))$$

is called an *adjacent pair* of P . It is the purpose of this section to show that certain adjacent pairs cannot occur in P . Excluding enough adjacent pairs for fixed τ_1 then proves that the type τ_1 cannot occur as the type of a 1-vertex.

Our main tools for achieving this will be the inequalities established in Proposition 3.5 (i) and (ii), that is

$$E(\tau_1) \stackrel{(i)}{<} K(\tau_1) - 1 \quad \text{and} \quad E(\tau_2) \stackrel{(ii)}{>} s - 2 - K(\tau_2),$$

where s is the number of elements in τ_2 . For a warmup, and as a template for further calculations, we prove that the adjacent pair $(\tau_1, \tau_2) = ((4, 6, 8), (6, 8, 8))$ will not occur in P .

Example 3.13. By Proposition 3.5(i) we have

$$(*) \quad \epsilon_2 + \epsilon_3 + \epsilon_4 = E(\tau_1) \stackrel{(i)}{<} K(\tau_1) - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - 1 = \frac{1}{12}.$$

On the other hand, by Proposition 3.5(ii) we have

$$(**) \quad \frac{2}{12} = 3 - 2 - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{4}\right) = s - 2 - K(\tau_2) \\ < \stackrel{(ii)}{E(\tau_2)} = \underbrace{\epsilon_3 + \epsilon_4}_{<1/12} + \underbrace{\epsilon_4}_{<1/12} < \frac{2}{12},$$

which is a contradiction. Hence this adjacent pair cannot occur. Note that we used $(*)$ to upperbound certain sums of ϵ_i in $(**)$.

An adjacent pair excluded by using the inequalities from Proposition 3.5(i) and (ii) as demonstrated in Example 3.13 will be called *infeasible*.

The argument applied in Example 3.13 will be repeated many times for many different adjacent pairs in the upcoming Sections 3.5, 3.4, 3.6, 3.8, and we shall therefore use a tabular form to abbreviate it. After fixing, $\tau_1 = (4, 6, 8)$, the argument to refute the adjacent pair $(\tau_1, \tau_2) = ((4, 6, 8), (6, 8, 8))$ is abbreviated in the first row of the following table:

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$
$(6, 8, 8)$	2/12	$\not<$	$(\epsilon_3 + \epsilon_4) + \epsilon_4 < 2/12$
$(6, 8, 6, 6)$	9/12	$\not<$	$(\epsilon_3 + \epsilon_4) + \epsilon_3 + \epsilon_3 < 3/12$

The second row displays the analogue argument for another example, namely, the pair $((4, 6, 8), (6, 8, 6, 6))$, showing that it is infeasible as well. Both rows will reappear in the table of Section 3.5 where we exclude $(4, 6, 8)$ as a type for 1-vertices entirely. Note that the terms in the column below $E(\tau_2)$ are grouped by parenthesis to indicate which subsums are upper bounded via Proposition 3.5(i). In this example, if there are n groups, then the sum is upper bounded by $n/12$.

The placeholders in an adjacent pair $((2k_1, 2k_2, 2k_3), (2k_i, 2k_j, *, \dots, *))$ can, in theory, be replaced by an arbitrary sequence of even numbers, and each such pair has to be refuted separately. The following fact will make this task tractable: write $\tau \subset \tau'$ if τ is a *subtype* of τ' , that is, a vertex type that can be obtained from τ' by removing some of its entries. We then can prove

Proposition 3.14. *If (τ_1, τ_2) is an infeasible adjacent pair, then the pair (τ_1, τ'_2) is infeasible as well, for every $\tau'_2 \supset \tau_2$.*

Proof. Suppose $\tau_2 = (2k_1, \dots, 2k_s)$, $\tau'_2 = (2k_1, \dots, 2k_s, 2k_{s+1}, \dots, 2k_{s'}) \supset \tau_2$, and that the pair (τ_1, τ'_2) is *not* infeasible. Then τ'_2 satisfies Proposition 3.5(ii)

$$\begin{aligned} & E(\tau'_2) > s' - 2 - K(\tau'_2) \\ \implies & E(\tau_2) > s - 2 - K(\tau_2) + \sum_{i=s+1}^{s'} \overbrace{\left(1 - \frac{1}{k_i} - \epsilon_{k_i}\right)}^{\alpha_2^{k_i}/\pi > 0} > s - 2 - K(\tau_2). \end{aligned}$$

But this is exactly the statement that τ_2 satisfies Proposition 3.5(ii) as well, *i.e.*, that the pair (τ_1, τ_2) is also not infeasible. □

By Proposition 3.14 it is sufficient to exclude so-called *minimal infeasible adjacent pairs*, that is, infeasible adjacent pairs (τ_1, τ_2) for which (τ_1, τ'_2) is not infeasible for any $\tau'_2 \subset \tau_2$.

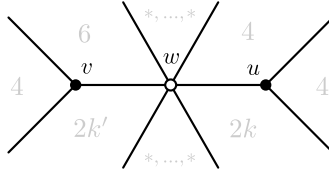
A second potential problem is, that we know little about the values that might replace the placeholders in $\tau_2 = (2k_i, 2k_j, *, \dots, *)$. For our immediate goal, dealing with the following special case is sufficient:

Proposition 3.15. *The placeholders in an adjacent pair $((4, 6, 2k'), (6, 2k', *, \dots, *))$ can only contain 4, 6 and $2k'$.*

Proof. Suppose that P contains an adjacent pair

$$(\tau_1, \tau_2) = ((4, 6, 2k'), (6, 2k', 2k, *, \dots, *))$$

induced by a 1-vertex $v \in V_1$ of type τ_1 with neighbor $w \in V_2$ of type τ_2 . Suppose further that $2k \notin \{4, 6, 2k'\}$. The vertex w is then incident to a $2k$ -face, and therefore also to a 1-vertex $u \in V_1$ of type $(4, 4, 2k)$ (u cannot be of type $(4, 6, 2k)$ because of $k \neq k'$ and Corollary 3.9). This configuration is depicted below:



Note that w is also incident to a 4-face, and thus $(6, 2k', 2k, 4) \subseteq \tau_2$.

