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Finitizable set of reductions for polyhedral quadrangulations of closed surfaces

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Abstract

In this paper, we discuss generating theorems of polyhedral quadrangulations of closed surfaces. We prove that the set of the eight reductional operations $\{R_1, \dots, R_8\}$ defined for polyhedral quadrangulations is finitizable for any closed surface F^2, that is, there exist finitely many minimal polyhedral quadrangulations of F^2 using such operations $R_1, \dots, R_7$ and R_8. Furthermore, we show that any proper subset of $\{R_1, \dots, R_8\}$ is not finitizable for polyhedral quadrangulations of the torus.

Keywords: Generating theorem, reduction, finitizable set, polyhedral quadrangulation.

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1 Introduction

In this paper, we consider simple connected graphs embedded on closed surfaces. Although we follow the standard graph theory terminology, for some technical terms without description here, refer to Section 2. Sometimes, such an embedded graph is expected to be a "*good*" one, that is, every facial walk is a cycle, and any two of them are disjoint, intersect in one vertex, or intersect in one edge. It is known that a graph G embedded on the sphere satisfies the above good conditions if and only if G is 3-connected. However, if G is embedded on a non-spherical closed surface, then G is required to be *polyhedral*, i.e., 3-connected and 3-representative; note that 3-connected graphs on the sphere are also polyhedral.

For example, a simple graph G cellularly embedded on a closed surface F^2 each of whose face is bounded by a cycle of length 3 is polyhedral if G is not a 3-cycle on the sphere. Such a graph triangulating a closed surface F^2 is known as a *triangulation* of

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F^2. On the other hand, following the convention in topological graph theory, a 4-cycle embedded on the sphere is regarded as a *quadrangulation*, which is a graph cellularly embedded on a closed surface F^2 so that each face is bounded by a cycle of length 4. In this paper, our main subject is the set of polyhedral quadrangulations of closed surfaces.

In topological graph theory, we sometimes discuss *generating theorems* of graphs embedded on closed surfaces (i.e., constructing all graphs in a certain class $\mathcal{C}$ from $\mathcal{C}_0 \subset \mathcal{C}$ by a repeated applications of certain expanding operations only through $\mathcal{C}$). This notion is equivalent to that every graph in $\mathcal{C}$ can be reduced to one in $\mathcal{C}_0$ by a repeated applications of the reductional operations (or *reductions*, simply), which are inverses of the above expanding operations; we denote the set of such reductions by X here. In a generating theorem of graphs, $|X|$ and $|\mathcal{C}_0|$ are expected to be small. In particular, X is called *finitizable* for $\mathcal{C}$ if $|\mathcal{C}_0|$ is finite. If X' is not finitizable for any proper subset $X' \subset X$, then the finitizable set X is *minimal*. For example, if $\mathcal{C}$ is the set of simple triangulations of the sphere, then $X = \{\text{contraction}\}$ is finitizable and $\mathcal{C}_0 = \{\text{tetrahedron}\}$. (See [19]. A *contraction* of e in a triangulation G is to remove e, identify the two ends of e and replace two pairs of multiple edges by two single edges respectively.) In fact, it was proved in [2, 3, 7, 16] that for every closed surface F^2, $\{\text{contraction}\}$ is finitizable for the set of simple triangulations of F^2. Furthermore, see [1, 9, 10, 20, 21] for the complete lists of minimal triangulations on fixed non-spherical closed surfaces with low genera. Moreover, finitizable sets of reductions for even triangulations, i.e., triangulations such that each vertex has even degree, are discussed in literatures; e.g., see [6, 18].

As mentioned above, in this paper, we focus on quadrangulations of closed surfaces. Figure 1 shows the eight reductions, denoted by $R_1, \dots, R_7$ and R_8 simply for our purpose, defined for quadrangulations of closed surfaces. In fact, R_1, R_2 and R_3 are typical ones which were first given by Batagelj [4] (see e.g., [23] for the formal definition); especially, R_1 and R_2 are called a *face-contraction* and a 4-*cycle removal*, respectively, in the literature. Further, the fourth reduction R_4 was defined and discussed in [22]; which is called a *cube-contraction* in the paper. The other four reductions will be defined in the next section.

