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A new family of additive designs

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Abstract

In this paper we construct a family of 2-(q^n, sp^2, λ) additive designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, where q is a power of a prime p and $\mathcal{P}$ is a n-dimensional vector space over $\mathrm{GF}(q)$, and we compute their parameters explicitly. These designs, except for some special cases, had not been considered in the previous literature on additive block designs.

Keywords: Block designs, additive designs.

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1 Additive designs

Point-flat designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ of an affine geometry $\mathrm{AG}(d, p)$ over $\mathrm{GF}(p)$, as well as of a projective geometry $\mathrm{PG}(d, 2)$ over $\mathrm{GF}(2)$, are basic examples of 2-(v, k, λ) designs for which, if $\mathcal{P}$ is taken to be $\mathrm{GF}(p^d)$, respectively $\mathrm{GF}(2^{d+1})^* = \mathrm{GF}(2^{d+1}) \setminus \{0\}$, then the blocks have the property that the sum of their points in $\mathcal{P}$ is zero.

As soon as $k > 4$, the family $\mathcal{B}$ of blocks of any of these designs is strictly contained in the family $\mathcal{B}_k$ (respectively, $\mathcal{B}_k^*$) of all the k-subsets of $\mathrm{GF}(p^d)$ (respectively, $\mathrm{GF}(2^{d+1})^*$) whose elements sum up to zero. In [19], and in [13] for the case $p = 2$, it is shown that the incidence structure $\mathcal{D}_k = (\mathcal{P}, \mathcal{B}_k)$ is a 2-(p^d, k, λ) design if and only if $k = mp$ for some integer m, and that, in such a case, the automorphism group of $\mathcal{D}_k$ is the group of invertible affine mappings $\phi(x) = \phi_0(x) + \phi(0)$ over $\mathrm{GF}(p)$, with $\phi_0 \in \mathrm{GL}(d, p)$. In this case, by applying a well-known result of Li and Wan [15] (see also [14, Theorem 2.4] and [20]), one finds that

$$\lambda = \frac{1}{p^d}\binom{p^d - 2}{k - 2} + c_k \, \frac{k-1}{p^d}\binom{p^{d-1} - 1}{m - 1},$$

where $c_k = (-1)^m$ if $p = 2$ and $c_k = 1$ otherwise.

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Moreover, for $p = 2$, the incidence structure $\mathcal{D}_k^* = (\mathrm{GF}(2^{d+1})^*, \mathcal{B}_k^*)$ is a 2-$(2^{d+1}-1, k, \lambda)$ design for any integer k, and, again, the parameter λ is given by an explicit formula [13, Proposition 2.6], whereas the automorphism group of $\mathcal{D}_k^*$ is the group $\mathrm{GL}(d+1, 2)$ of invertible linear mappings on $\mathrm{GF}(2^{d+1})$ over $\mathrm{GF}(2)$. Among the subdesigns of the latter designs one finds the only known Steiner 2-design over a finite field, found by Braun et al. [2] and revisited in [6], when seen as a 2-$(8191, 7, 1)$ design (note that $8191 = 2^{13} - 1$), as well as the 2-$(2v-1, 7, 7)$ designs over $\mathrm{GF}(2)$ considered in [4], [21].

More generally, in [8] and [9] a 2-(v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is said to be *additive* if $\mathcal{P}$ can be embedded in a finite commutative group G in such a way that the sum of the elements in every block is zero. Moreover, it is shown that symmetric and affine resolvable 2-designs are additive and that, for these designs, and for a suitable choice of G, the blocks are *exactly* the (unordered) k-tuples of elements in $\mathcal{P}$ which sum up to zero, so that the automorphism group of $\mathcal{D}$ coincides with the stabilizer of $\mathcal{P}$ in the automorphism group of G. On the contrary, it is shown that the only additive Steiner triple systems are the point-line designs of $\mathrm{AG}(d, 3)$ and $\mathrm{PG}(d, 2)$ (cf. also [11] and [12]).

With a similar construction to that considered in the present paper, in [18] an additive 2-design is provided, for which no embedding can be found in such a way that the blocks are characterized as the k-sets of elements of $\mathcal{P}$ summing up to zero, thereby settling an open question posed in [9].

Interestingly enough, the search for new additive designs occasionally produces new designs which, in addition to being additive, turn out to be also the first known examples of designs with a certain set of parameters. For instance, in [16] an additive 2-$(81, 6, 2)$ design is constructed, which is also the first known example of a simple 2-design (that is, with no repeated blocks) with these parameters, whereas in [17] an additive Steiner 2-$(124, 4, 1)$ design is presented. More generally, some infinite classes of additive Steiner 2-designs are presented in [5] and [3], in the latter case as a notable application of the method of partial differences.