By Proposition 3.5(i) the existence of 1-vertices of type $(4, 4, 2k)$ and $(4, 6, 2k')$ yields inequalities

$$\epsilon_2 + \epsilon_2 + \epsilon_k < \frac{1}{k} \quad \text{and} \quad \epsilon_2 + \epsilon_3 + \epsilon_{k'} < \frac{1}{k'} - \frac{1}{6}. \quad (3.2)$$

Since τ_2 has $\tau := (6, 2k', 2k, 4)$ as a subtype, by Proposition 3.14 it suffices to show that the pair $((4, 6, 2k'), (6, 2k', 2k, 4))$ is infeasible. This follows via Proposition 3.5(ii):

$$\begin{aligned} \frac{7}{6} - \frac{1}{k} - \frac{1}{k'} &= 4 - 2 - K(\tau) \\ &\stackrel{(ii)}{<} E(\tau) = \underbrace{\epsilon_2 + \epsilon_3 + \epsilon_{k'}}_{< 1/k' - 1/6} + \underbrace{\epsilon_k}_{< 1/k} &< \frac{1}{k} + \frac{1}{k'} - \frac{1}{6}, \end{aligned}$$

which rearranges to $1/k + 1/k' > 2/3$. Recalling $2k' \in \{6, 8, 10\} \implies k' \geq 3$ (from Corollary 3.9) and $2k \notin \{4, 6, 2k'\} \implies k \geq 4$ shows that this is not possible. \square

3.4 The case $\tau_1 = (4, 6, 10)$

If P contains a 1-vertex of type $(4, 6, 10)$, then it contains an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 6, 10), (6, 10, *, \dots, *)).$$

We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_3 + \epsilon_5 < 1/30$. By Proposition 3.15 the placeholders can only take on values 4, 6 or 10. The following table lists the minimally infeasible adjacent pairs and proves their infeasibility.

τ_2	$s - 2 - K(\tau_2)$	$\stackrel{?}{<}$	$E(\tau_2)$
$(6, 10, 6)$	4/30	$\not<$	$(\epsilon_3 + \epsilon_5) + \epsilon_3 < 2/30$
$(6, 10, 10)$	8/30	$\not<$	$(\epsilon_3 + \epsilon_5) + \epsilon_5 < 2/30$
$(6, 10, 4, 4)$	14/30	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_5) + \epsilon_2 < 2/30$

By Proposition 3.14 we conclude: the placeholder in $\tau_2 = (6, 10, *, \dots, *)$ can contain no 6 or 10, and at most one 4. This leaves us with the option $\tau_2 = (4, 6, 10)$, which is the same as τ_1 and therefore not possible by Corollary 3.12. Therefore, P cannot contain a 1-vertex of type $(4, 6, 10)$.

3.5 The case $\tau_1 = (4, 6, 8)$

If P contains a 1-vertex of type $(4, 6, 8)$, then it also contains an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 6, 8), (6, 8, *, \dots, *)).$$

We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_3 + \epsilon_4 < 1/12$. By Proposition 3.15 the placeholders can only take on values 4, 6 or 8. The following table lists the minimally infeasible adjacent pairs and proves their infeasibility.

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$
(6, 8, 8)	2/12	$\not<$	$(\epsilon_3 + \epsilon_4) + \epsilon_3 < 2/12$
(6, 8, 4, 4)	5/12	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_4) + \epsilon_2 < 2/12$
(6, 8, 4, 6)	7/12	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_4) + \epsilon_3 < 2/12$
(6, 8, 6, 6)	9/12	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_4) + \epsilon_3 + \epsilon_3 < 3/12$

By Proposition 3.14 we conclude: the placeholder in $\tau_2 = (6, 8, *, \dots, *)$ can contain no 8, and at most one 4 or 6, but not both at the same time.

This leaves us with the options $\tau_2 = (4, 6, 8)$ and $\tau_2 = (6, 6, 8)$. In the first case, $\tau_1 = \tau_2$ which not possible by Corollary 3.12. In the second case, there would be two adjacent 6-faces, but P does not contain (6, 6)-edges by Observation 3.10 with $2k' = 8$. Therefore, P cannot contain a 1-vertex of type (4, 6, 8).

3.6 The case $\tau_1 = (4, 6, 6)$

If P contains a 1-vertex of type (4, 6, 6), then it also contains an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 6, 6), (6, 6, *, \dots, *)).$$

We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_3 + \epsilon_3 < 1/6$. By Proposition 3.15 the placeholders can only take on values 4 or 6. The following table lists the minimally infeasible adjacent pairs and proves their infeasibility.