Let $\mathcal{C}$ be a set of quadrangulations of a closed surface F^2 with some certain conditions, and let $G \in \mathcal{C}$. For a subset $X \subseteq \{R_1, \dots, R_8\}$, G is X-*irreducible* if we cannot apply any reduction in X without violating the condition of $\mathcal{C}$; i.e., the resulting graph is no longer in $\mathcal{C}$. In particular, an $\{R_1\}$-irreducible quadrangulation in the set of simple quadrangulations of a closed surface F^2 is known as just a *irreducible* quadrangulation of F^2. In [16], it was proved that for any closed surface F^2 there exist only finitely many irreducible quadrangulations of F^2, that is, $\{R_1\}$ is finitizable for the set of simple quadrangulations of every closed surface. Actually, the complete lists of irreducible quadrangulations of the sphere, the projective plane, the torus and the Klein bottle were obtained in [4, 5, 14, 17] and [13], respectively; for example, a 4-cycle is the unique irreducible quadrangulation of the sphere, and the unique quadrangular embeddings of K_4 and $K_{3,4}$ are irreducible quadrangulations of the projective plane. (Note that a restricted R_1 was used in [5].)

The situation for 3-connected (and simple) quadrangulations of closed surfaces is a little bit complicated in comparison with the above case of irreducible quadrangulations. Throughout the researches in [4, 5, 12, 15], it had been proved that for any closed surface F^2, $\{R_1, R_2, R_3\}$ is finitizable for 3-connected quadrangulations of F^2; note that the minimal one on the sphere is the cube, and for any non-spherical closed surface F^2, the set of the minimal graphs coincides with the set of irreducible quadrangulations of F^2. Further-

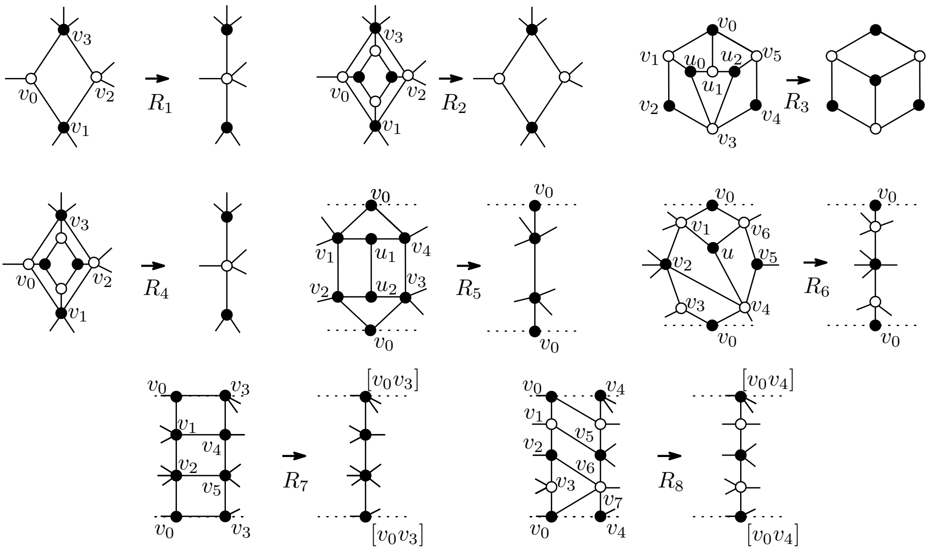

Figure 1: Reductional operations for quadrangulations.

more, it was shown that $\{R_1, R_2, R_3\}$ is minimal for those graphs on the sphere and the projective plane while it is not minimal on the other closed surfaces; in fact, R_3 is unnecessary and hence $\{R_1, R_2\}$ is minimal and finitizable for those closed surfaces. Moreover, it was proved in [22] that $\{R_1, R_3, R_4\}$ is minimal and finitizable for 3-connected quadrangulations of the sphere and the projective plane, and $\{R_1, R_4\}$ is minimal and finitizable for those graphs on the other closed surfaces.