The goal of this paper is to introduce a class of (additive) block designs that are subdesigns of $\mathcal{D} = (\mathrm{GF}(p^d), \mathcal{B}_k)$ and which seem not to have appeared so far in the literature.

2 Some new designs

In [7] we considered the 2-$(n^2, 2n, 2n-1)$ design obtained by taking the points and the (unordered) pairs of distinct parallel lines of a finite affine plane of order $n > 2$. Similarly, in this paper we consider an incidence structure whose blocks are unions of suitable parallel lines in an affine geometry over $\mathrm{GF}(p)$. We obtain an additive subdesign of the design $\mathcal{D} = (\mathrm{GF}(p^d), \mathcal{B}_k)$ considered here in Section 1, for which we are able to compute the parameters.

Note that one finds, among these designs, the classical point-flat designs $\mathrm{AG}_2(n, 3)$, $n \geq 2$, and $\mathrm{AG}_3(n, 2)$, $n \geq 3$. Interestingly enough, in some special cases the 2-(v, k, λ) designs that we construct have a smaller λ than that of the corresponding point-flat designs of $\mathrm{AG}(d, p)$ with the same parameters v and k.

As usual, we say that m vectors $x_1, x_2, \ldots, x_m$ are *affinely independent* if the $m-1$ vectors $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent.

Theorem 2.1. *Let q be a power of a prime p, and let $\mathcal{P}$ be a n-dimensional vector space over $\mathrm{GF}(q)$. Let m be divisible by p, with $3 \leq m \leq n+1$, and let $\mathcal{B}$ consist of all* *subsets* $\mathfrak{b}(x_1, x_2, \ldots, x_m)$ *of* $\mathcal{P}$ *of the form*

$$\mathfrak{b}(x_1, x_2, \ldots, x_m) = \{x_j + s(x_1 + x_2 + \cdots + x_m) |\, 1 \leq j \leq m \text{ and } s \in \mathrm{GF}(p)\},$$

where $x_1, x_2, \ldots, x_m \in \mathcal{P}$ *are affinely independent vectors over* $\mathrm{GF}(q)$, *and* $\mathrm{GF}(p)$ *is the fundamental subfield of* $\mathrm{GF}(q)$. *Then* $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ *is a* $2 - (q^n, mp, \lambda)$ *additive design, with*

$$\lambda = \begin{cases} \frac{(q^n - q)\cdots(q^n - q^{m-2})}{(m-1)!\, p^{m-2}\, (p-1)}\, (mp - 1) & \text{if } m > 4, \\ \frac{(q^n - q)(q^n - q^2)}{24} & \text{if } m = 4, \\ \frac{q^n - q}{6} & \text{if } m = 3. \end{cases} \tag{2.1}$$

Proof. Suppose $\mathfrak{b}(y_1, y_2, \ldots, y_m) \in \mathcal{B}$. Since the vectors $y_1, y_2, \ldots, y_m \in \mathcal{P}$ are affinely independent, the sum $(y_2 - y_1) + \cdots + (y_m - y_1)$ is not zero and, since m is divisible by p and $my_1 = 0$, we deduce that $y_1 + y_2 + \cdots + y_m$ is not zero, as well. Since the case $m = 2$ is excluded by hypothesis, the sets $\{y_i + s(y_1 + y_2 + \cdots + y_m) |\, s \in \mathrm{GF}(p)\}$ and $\{y_j + s(y_1 + y_2 + \cdots + y_m) |\, s \in \mathrm{GF}(p)\}$ are disjoint, for $i \neq j$, thus $\mathfrak{b}(y_1, y_2, \ldots, y_m)$ contains exactly mp elements (note that in the excluded case where $m = 2$ the two sets are coincident). Because $m \leq n + 1$ and because $G = \mathrm{Aff}(\mathcal{P})$ (the affine group of $\mathcal{P}$ over $\mathrm{GF}(q)$) acts 2-homogeneously on $\mathcal{P}$ and permutes the subsets $\{w_1, w_2, \ldots, w_m\}$ of $\mathcal{P}$ consisting of m affinely independent vectors, the block-set $\mathcal{B}$ may be written as $\mathcal{B} = \mathfrak{b}_0^G$ (the G-orbit of a fixed block $\mathfrak{b}_0 = \mathfrak{b}(x_1, x_2, \ldots, x_m)$), and it follows from [1, Proposition 4.6, page 175] (or from [10, Remark 4.29, page 82]) that $\mathcal{D}$ is a $2 - (v, k, \lambda)$ design with parameters $v = q^n$, $k = mp$ and

$$b = |\mathcal{B}| = \frac{|G|}{|S_{\mathfrak{b}_0}|},$$

where $S_{\mathfrak{b}_0} = \{f \in \mathrm{Aff}(\mathcal{P}) |\, f(\mathfrak{b}_0) = \mathfrak{b}_0\}$ is the setwise stabilizer of the base block $\mathfrak{b}_0$.