τ_2	$s - 2 - K(\tau_2)$	$\overset{?}{<}$	$E(\tau_2)$
(6, 6, 4, 4)	2/6	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_3) + \epsilon_2 < 2/6$
(6, 6, 6, 4)	3/6	$\not<$	$(\epsilon_2 + \epsilon_3 + \epsilon_3) + \epsilon_3 < 2/6$
(6, 6, 6, 6)	4/6	$\not<$	$(\epsilon_3 + \epsilon_3) + (\epsilon_3 + \epsilon_3) < 2/6$

By Proposition 3.14 we conclude: the placeholder in $\tau_2 = (6, 6, *, \dots, *)$ can contain at most one 4 or 6, but not both at the same time.

This leaves us with the options $\tau_2 = (4, 6, 6)$ and $\tau_2 = (6, 6, 6)$. In the first case we have $\tau_1 = \tau_2$, which is not possible by Corollary 3.12. Excluding (6, 6, 6) needs more work: fix a 6-gon $\sigma \in \mathcal{F}_2(P)$. Each edge of σ is either of type (4, 6) or of type (6, 6) (by Observation 3.10). Each 1-vertex of σ (which must be of type (4, 6, 6)) is then incident to exactly one of these (6, 6)-edges of σ . Thus, there are exactly *three* (6, 6)-edges incident to σ (see Figure 3). On the other hand, each 2-vertex of σ is incident to an even number of (6, 6)-edges of σ (since if a 2-vertex is incident to at least one (6, 6)-edge, then we have previously shown that its type must be (6, 6, 6), implying another incident (6, 6)-edge). Therefore the number of (6, 6)-edges incident to σ must be *even* (see Figure 3), in contradiction to the previously obtained number three of such edges.

Consequently, P cannot contain a 1-vertex of type (4, 6, 6).

Observation 3.16. It is a consequence of Sections 3.6, 3.5, 3.4 that P cannot have a 1-vertex of a type (4, 6, $2k'$) for a $2k' \in \{6, 8, 10\}$. By Corollary 3.9 this means that *all* 1-vertices of P are of the same type $\tau_1 = (4, 4, 2k)$ for some fixed $2k \geq 4$.

It is worth to distinguish the case (4, 4, 4) from the cases (4, 4, $2k$) with $2k \geq 6$.

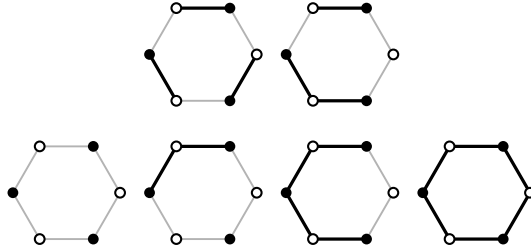


Figure 3: Possible distributions of (4, 6)-edges (gray) and (6, 6)-edges (thick) around a 6-gon as discussed in Section 3.6. The top row shows configurations compatible with the conditions set by 1-vertices (black), and the bottom row shows the configurations compatible with the conditions set by the 2-vertices (white).

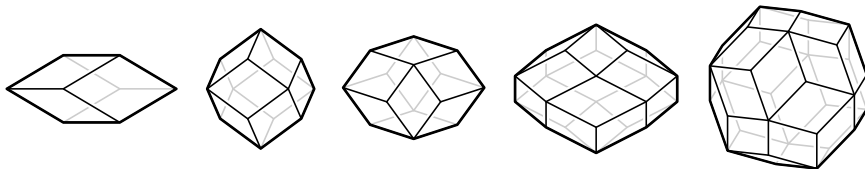
3.7 The case $\tau_1 = (4, 4, 4)$

In this case, all 2-faces are 4-gons, and all 4-gons are congruent by Proposition 3.2. A 4-gon with all edges of the same length is known as a *rhombus*, and the polyhedra with congruent rhombic faces are known as *rhombic isohedra* (from german *Rhombenisoeder*). These have a known classification:

Theorem 3.17 (S. Bilinski, 1960 [2]). *If P is a polyhedron with congruent rhombic faces, then P is one of the following:*

- (i) *a member of the infinite family of rhombic hexahedra, i.e., P can be obtained from a cube by stretching or squeezing it along a long diagonal,*
- (ii) *the rhombic dodecahedron,*
- (iii) *the Bilinski dodecahedron,*
- (iv) *the rhombic icosahedron, or*
- (v) *the rhombic triacontahedron.*

The figure below depicts these polyhedra in the given order (from left to right; including only one instance from the family (i)):



The rhombic dodecahedron and triacontahedron are known edge- but not vertex-transitive polytopes. We show that the others are not even strictly bipartite.

Corollary 3.18. *If P is strictly bipartite with all 1-vertices of type $(4, 4, 4)$, then P is one of the following:*

- (i) *the rhombic dodecahedron,*

(ii) *the rhombic triacontahedron.*

Proof. The listed ones are edge-transitive but not vertex-transitive. Also they are not inscribed. By Proposition 2.4 they are therefore *strictly* bipartite.

We then have to exclude the other polyhedra listed in Theorem 3.17. The rhombic hexahedra include the cube, which is inscribed, hence not strictly bipartite. In all the other cases, there exist vertices where acute and obtuse angles meet (see the figure). So this vertex cannot be assigned to either V_1 or V_2 (cf. Observation 2.9), and the polyhedron cannot be bipartite. \square

These are the only strictly bipartite polyhedra we will find, and both are edge-transitive without being vertex-transitive.