As mentioned above, in this paper, we deal with polyhedral quadrangulations of closed surfaces. Recently in [23], the generating theorem for such polyhedral quadrangulations of the projective plane was discussed using three reductions R_1, R_2 and R_3, and they obtained 26 families of $\{R_1, R_2, R_3\}$-irreducible quadrangulations of the projective plane. However, such families contains infinite series of graphs; i.e., unfortunately, $\{R_1, R_2, R_3\}$ is not finitizable for those graphs. The following is our main result in the paper:

Theorem 1.1. *For every closed surface F^2, $\{R_1, \ldots, R_8\}$ is finitizable for polyhedral quadrangulations of F^2.*

Since every reduction in the above theorem preserves bipartiteness of quadrangulations and each of R_5 and R_7 requires an essential cycle of length 3, we obtain the following corollary.

Corollary 1.2. *For every closed surface F^2, $\{R_1, R_2, R_3, R_4, R_6, R_8\}$ is finitizable for bipartite polyhedral quadrangulations of F^2.*

One might think that the eight reductions in Theorem 1.1 are a little bit too many. However, at least those on the torus, we can show the necessity of such eight reductions as follows.

Theorem 1.3. *For polyhedral quadrangulations of the torus, $\{R_1, \ldots, R_8\}$ is minimal finitizable.*

Furthermore, R_7 (resp., R_8) requires an annular region on the closed surface which is bounded by two 2-sided 3-cycles (resp., 4-cycles). Therefore, in particular on the projective plane, $\{R_1, \ldots, R_6\}$ is finitizable by Theorem 1.1. As well as the previous case on the torus, we can show the following.

Theorem 1.4. *For polyhedral quadrangulations of the projective plane,* $\{R_1, \ldots, R_6\}$ *is minimal finitizable.*

This paper is organized as follows. In the next section, we define terminology and the remaining four new reductions for our argument in the paper. Next, we show some propositions and lemmas holding for polyhedral quadrangulations for our purpose, some of which are quoted from [23]. Section 4 is devoted to prove our main result in the paper. In Section 5, we discuss the minimality of the set of eight reductions by showing some infinite series of polyhedral quadrangulations.

2 Basic definitions

We denote the vertex set and the edge set of a graph G embedded on a closed surface F^2 by $V(G)$ and $E(G)$, respectively. A *k-path* (resp., *k-cycle*) in a graph G is a path (resp., cycle) of length k. (The *length* of a path (or cycle) is the number of its edges in this paper.) We denote the set of vertices of degree 3 by V_3 in our argument, and $\langle V_3 \rangle_G$ represents the subgraph induced by V_3 in G.

Let G be a graph embedded on a closed surface F^2. Then, a connected component of $F^2 - G$ is a *face* of G, and we denote the face set of G by $F(G)$. If every face of G is homeomorphic to an open 2-cell (or an open disc), then, G is a 2-*cell embedding* or 2-*cell embedded graph* on F^2. Clearly, every quadrangulation (or triangulation) of a closed surface is a 2-cell embedded graph. A *facial cycle* C of a face f is a cycle bounding f in G; i.e., $C = \partial f$. Then, $\bar{f}$ denotes a closure of f, i.e., $\bar{f} = f \cup \partial f$. For brevity, we sometimes denote like $f = v_0v_1v_2v_3$ where $v_0v_1v_2v_3$ is a facial cycle of $f \in F(G)$. Furthermore in our argument, we often discuss the interior of a 2-cell region D bounded by a closed walk W of G, i.e., $W = \partial D$, which contains some vertices and edges. (Note that a 2-cell region implies an "open" 2-cell region in this paper.) Similarly, $\bar{D}$ denotes a closure of D, i.e., $\bar{D} = D \cup \partial D$. Let $f_1, \ldots, f_k$ denote the faces of G incident to $v \in V(G)$ where $\deg(v) = k$. Then, the boundary walk of $\bar{f}_1 \cup \cdots \cup \bar{f}_k$ is the *link walk* of v and denoted by $lw(v)$. Clearly, $lw(v)$ bounds a 2-cell region containing a unique vertex v.