Since, for every block $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m)$ of $\mathcal{D}$,

$$\sum_{y \in \mathfrak{b}} y = \begin{cases} \left(p + m\binom{p}{2}\right)(y_1 + y_2 + \cdots + y_m) & \text{for } p > 2 \\ m(y_1 + y_2 + \cdots + y_m) & \text{for } p = 2 \end{cases},$$

which is the zero vector in either case, the design $\mathcal{D}$ is additive by [8, Proposition 2.7, page 277].

In order to determine the number b of blocks of $\mathcal{D}$, we claim that, if $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m)$ is any block of the 2-design $\mathcal{D}$ and if we denote by $R_{\mathfrak{b}}$ the number of (unordered) sets $\{z_1, z_2, \ldots, z_m\} \subset \mathfrak{b}$ consisting of affinely independent vectors $z_1, z_2, \ldots, z_m$ having the property that $\mathfrak{b}(z_1, z_2, \ldots, z_m) = \mathfrak{b}(y_1, y_2, \ldots, y_m)$, then $R_{\mathfrak{b}}$ does not depend on $\mathfrak{b}$ and we have

$$R_{\mathfrak{b}} = \begin{cases} p^{m-1}(p-1), \text{ if } m > 4, \\ 56, \text{ if } m = 4, \\ 72, \text{ if } m = 3. \end{cases}$$

Indeed, if $t_1, t_2, \ldots, t_m \in \mathrm{GF}(p)$ are chosen in such a way that $t_1 + t_2 + \cdots + t_m \neq -1 \in \mathrm{GF}(p)$, then the m (distinct) vectors $z_i = y_i + t_i(y_1 + y_2 + \cdots + y_m)$ of $\mathcal{P}$ (belonging to $\mathfrak{b}$) are affinely independent and have the property that $\mathfrak{b}(z_1, z_2, \ldots, z_m) = \mathfrak{b}$. Hence $R_\mathfrak{b} \geq p^m - p^{m-1} = p^{m-1}(p-1)$.

On the other hand, since

$$l_j = \{y_j + \tau(y_1 + y_2 + \cdots + y_m) | \, \tau \in \mathrm{GF}(q)\} \qquad (j = 1, 2, \ldots, m)$$

are m distinct parallel lines of $\mathcal{P}$ such that $\mathfrak{b} \subseteq l_1 \cup l_2 \cup \cdots \cup l_m$, we infer: if $\mathfrak{b}(w_1, w_2, \ldots, w_m) = \mathfrak{b}$ for suitable affinely independent vectors $w_1, w_2, \ldots, w_m \in \mathfrak{b}$, and if $m > 4$, then the block $\mathfrak{b}$ is strictly contained in the affine subspace over $\mathrm{GF}(p)$ through the m given affinely independent points and defines uniquely the direction $y_1 + y_2 + \cdots + y_m$ of the parallel lines, thus the m-set $\{w_1, w_2, \ldots, w_m\}$ meets each of the m lines l_j ($j = 1, 2, \ldots, m$) in just one point (vector), otherwise some of the y_j would not belong to $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m) = \mathfrak{b}(w_1, w_2, \ldots, w_m)$. Hence there are $c_1, c_2, \ldots, c_m \in \mathrm{GF}(p)$ such that $w_j = y_j + c_j(y_1 + y_2 + \cdots + y_m)$ for $j = 1, 2, \ldots, m$. Therefore we must have $R_\mathfrak{b} \leq p^{m-1}(p-1)$, if $m > 4$. Thus we proved that, if $m > 4$, then $R_\mathfrak{b} \leq p^{m-1}(p-1) \leq R_\mathfrak{b}$, that is, $R_\mathfrak{b} = p^{m-1}(p-1)$.

Suppose now $m = 4$. Thus $p = 2$ and the four lines $y_i + \langle y_1 + y_2 + y_3 + y_4 \rangle$ (with $i = 1, 2, 3, 4$), whose union is $\mathfrak{b}$, fill a whole 3-dimensional space over $\mathrm{GF}(2)$. Then four vectors (points) $z_1, z_2, z_3, z_4 \in \mathfrak{b}$ have the property that $\mathfrak{b}(z_1, z_2, z_3, z_4) = \mathfrak{b}$ if and only if z_1, z_2, z_3, z_4 are non-coplanar points of (the affine space) $\mathfrak{b}$: choosing 3 points out of the 8, and a further point not in the plane through them, we obtain 4 non-coplanar points, in $\binom{4}{3}$ different ways, hence $R_\mathfrak{b} = 4 \times \binom{8}{3} / \binom{4}{3} = 56$, if $m = 4$.