3.8 The case $\tau_1 = (4, 4, 2k), 2k \geq 6$

If P contains a 1-vertex of type $(4, 4, 2k)$ with $2k \geq 6$, then it also has an adjacent pair of the form

$$(\tau_1, \tau_2) = ((4, 4, 2k), (4, 2k, *, \dots, *)).$$

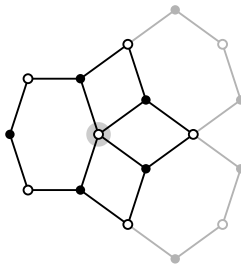
We proceed as demonstrated in Example 3.13. Proposition 3.5(i) yields $\epsilon_2 + \epsilon_2 + \epsilon_k < 1/k$. Since $(4, 4, 2k)$ is the only type of 1-vertex of P , there are only 4-faces and $2k$ -faces and the placeholders can only take on the values 4 and $2k$ (note that we do *not* use Proposition 3.15 for this). The following table lists some inequalities derived for infeasible pairs:

τ_2	$s - 2 - K(\tau_2)$	$\stackrel{?}{<}$	$E(\tau_2)$
$(4, 2k, 4, 4, 4)$	$1 - 1/k$	$<$	$(\epsilon_2 + \epsilon_2 + \epsilon_k) + (\epsilon_2 + \epsilon_2) < 2/k$
$(4, 2k, 4, 4, 2k)$	$3/2 - 2/k$	$<$	$(\epsilon_2 + \epsilon_2 + \epsilon_k) + (\epsilon_2 + \epsilon_k) < 2/k$

One checks that these inequalities are not satisfied for $2k \geq 6$. Proposition 3.14 then states that the placeholders can contain at most two 4-s, and if exactly two, then nothing else. Moreover, τ_2 must contain at least as many 4-s as it contains $2k$ -s, as otherwise we would find two adjacent $2k$ -faces while P cannot contain a $(2k, 2k)$ -edge by Observation 3.10. We are therefore left with the following options for τ_2 :

$$(4, 4, 2k), (4, 4, 4, 2k) \quad \text{and} \quad (4, 2k, 4, 2k).$$

The case $\tau_2 = (4, 4, 2k)$ is impossible by Corollary 3.12. We show that $\tau_2 = (4, 4, 4, 2k)$ is also not possible: consider the local neighborhood of a $(4, 4, 4, 2k)$ -vertex (the highlighted vertex) in the following figure:



Since the 1-vertices (black dots) are of type $(4, 4, 6)$, this configuration forces on us the existence of the two gray 6-faces. These two faces intersect in a 2-vertex, which is then incident to two $2k$ -faces and must be of type $(4, 2k, 4, 2k)$. But we can show that the types $(4, 4, 4, 2k)$ and $(4, 2k, 4, 2k)$ are incompatible by Observation 3.8(i):

$$\beta_2^2 + \beta_2^2 + \beta_2^2 + \beta_2^k \stackrel{(i)}{=} \beta_2^2 + \beta_2^k + \beta_2^2 + \beta_2^k \implies \beta_2^2 = \beta_2^k \stackrel{(3.1)}{\implies} 4 = 2k \geq 6.$$

Thus, $(4, 4, 4, 2k)$ cannot occur.

We conclude that every 2-vertex incident to a $2k$ -face must be of type $(4, 2k, 4, 2k)$. Consider then the following table:

τ_2	$s - 2 - K(\tau_2)$	$\stackrel{?}{<}$	$E(\tau_2)$
$(4, 2k, 4, 2k)$	$1 - 2/k$	$<$	$(\epsilon_2 + \epsilon_2 + \epsilon_k) + \epsilon_2 < 2/k$

The established inequality yields $2k \leq 6$, and hence $2k = 6$. We found that then all 1-vertices must be of type $(4, 4, 6)$, and all 2-vertices incident to a 6-face must be of type $(4, 6, 4, 6)$.

3.9 The case $\tau_1 = (4, 4, 6)$

At this point we can now assume that all 1-vertices of P are of type $(4, 4, 6)$ and that each 2-vertex of P that is incident to a 6-face is of type $(4, 6, 4, 6)$. In particular, P contains a 2-vertex $w \in V_2$ of this type. Since there is no $(6, 6)$ -edge in P , the two 6-faces incident to w cannot be adjacent. In other words, the faces around w must occur alternatingly of type 4 and type 6, which is the reason for writing the type $(4, 6, 4, 6)$ with alternating entries.

On the other hand, P contains $(4, 4)$ -edges, and none of these is incident to a $(4, 6, 4, 6)$ -vertex surrounded by alternating faces. Thus, there must be further 2-vertices of a type other than $(4, 6, 4, 6)$, necessarily *not* incident to any 6-face. These must then be of type

$$(4^r) := \underbrace{(4, \dots, 4)}_r, \quad \text{for some } r \geq 3.$$

Proposition 3.19. $r = 5$.

Proof. If there is a (4^r) -vertex, Observation 3.8(i) yields $\beta_2^2 = 2\pi/r$. Analogously, from the existence of a $(4, 6, 4, 6)$ -vertex follows

$$2\beta_2^2 + 2\beta_2^3 \stackrel{(i)}{=} 2\pi \implies \beta_2^3 = \frac{2\pi - 2\beta_2^2}{2} = \left(1 - \frac{2}{r}\right)\pi.$$

Recall $k\beta_1^k + k\beta_2^k > 2\pi(k - 1)$ from Observation 3.8(ii). Together with the previously established values for β_2^2 and β_2^3 , this yields

$$\begin{aligned} \beta_1^2 &> \frac{2\pi(2 - 1) - 2\beta_2^2}{2} = \left(1 - \frac{2}{r}\right)\pi, \quad \text{and} \\ \beta_1^3 &> \frac{2\pi(3 - 1) - 3\beta_2^3}{3} = \left(\frac{1}{3} + \frac{2}{r}\right)\pi. \end{aligned} \tag{3.3}$$