A simple closed curve γ on a closed surface F^2 is *trivial* if γ bounds a 2-cell region on F^2, and *essential* otherwise. Among essential simple closed curves, one with an annular neighborhood is called 2-*sided* while one whose tubular neighborhood forms a Möbius band is called 1-*sided*. Since cycles in graphs embedded on surfaces can be regarded as simple closed curves, we use the above terminology for them; e.g., we say that a cycle is essential and 2-sided.

The *representativity* of G, denoted by $r(G)$, is the minimum number of intersecting points of G and γ, where γ ranges over all essential simple closed curves on the surface. A graph G embedded on F^2 is *r-representative* if $r(G) \geq r$. Note that the "representativity" is also called the "*face-width*" in the literature; see e.g., [11] for the details. A graph G embedded on a non-spherical closed surface F^2 is *polyhedral* if G is 3-connected and 3-representative. Observe that for every vertex v of a polyhedral graph, the link walk of v forms a cycle.

Let G be a quadrangulation of a closed surface F^2 and let $f = v_0v_1v_2v_3$ be a face of G. Then a pair $\{v_i, v_{i+2}\}$ is called a *diagonal pair* of f in G for each $i \in \{0, 1\}$. A closed curve γ on F^2 is a *diagonal k-curve* for G if γ passes only through distinct k faces $f_0, \dots, f_{k-1}$ and distinct k vertices $x_0, \dots, x_{k-1}$ of G such that for each i, f_i and f_{i+1} share x_i, and that for each i, $\{x_{i-1}, x_i\}$ forms a diagonal pair of f_i of G, where the subscripts are taken modulo k. Furthermore, we call a simple closed curve γ on F^2 a *semi-diagonal k-curve* if in the above definition $\{x_{i-1}, x_i\}$ is not a diagonal pair for exactly one i; note that $x_{i-1}x_i$ is an edge of ∂f_i in this case. Each simple curve β_i along γ joining x_{i-1} and x_i in f_i is called a *γ-segment*; where $\bigcup_{i=0}^{k-1} \beta_i = \gamma$.

For a simple closed curve ℓ on F^2, when ℓ intersects with G at only vertices of G, that is, $G \cap \ell$ is a subset $S \subset V(G)$, then we say that ℓ *passes* S; observe that ℓ does not pass through any vertex in $V(G) \setminus S$ in this case. For example, in the above definition of a diagonal (or semi-diagonal) k-curve, we say that γ passes $\{x_0, \dots, x_{k-1}\}$. On the other hand, when we say that ℓ *passes through* a vertex v (or some vertices) of G, then ℓ probably passes through other vertices of G.

Let G be a simple quadrangulation of a non-spherical closed surface F^2. Assume that G has a hexagonal 2-cell region D bounded by a closed walk $\partial D = v_0v_1v_2v_0v_3v_4$ containing exactly two vertices u_1 and u_2 such that $v_0v_1u_1v_4, v_1v_2u_2u_1, v_3v_4u_1u_2$ and $v_2v_0v_3u_2$ are faces of G in D, and that $v_0v_1v_2$ is an essential cycle of length 3. Furthermore, we assume that v_0, v_1, v_2, v_3 and v_4 are different vertices, and that each of v_1, v_2, v_3 and v_4 has degree at least 4 ; otherwise, G would not be polyhedral under the condition. A reduction R_5 of D is to eliminate u_1 and u_2, and identify v_1 (resp., v_2) and v_4 (resp., v_3), and replace three pairs of multiple edges by three single edges, respectively, as shown in Figure 1. Throughout the paper, the vertex obtained by the identification of two vertices a and b is denoted by $[ab]$. That is, $v_0[v_1v_4][v_2v_3]$ is an essential 3-cycle in the resulting graph.