Finally, suppose $m = 3$. Then $p = 3$ and the three lines $y_i + \langle y_1 + y_2 + y_3 \rangle$ (with $i = 1, 2, 3$), whose union is $\mathfrak{b}$, are coplanar, hence $\mathfrak{b}$ is a finite affine plane of order 3. Then three vectors (points) $z_1, z_2, z_3 \in \mathfrak{b}$ are affinely independent (and have the property that $\mathfrak{b}(z_1, z_2, z_3) = \mathfrak{b}$) if and only if z_1, z_2, z_3 are non-collinear points of (the affine plane) $\mathfrak{b}$. Therefore $R_\mathfrak{b} = \binom{9}{3} - 12 = 72$, if $m = 3$, and the claim is proved.

Since $\frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{1\cdot 2\cdot 3\cdots m}$ is the number of all the m-subsets of $\mathcal{P}$ consisting of affinely independent vectors, counting in two ways the number of flags $(W, \mathfrak{b})$, where $W = \{w_1, w_2, \ldots, w_m\}$ is an m-subset of $\mathcal{P}$ consisting of affinely independent vectors and $\mathfrak{b} = \mathfrak{b}(y_1, y_2, \ldots, y_m)$ is a block of $\mathcal{D}$ through W, we obtain by the above argument

$$\begin{cases} \frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{1\cdot 2\cdot 3\cdots m} = p^{m-1}(p-1)b \ , \text{ if } m > 4, \\ \\ \frac{q^n(q^n-1)(q^n-q)(q^n-q^2)}{24} = 56b \ , \text{ if } m = 4, \\ \\ \frac{q^n(q^n-1)(q^n-q)}{6} = 72b \ , \text{ if } m = 3, \end{cases}$$

and this gives the number b of blocks. The parameter λ follows consequently. □

Remark 2.2. It is worth noting that the cases where $m = 3, 4$ are sensibly different from those where $m > 4$.

Let us first point out that, since the $2\!-\!(q^n, mp, \lambda)$ designs $\mathcal{D}$ considered in Theorem 2.1 have $v = q^n$ points, it is natural to ask in what cases such designs arise just as classical point-flat designs $\mathrm{AG}_\mu(n, q)$ of the affine geometries $\mathrm{AG}(n, q)$. It turns out that this is the case only for $\mathrm{AG}_2(n, 3)$, $n \geq 2$, and $\mathrm{AG}_3(n, 2)$, $n \geq 3$. Indeed, the μ-flat through m affinely independent points has $k = q^{m-1}$ points, and this equals $k = mp$ only in the cases where $m = 2$ and $q = 4$ (which is excluded), $m = 3$ and $q = 3$, and $m = 4$ and $q = 2$. The fact that in these two cases the blocks turn out to be affine subspaces has already been pointed out in the above proof.

In all the remaining cases, the designs $\mathcal{D}$ in Theorem 2.1 are not point-flat designs $\mathrm{AG}_\mu(n, q)$. For $q = p^c, m = p^h$, such designs $\mathcal{D}$ are $2-(p^{cn}, p^{h+1}, \lambda)$ designs, hence they have the same parameters v and k as the point-flat designs $\mathrm{AG}_{h+1}(cn, p)$ of the affine geometries $\mathrm{AG}(cn, p)$, thus it is appropriate to compare the value of the parameter λ in (2.1) for $\mathcal{D}$ with the value of λ for $\mathrm{AG}_{h+1}(cn, p)$. As we will now see, for $m = p = 3$, $q = 3^c$ (resp., for $m = 4$, $p = 2$, $q = 2^c$), with $c > 1$, the value of λ in (2.1) is smaller than the corresponding value of λ for the point-plane design $\mathrm{AG}_2(cn, 3)$ (resp., for the point-flat design $\mathrm{AG}_3(cn, 2)$). In either case, the design $\mathcal{D}$ has a $\mathrm{GF}(p)$-structure, but not a $\mathrm{GF}(q)$-structure.