Since the 1-vertices are of type $(4, 4, 6)$, Observation 3.8(i) yields

$$2\pi \stackrel{(i)}{=} 2\beta_1^2 + \beta_1^3 \stackrel{(3.3)}{>} 2\left(1 - \frac{2}{r}\right)\pi + \left(\frac{1}{3} + \frac{2}{r}\right)\pi = \left(\frac{7}{3} - \frac{2}{r}\right)\pi.$$

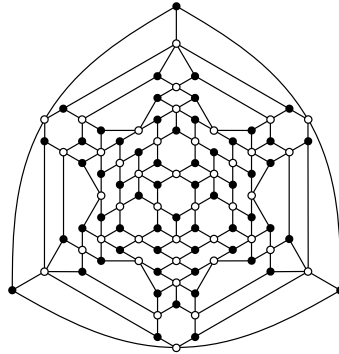


Figure 4: The edge-graph of the final candidate polyhedron.

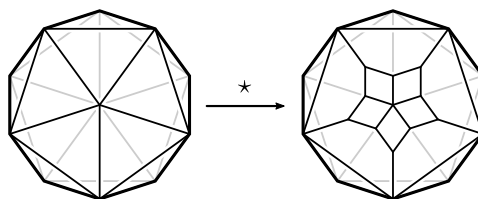
And one checks that this rearranges to $r < 6$.

This leaves us with the options $r \in \{3, 4, 5\}$. If $r = 4$, then $\beta_2^3 = \pi/2 = \beta_2^2$, which is impossible by Equation (3.1). And if $r = 3$, then (3.3) yields $\beta_1^3 > \pi$, which is also impossible for a convex face of a spherical polyhedron. We are left with $r = 5$. \square

To summarize: P is a strictly bipartite polyhedron in which all 1-vertices are of type $(4, 4, 6)$, and all 2-vertices are of types $(4, 6, 4, 6)$ or (4^5) , and both types actually occur in P . This information turns out to be sufficient to uniquely determine the edge-graph of P , which is shown in Figure 4.

This graph can be constructed by starting with a hexagon in the center with vertices of alternating colors (indicating the partition classes). One then successively adds further faces (according to the structural properties determined above), layer by layer. This process involves no choice and thus the result is unique.

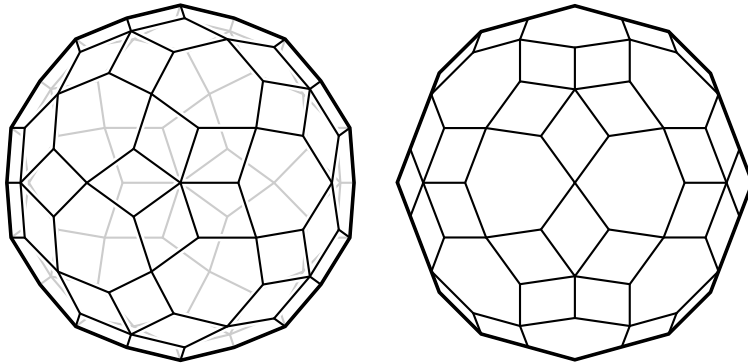
As mentioned in Remark 2.3, a bipartite polyhedron has an edge in-sphere. Thus, P is a polyhedral realization of the graph in Figure 4 with an edge in-sphere. It is known that any two such realizations are related by a projective transformation [8]. One representative $Q \subset \mathbb{R}^3$ from this class (which we do not yet claim to coincide with P) can be constructed by applying the following operation \star to each vertex of the regular icosahedron:



The operation is performed in such a way, so that

- the five new “outer” vertices of the new 4-gons are positioned in the centers of edges of the icosahedron.
- the edges of each new 4-gon are tangent to a common sphere centered at the center of the icosahedron

The resulting polyhedron Q looks as follows:



One can verify that Q has indeed the desired edge-graph.

It is clear from the construction that Q has an edge in-sphere, and any two of its 4-gonal or 6-gonal faces are congruent (as we would expect from a bipartite polyhedron). Like-wise, P has an edge in-sphere and the same edge-graph. Hence, P must be a projective transformation of Q . However, any projective transformation that is not just a re-orientation or a uniform rescaling will inevitably destroy the property of congruent faces. In conclusion, we can assume that P is identical to Q (up to scale and orientation).

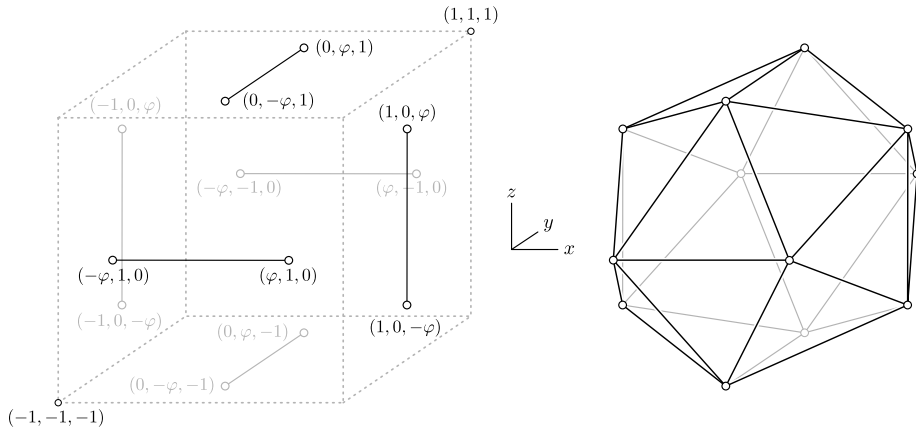
It remains to check whether Q is indeed a bipartite polyhedron. For this, recall that any two of the following properties imply the third (*cf.* Remark 2.3):

- (i) Q has an edge in-sphere.
- (ii) Q has all edges of the same length.
- (iii) for each vertex $v \in \mathcal{F}_0(Q)$, the distance $\|v\|$ only depends on the partition class of the vertex.