Secondly, assume that G has an octagonal 2-cell region D bounded by a closed walk $W = v_0v_1v_2v_3v_0v_4v_5v_6$ containing exactly one vertex u such that $v_0v_1uv_6, v_1v_2v_4u$, $v_4v_5v_6u$ and $v_2v_3v_0v_4$ are faces of G in D, and that $v_0v_1v_2v_3$ is an essential cycle of length 4. Furthermore, we assume that $v_0, v_1, v_2, v_3, v_4, v_5$ and v_6 are different vertices. Note that v_1 and v_4 has degree at least 4 under the condition. (If $\deg(v_1) = 3$, then G is representativity at most 2. On the other hand, $\deg(v_4) = 3$ implies that $v_0 = v_5$, a contradiction.) A reduction R_6 of D is to eliminate u and an edge v_2v_4, and identify v_1 (resp., v_2, v_3) and v_6 (resp., v_5, v_4), and replace four pairs of multiple edges by four single edges, respectively, as shown in Figure 1. Then, $v_0[v_1v_6][v_2v_5][v_3v_4]$ is an essential 4-cycle in the resulting graph.

Thirdly, assume that G has an annular region A bounded by two essential cycles $C = v_0v_1v_2$ and $C' = v_3v_4v_5$ such that $f_1 = v_0v_1v_4v_3, f_2 = v_1v_2v_5v_4$ and $f_3 = v_2v_0v_3v_5$ are faces of G in A. (Sometimes, $f_1f_2f_3 (= W_F)$ is called a *face walk* of length 3 in G, which corresponds to a 3-cycle in the dual of G.) Here, note that C_1 and C_2 are essential 2-sided cycles of G on F^2; if C_1 is trivial, then it contradicts Proposition 3.2 in the next section. The seventh reduction R_7 of A (or the above face walk W_F) is to contract edges v_0v_3, v_1v_4 and v_2v_5 simultaneously, and replace three pairs of multiple edges by three single edges, respectively, as shown in Figure 1. Note that $C = [v_0v_3][v_1v_4][v_2v_5]$ is also an essential 2-sided 3-cycle in the resulting graph.

Fourthly, assume that G has an annular region A bounded by two essential cycles $C_1 = v_0v_1v_2v_3$ and $C_2 = v_4v_5v_6v_7$ such that $f_1 = v_0v_1v_6v_5, f_2 = v_1v_2v_7v_6, f_3 = v_2v_3v_0v_7$ and $f_4 = v_0v_5v_4v_7$ are faces of G in A. (As well as the previous reduction,

$f_1f_2f_3f_4(= W_F)$ is a face walk of length 4.) Furthermore, we assume that C_1 and C_2 are essential cycles of G on F^2; observe that they are 2-sided. The eighth reduction R_8 of A (or the face walk W_F) is to eliminate edges v_0v_5, v_1v_6, v_2v_7 and v_0v_7, and identify v_i and v_{i+4} for each $i \in \{0, 1, 2, 3\}$, and replace four pairs of multiple edges by four single edges, respectively, as shown in Figure 1. Note that $C = [v_0v_4][v_1v_5][v_2v_6][v_3v_7]$ is also an essential 2-sided 4-cycle in the resulting graph.

As mentioned in the introduction, for R_1, R_2, R_3 and R_4, see e.g., [22, 23] for formal definitions. Note that the boundary of the hexagon of the graph in R_3 in the figure is a cycle. Furthermore, every quadrangulation of a closed surface is locally bipartite, and hence we color vertices of graphs in R_1, R_2, R_3, R_4, R_6 and R_8 by black and white; however, graphs in the reductions R_5 and R_7 contain short odd cycles, and hence we cannot do so.