(i) For $m = 3$ and $q = 3^c$, $c > 1$, $\mathcal{D}$ is a $2-\left(3^{cn}, 9, \frac{3^{cn}-3^c}{6}\right)$ design, whereas the point-plane design $\mathrm{AG}_2(cn, 3)$ has a larger $\lambda = \frac{3^{cn}-3}{6}$, whose difference with the parameter λ of $\mathcal{D}$ is $\frac{3^c-3}{6}$, which increases exponentially with c. The smallest example is the case $n = c = 2$: in this case, $\mathcal{D}$ is a $2-(81, 9, 12)$ design, whereas the point-plane design $\mathrm{AG}_2(4, 3)$ is a $2-(81, 9, 13)$ design.

(ii) For $m = 4$ and $q = 2^c$, $c > 1$, $\mathcal{D}$ is a $2-\left(2^{cn}, 8, \frac{(2^{cn}-2^c)(2^{cn}-2^{2c})}{24}\right)$ design, whereas the point-flat design $\mathrm{AG}_3(cn, 2)$ has a larger value of $\lambda = \frac{(2^{cn}-2)(2^{cn}-4)}{24}$.

On the contrary, for $m = p = q > 3$ the parameter λ for $\mathcal{D}$ becomes much larger than that for the point-plane design $\mathrm{AG}_2(n, p)$. For instance, for the smallest case $m = p = q = 5$, $n = 4$, $\mathcal{D}$ is a $2-(625, 25, 372000)$ design, whereas the point-plane design $\mathrm{AG}_2(4, 5)$ is a $2-(625, 25, 31)$ design. And the situation in the cases that do not have a corresponding $\mathrm{AG}_\mu(n, q)$ to be compared with is not different: for $q = 3$, $n = 5$, and $m = 6$, $\mathcal{D}$ is a $2-(243, 18, \lambda)$ design, with $\lambda = 1718496$.

Remark 2.3. *As the affine group* $\mathrm{Aff}(\mathcal{P})$ *has order* $|\mathrm{Aff}(\mathcal{P})| = q^n(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$ *and* $b = \frac{|\mathrm{Aff}(\mathcal{P})|}{|S_{\mathfrak{b}_0}|}$, *we may conclude that the stabilizer* $S_{\mathfrak{b}_0}$ *is a group of order*

$$|S_{\mathfrak{b}_0}| = \begin{cases} (1\cdot 2\cdot 3\cdots m)p^{m-1}(p-1)(q^n-q^{n-1})(q^n-q^{n-2})\cdots(q^n-q^{m-1}), \textit{ if } m > 4,\\ 1344(q^n-q^3)\cdots(q^n-q^{n-1}), \textit{ if } m = 4,\\ 432(q^n-q^{n-1})(q^n-q^{n-2})\cdots(q^n-q^2), \textit{ if } m = 3. \end{cases}.$$

Remark 2.4. The design $\mathcal{D}_k = (\mathrm{GF}(2^n), \mathcal{B}_k)$, considered in [13, Proposition 2.5], is a 3-design for any even k. Similarly, for $p = 2$, the 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ considered in Theorem 2.1 is a 3-design if and only if $q = 2$. Indeed, let $q = 2$, and let $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$ be two 3-subsets of $\mathcal{P}$. Since the group of affinities of $\mathcal{P}$ acts 3-transitively on $\mathcal{P}$, there exists an (invertible) affinity ρ such that $\rho(P_i) = Q_i$, $i = 1, 2, 3$. Moreover, $\rho(\mathfrak{b}(y_1, y_2, \ldots, y_m)) = \mathfrak{b}(\rho(y_1), \rho(y_2), \ldots, \rho(y_m))$ for any subset $\{y_1, y_2, \ldots, y_m\}$ of $\mathcal{P}$ consisting of m affinely independent vectors, hence P_1, P_2, P_3 belong to a block $\mathfrak{b}$ if and only if Q_1, Q_2, Q_3 belong to the block $\rho(\mathfrak{b})$. Therefore $\mathcal{D}$ is a 3-design.

Now let $q \neq 2$. If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ were a 3-design, then the corresponding derived design at the point 0 would be a 2-design. By definition, every block of the latter design is of the form $\mathfrak{b}(x_1 = 0, x_2, \ldots, x_m) \setminus \{0\} = \{x_j + s(x_1 + x_2 + \cdots + x_m) | 1 \leq j \leq m$ and

$s \in \mathrm{GF}(2)\} \setminus \{0\}$, where $x_2, \ldots, x_m$ are linearly independent vectors over $\mathrm{GF}(q)$, hence one can prove that, for any nonzero x in $\mathcal{P}$, and for any scalar c in $\mathrm{GF}(q) \setminus \mathrm{GF}(2)$, the two vectors x and cx cannot lie in a common block. Therefore $\mathcal{D}$ is not a 3-design for $p = 2$ and $q \neq 2$.

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