Now, Q satisfies (i) by construction, and it would need to satisfy both (ii) and (iii) in order to be bipartite. The figure certainly suggests that all edges of Q are of the same length. However, as we shall show now, Q cannot satisfy both (ii) and (iii) at the same time, and thus, can satisfy neither. In particular, the edges must have a tiny difference in length that cannot be spotted visually, making Q into a remarkable near-miss (we will quantify this below).

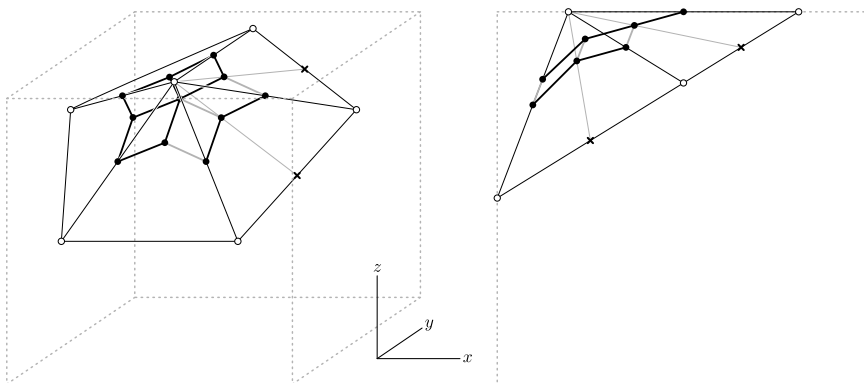
For what follows, let us assume that (ii) holds, that is, that all edges of Q are of the same length, in particular, that all 4-gons are rhombuses. Our goal is to show that $\|v\|$ depends on the type of the vertex $v \in V_2$ (not only its partition class), establishing that (iii) does not hold.

For this, start from the following well-known construction of the regular icosahedron from the cube of edge-length 2 centered at the origin.

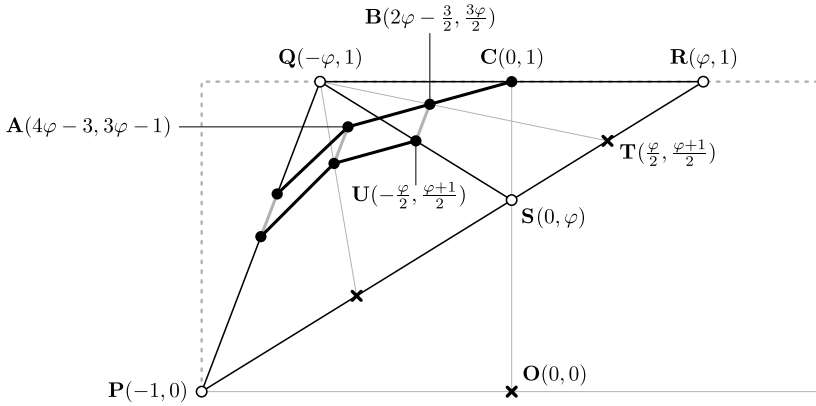


The construction is as follows: insert a line segment in the center of each face of the cube as shown in the left image. Each line segment is of length 2φ , where $\varphi \approx 0.61803$ is the positive solution of $\varphi^2 = 1 - \varphi$ (one of the numbers commonly known as the *golden ratio*). The convex hull of these line segments gives the icosahedron with edge length 2φ .

It is now sufficient to consider a single vertex of the icosahedron together with its incident faces. The image below shows this vertex after we applied \star .



The image on the right is the orthogonal projection of the configuration on the left onto the yz -plane. This projection makes it especially easy to give 2D-coordinates for several important points:



The points **A** and **C** are 2-vertices of Q of type (4^5) and $(4, 6, 4, 6)$ respectively. Both points and the origin **O** are contained in the yz -plane onto which we projected. Consequently, distances between these points are preserved during the projection, and assuming that Q is bipartite, we would expect to find $|\overline{\mathbf{OA}}| = |\overline{\mathbf{OC}}| = r_2$. We shall see that this is not the case, by explicitly computing the coordinates of **A** and **C** in the new coordinate system (y, z) .

By construction, $\mathbf{C} = (0, 1)$ and $|\overline{\mathbf{OC}}| = 1$. Other points with easily determined coordinates are **P**, **Q**, **R**, **S**, **T** (the midpoint of **R** and **S**) and **U** (the midpoint of **Q** and **S**).

By construction, the point **B** lies on the line segment $\overline{\mathbf{QT}}$. The parallel projection of a rhombus is a (potentially degenerated) parallelogram, and thus, opposite edges in the projection are still parallel. Hence, the gray edges in the figure are parallel. For that reason, the segment $\overline{\mathbf{UB}}$ is parallel to $\overline{\mathbf{PQ}}$. This information suffices to determine the coordinates of **B**, which is now the intersection of $\overline{\mathbf{QT}}$ with the parallel of $\overline{\mathbf{PQ}}$ through **U**. The coordinates are given in the figure.