3 Lemmas

First of all, we introduce the following two propositions for quadrangulations of closed surfaces; these are well-known in topological graph theory, and hence we omit the proofs.

Proposition 3.1. *The length of two essential cycles in a quadrangulation of a closed surface have the same parity if they are homotopic to each other on F^2.*

Proposition 3.2. *A quadrangulation of a closed surface has no separating odd cycle.*

It was shown in [23] that many facts hold for $\{R_1, R_2, R_3\}$-irreducible polyhedral quadrangulations of non-spherical closed surfaces. First, we show some of them, which will be used in our later argument in the paper. In the following lemmas, G represents a $\{R_1, R_2, R_3\}$-irreducible polyhedral quadrangulations of a non-spherical closed surface F^2 otherwise specified. (The assertions are a little bit changed so as to suit for this paper.)

Lemma 3.3 (Lemmas 3.5, 3.13 and 3.15 in [23])**.** *Every connected component of $\langle V_3 \rangle_G$ is a 4-cycle bounding a face of G or a path of length at most 2.*

Lemma 3.4 (Lemmas 3.8, 3.10 and 3.12 in [23])**.** *Let $f = v_0v_1v_2v_3$ be a face of G with $\deg(v_0), \deg(v_2) \geq 4$. Then, there exists*

(i) *an essential 4-cycle $v_0v_1xv_3$ for $x \notin \{v_0, v_1, v_2, v_3\}$,*

(ii) *an essential diagonal 3-curve passing through v_1 and v_3, or*

(iii) *an essential semi-diagonal 3-curve passing through v_1 and v_3.*

Lemma 3.5. *Let $f = v_0v_1v_2v_3$ be a face of G with $\deg(v_0), \deg(v_2) \geq 4$. Then, there exists an essential cycle passing through v_0, v_1 and v_3 with length 4, 5 or 6.*

Proof. It is clear by Lemma 3.4. (For example, if (ii) in the previous lemma holds, then there exists an essential cycle of length 6 along the essential diagonal 3-curve.) □

Lemma 3.6 (Lemma 3.14 in [23])**.** *Let $P = u_0u_1u_2$ be a 2-path in $\langle V_3 \rangle_G$ as shown in the left-hand side of R_3 in* Figure 1 *where $\deg(v_4) \geq 4$. Then, there is an essential diagonal 3-curve or an essential semi-diagonal 3-curve passing $\{v_1, u_1, v_5\}$.*

Assume that G has a 4-cycle $C = u_0u_1u_2u_3$ in $\langle V_3\rangle_G$ bounding a face of G such that u_i is adjacent to a third vertex $v_i \notin \{u_0, u_1, u_2, u_3\}$ for each $i \in \{0, 1, 2, 3\}$. Under the situation, a 4-cycle $v_0v_1v_2v_3$ bounds a 2-cell region which contains exactly four vertices u_0, u_1, u_2 and u_3. We call the subgraph H isomorphic to a cube with eight vertices u_i, v_i for $i \in \{0, 1, 2, 3\}$ an *attached cube*. We denote $\partial(H) = v_0v_1v_2v_3$, and we call C an *attached* 4*-cycle* of H.

Lemma 3.7 (Lemma 3.16 in [23]). *Assume that G has an attached cube H with $\partial(H) = v_0v_1v_2v_3$, an attached 4-cycle $C = u_0u_1u_2u_3$ and $u_iv_i \in E(G)$ for each $i \in \{0, 1, 2, 3\}$. Then there is an essential diagonal (or semi-diagonal) 3-curve γ passing $\{v_0, u_1, v_2\}$ or $\{v_1, u_2, v_3\}$.*

Next, we show three lemmas holding for $\{R_1, R_2, R_3, R_4\}$-irreducible polyhedral quadrangulations of non-spherical closed surfaces.