The rhombus containing the vertices **A**, **B** and **C** degenerated to a line. Its fourth vertex is also located at **B**. Therefore, the segments $\overline{\mathbf{CB}}$ and $\overline{\mathbf{BA}}$ are translates of each other. Since the point **B** and the segment $\overline{\mathbf{CB}}$ are known, this allows the computation of the coordinates of **A** as given in the figure.

We can finally compute $|\overline{\mathbf{OA}}|$. For this, recall $(*) \varphi^{2n} = F_{2n-2} - \varphi F_{2n-1}$, where F_n denotes the n -th *Fibonacci number* with initial conditions $F_0 = F_1 = 1$. Then

$$\begin{aligned} |\overline{\mathbf{OA}}|^2 &= (4\varphi - 3)^2 + (3\varphi - 1)^2 \\ &= 25\varphi^2 - 30\varphi + 10 \\ &\stackrel{(*)}{=} 25(1 - \varphi) - 30\varphi + 10 \\ &= 35 - 55\varphi \\ &= 1 + (34 - 55\varphi) \stackrel{(*)}{=} 1 + \varphi^{10} > 1, \end{aligned}$$

and thus, Q cannot be bipartite. Remarkably, we find that

$$|\overline{\mathbf{OA}}| = \sqrt{1 + \varphi^{10}} \approx 1.00405707$$

is only about 0.4% larger than $|\overline{\mathbf{OC}}| = 1$, and so while Q is not bipartite, it is a remarkable near-miss.

Since P was assumed to be bipartite, but was also shown to be identical to Q , we reached a contradiction, which finally proves Theorem 2.13, and the goal of the paper is achieved.

4 Conclusions and open questions

In this paper we have shown that any edge-transitive (convex) polytope in four or more dimensions is necessarily vertex-transitive. We have done this by classifying all polytopes which simultaneously have all edges of the same length, an edge in-sphere and a bipartite edge graph (which we named *bipartite* polytopes).

The obstructions we derived for being edge-transitive without being vertex-transitive have been primarily geometric and less a matter of symmetry (a detailed investigation of the Euclidean symmetry groups was not necessary, but it might be interesting to view the problem from this perspective). We suspect that dropping convexity or considering combinatorial symmetries instead of geometric ones will quickly lead to examples of “just edge-transitive structures”. For example, it is easy to find embeddings of graphs into \mathbb{R}^d with these properties.

Slightly stronger than being simultaneously vertex- and edge-transitive is being transitive on *arcs*, that is, on incident vertex-edge pairs. This additional degree of symmetry allows an edge to be not only mapped onto any other edge, but also onto itself with reversed orientation. While there are graphs that are vertex- and edge-transitive without being arc-transitive (the so-called *half-transitive* graphs, see [7]), we believe it is unlikely that this distinction is necessary for convex polytopes.

Question 4.1. Is there a polytope $P \subset \mathbb{R}^d$ that is edge-transitive and vertex-transitive, but not arc-transitive?

In a different direction, the questions of this paper naturally generalize to faces of higher dimensions. In general, the interactions between transitivity of faces of different dimensions have been little investigated. For example, already the following question seems to be open:

Question 4.2. For fixed $k \in \{2, \dots, d - 3\}$, are there convex d -polytopes for arbitrarily large $d \in \mathbb{N}$ that are transitive on k -dimensional faces without being transitive on either vertices or facets?

Of course, any such question could be attacked by attempting to classify the k -face-transitive (convex) polytopes for some $k \in \{1, \dots, d - 2\}$. It seems to be unclear for which k this problem is tractable (for comparison, $k = 0$ is intractable, see [1]), and it appears that there are no techniques applicable to all (or many) k at the same time.

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A

A.1 Geometry

Proposition A.1. *Given a set $x_0, \dots, x_d \in \mathbb{R}^d \setminus \{0\}$ of $d+1$ vectors with pair-wise negative inner product, then there are positive coefficients $\alpha_0, \dots, \alpha_d > 0$ with*

$$\alpha_0 x_0 + \dots + \alpha_d x_d = 0.$$

Proof. We proceed by induction. The induction base $d = 1$ which is trivially true.

Now suppose $d \geq 2$, and, W.l.o.g. assume $\|x_0\| = 1$. Let π_0 be the orthogonal projection onto x_0^\perp , that is, $\pi_0(u) := u - x_0 \langle x_0, u \rangle$. In particular, for $i \neq j$ and $i, j > 0$

$$\langle \pi_0(x_i), \pi_0(x_j) \rangle = \underbrace{\langle x_i, x_j \rangle}_{<0} - \underbrace{\langle x_0, x_i \rangle}_{<0} \underbrace{\langle x_0, x_j \rangle}_{<0} < 0.$$

Then $\{\pi(x_1), \dots, \pi_0(x_d)\}$ is a set of d vectors in $x_0^\perp \cong \mathbb{R}^{d-1}$ with pair-wise negative inner product. By induction assumption there are positive coefficients $\alpha_1, \dots, \alpha_d > 0$ so that $\alpha_1 \pi_0(x_1) + \dots + \alpha_d \pi_0(x_d) = 0$.