Lemma 3.8. *Let G be an $\{R_1, R_2, R_3, R_4\}$-irreducible polyhedral quadrangulation of a non-spherical closed surface F^2 having an attached cube H with $\partial(H) = v_0v_1v_2v_3$, an attached 4-cycle $C = u_0u_1u_2u_3$ and $u_iv_i \in E(G)$ for each $i \in \{0, 1, 2, 3\}$. By* Lemma 3.7*, we may assume that there exists an essential simple closed curve γ_1 passing $\{v_0, u_1, v_2\}$. Then, there exists an essential simple closed curve γ_2 passing either $\{v_1, u_2, v_3\}$ or $\{v_1, u_2, v_3, x\}$ where $x \notin V(H)$. In particular, if γ_1 is 2-sided, then γ_2 is not homotopic to γ_1.*

Proof. Let G' denote the quadrangulation obtained from G by applying an R_4 of H so as to identify v_1 and v_3. We denote the 2-path $v_0[v_1v_3]v_2$ in G' by P. By our assumption, G' is not polyhedral. If G' has a loop e, then e is incident to $[v_1v_3]$ such that e and P cross transversally at $[v_1v_3]$; otherwise, G would have a loop, a contradiction. Further, this e is essential by Proposition 3.2. Thus in this case, we find an essential semi-diagonal 3-curve γ_2 passing $\{v_1, u_2, v_3\}$ in G, half of which is along e.

Secondly, we suppose that G' has a pair of multiple edges. Similar to the previous case, we may assume that such multiple edges join $[v_1v_3]$ and another vertex $x \notin \{v_0, v_2\}$; otherwise, G would have multiple edges. Then, the 2-cycle $C = [v_1v_3]x$ formed by the above multiple edges crosses P transversally, similar to the previous case. Thus, C cannot be trivial by the above observation and the existence of γ_1, and hence we have our desired simple closed curve γ_2 passing $\{v_1, u_2, v_3, x\}$ in G; note that if v_1xv_3 forms a corner of a face of G, then we can take an essential diagonal 3-curve passing $\{v_1, u_2, v_3\}$. In the following argument, we assume that G' is simple and hence G' is 2-connected and 2-representative.

By the above argument, we may assume that G' has a diagonal (or semi-diagonal) 2-curve γ' passing $\{[v_1v_3], x\}$ such that γ' and P cross at $[v_1v_3]$ transversally; note that if G' has a 2-cut, then G' also has a surface separating diagonal 2-curve by Lemma 3.6 in [23]. Observe that at least one of two γ'-segments β_0 and β_1, say β_0 without loss of generality, joins the diagonal pair of $f_0 = [v_1v_3]sxt$ for $s, t \in V(G')$. Here, suppose that x is either v_0 or v_2, say v_0. Then, let $\tilde{\beta}_0$ denote a simple closed curve obtained from β_0 by joining $[v_1v_3]$ and v_0 by a simple curve along the edge $[v_1v_3]v_0$. In this case, $\tilde{\beta}_0$ must be essential by Proposition 3.2. Under the situation, we can take an essential simple closed curve intersecting with G at exactly two vertices v_0 and either v_1 or v_3, which corresponds to $\tilde{\beta}_0$, a contradiction. Thus, we conclude that x is neither v_0 nor v_2.

Observe that even when γ_1 is an essential diagonal 3-curve passing through a face $f = v_0pv_2q$ for $p, q \in V(G)$, we have $\{v_0, v_2\} \cap \{p, q\} = \emptyset$ since G is simple. This implies that the γ_1-segment in f and γ' cannot cross transversally, and hence we conclude that γ' is essential. Therefore, we have an essential diagonal (or semi-diagonal) 4-curve γ_2 passing $\{v_1, u_2, v_3, x\}$ in the statement, half of which is along γ', and the other half is inside the quadrangular region bounded by $\partial(H)$.