Set $\alpha_0 := -\langle x_0, \alpha_1 x_1 + \dots + \alpha_d x_d \rangle > 0$. We claim that $x := x_0 \alpha_0 + \dots + \alpha_d x_d = 0$. Since $\mathbb{R}^d = \text{span}\{x_0\} \oplus x_0^\perp$, it suffices to check that $\langle x_0, x \rangle = 0$ as well as $\pi_0(x) = 0$. This follows:

$$\begin{aligned} \langle x_0, x \rangle &= \alpha_0 \underbrace{\langle x_0, x_0 \rangle}_{=1} + \underbrace{\langle x_0, \alpha_1 x_1 + \dots + \alpha_d x_d \rangle}_{=-\alpha_0} = 0, \\ \pi_0(x) &= \alpha_0 \underbrace{\pi_0(x_0)}_{=0} + \underbrace{\alpha_1 \pi_0(x_1) + \dots + \alpha_d \pi_0(x_d)}_{=0} = 0. \end{aligned} \quad \square$$

Proposition A.2. *Let $P \subset \mathbb{R}^3$ be a polyhedron with $v \in \mathcal{F}_0(P)$ a vertex of degree three. The interior angles of the faces incident to v determine the dihedral angles at the edges incident to v and vice versa.*

Proof. For $w_1, w_2, w_3 \in \mathcal{F}_0(P)$ the neighbors of v , let $u_i := w_i - v$ denote the direction of the edge e_i from v to w_i . Let σ_{ij} be the face that contains v, w_i and w_j . Then $\angle(u_i, u_j)$ is the interior angle of σ_{ij} at v .

The set $\{u_1, u_2, u_3\}$ is uniquely determined (up to some orthogonal transformation) by the angles $\angle(u_i, u_j)$. Furthermore, since P is convex, $\{u_1, u_2, u_3\}$ forms a basis of \mathbb{R}^3 , and this uniquely determines the *dual basis* $\{n_{12}, n_{23}, n_{31}\}$ for which $\langle n_{ij}, u_i \rangle = \langle n_{ij}, u_j \rangle = 0$. In other words, n_{ij} is a normal vector to σ_{ij} . The dihedral angle at the edge e_j is then $\pi - \angle(n_{ij}, n_{jk})$, hence uniquely determined. The other direction is analogous, via constructing $\{u_1, u_2, u_3\}$ as the dual basis to the set of normal vectors. \square

A.2 Computations

The edge lengths in a spherical polyhedron are measured as angles between its end vertices. Consider adjacent vertices $v_1^S, v_2^S \in \mathcal{F}_0(P^S)$, then the incident edge has (arc-)length $\ell^S := \angle(v_1^S, v_2^S) = \angle(v_1, v_2)$.

It follows from Observation 2.6 that these angles are completely determined by the parameters, hence the same for all edges of P^S .

Proposition A.3. For a face $\sigma \in \mathcal{F}_2(P)$ and a vertex $v \in \mathcal{F}_0(\sigma)$, there is a direct relationship between the value of $\alpha(\sigma, v)$ and the value of $\beta(\sigma, v)$.

Proof. Let $w_1, w_2 \in V_2$ be the neighbors of v in the $2k$ -face σ , and set $u_i := w_i - v$. Then $\angle(u_1, u_2) = \alpha(\sigma, v)$. W.l.o.g. assume that v is a 1-vertex (the argument is equivalent for a 2-vertex).

For convenience, we introduce the notation $\chi(\theta) := 1 - \cos(\theta)$. We find that

$$\begin{aligned}
 (*) \quad 2\ell^2 \cdot \chi(\alpha(\sigma, v)) &= \ell^2 + \ell^2 - 2\ell^2 \cos(\angle(u_1, u_2)) \\
 &= \|u_1\|^2 + \|u_2\|^2 - 2\langle u_1, u_2 \rangle \\
 &= \|u_1 - u_2\|^2 = \|w_1 - w_2\|^2 \\
 &= \|w_1\|^2 + \|w_2\|^2 - 2\langle w_1, w_2 \rangle \\
 &= r_2^2 + r_2^2 - r_2^2 \cos \angle(w_1, w_2) = 2r_2^2 \cdot \chi(\angle(w_1, w_2)).
 \end{aligned}$$

The side lengths of the spherical triangle $w_1^S v^S w_2^S$ are $\angle(w_1, w_2)$, ℓ^S and ℓ^S . By the spherical law of cosine⁵ we obtain

$$\begin{aligned}
 \cos \angle(w_1, w_2) &= \cos(\ell^S) \cos(\ell^S) + \sin(\ell^S) \sin(\ell^S) \cos(\beta(\sigma, v)) \\
 &= \cos^2(\ell^S) + \sin^2(\ell^S)(\cos(\beta(\sigma, v)) - 1 + 1) \\
 &= [\cos^2(\ell^S) + \sin^2(\ell^S)] + \sin^2(\ell^S)(\cos(\beta(\sigma, v)) - 1) \\
 &= 1 - \sin^2(\ell^S) \cdot \chi(\beta(\sigma, v)) \\
 \implies \sin^2(\ell^S) \cdot \chi(\beta(\sigma, v)) &= \chi(\angle(w_1, w_2)) \stackrel{(*)}{=} \left(\frac{\ell}{r_2}\right)^2 \cdot \chi(\alpha(\sigma, v)). \quad \square
 \end{aligned}$$

⁵ $\cos(c) = \cos(a) \cos(b) + \sin(a) \sin(b) \cos(\gamma)$, where a, b and c are the side lengths (arc-lengths) of a spherical triangle, and γ is the interior angle opposite to the side of length c .