Finally, assume that γ_1 is 2-sided. Suppose, for a contradiction, that γ_2 is homotopic to γ_1. Under the condition, γ_2 must cross γ_1 even times, i.e., twice here. However, this is not the case by the above argument. □

Lemma 3.9. *Let G be an $\{R_1, R_2, R_3, R_4\}$-irreducible polyhedral quadrangulation of non-spherical closed surface. Then any 2-cell region bounded by a 4-cycle is either a face of G or contains exactly four vertices which is of an attached cube.*

Proof. Using the above Lemma 3.8 and Lemma 4.3 in [23], we immediately have the conclusion of the lemma. □

Furthermore in [23], Suzuki determined configurations in a 2-cell region bounded by a 6-cycle in $\{R_1, R_2, R_3\}$-irreducible polyhedral quadrangulations of non-spherical closed surfaces. By combining the results of Lemmas 3.7, 3.8 and 3.9, we can easily obtain the following lemma; so, we omit the proof.

Lemma 3.10. *Let G be an $\{R_1, R_2, R_3, R_4\}$-irreducible polyhedral quadrangulation of a non-spherical closed surface F^2. Then the number of vertices inside a 2-cell region bounded by a 6-cycle (resp., 4-cycle) is at most 16 (resp., 4).*

In the latter half of the section, we discuss reductions R_5, R_6, R_7 and R_8 applied to polyhedral quadrangulations in turn.

Lemma 3.11. *Let G be a polyhedral quadrangulation of a closed surface F^2 having a 2-cell region D with $\partial D = v_0v_1v_2v_0v_3v_4$ containing two vertices u_1 and u_2 as shown in the left-hand side of R_5 in* Figure 1*, and let G' denote a quadrangulation obtained from G by an R_5 of D. If G' is not polyhedral, then there exists an essential simple closed curve γ' such that*

(i) *γ' intersects exactly two vertices of G',*

(ii) *γ' passes through at least one vertex of $[v_1v_4]$ and $[v_2v_3]$, and*

(iii) *γ' does not pass through v_0.*

In particular, if $C = v_0[v_1v_4][v_2v_3]$ is 2-sided, then γ' is not homotopic to C.

Proof. Some similar arguments as in Lemma 3.8 will appear, and we omit the long explanation at that time for brevity. If G' has a loop e with a vertex u, then u must be one of $[v_1v_4]$ and $[v_2v_3]$, say $[v_1v_4]$ up to symmetry, such that e and $C = v_0[v_1v_4]v_2v_3$ cross transversally at $[v_1v_4]$. Clearly e is essential, and we can take an essential simple closed curve intersecting G at only v_1 and v_4, a contradiction.

Next, assume that G' has a pair of multiple edges, which joins $[v_1v_4]$ and another vertex $x \neq v_0$. If the 2-cycle $C' = [v_1v_4]x$ formed by the multiple edges is essential, then we can take our desired simple closed curve along C'. Thus, we suppose that C' is trivial below.

If $x \notin V(C)$, then G would have multiple edges joining x and either v_1 or v_4; observe that C and C' do not cross transversally, otherwise $x \in V(C)$ since C' is trivial. If $x \in V(C)$, then x must be $[v_2v_3]$. Also in this case, G would have multiple edges joining either v_1 and v_2 or v_3 and v_4, a contradiction. Therefore, we assume that G' is 2-connected and 2-representative below.

Now, G' has a diagonal (or semi-diagonal) 2-curve γ' passing $\{[v_1v_4], x\}$ such that γ' and C cross at $[v_1v_4]$ transversally. We consider the γ'-segment β_0 and $\tilde{\beta}_0$ which play the same role as in the argument in Lemma 3.8. If $x = v_0$, then $\tilde{\beta}_0$ is essential by Proposition 3.2, and hence G is not polyhedral as well, a contradiction. If γ' is trivial, then x must be $[v_2v_3]$ since $x \neq v_0$. However, this contradicts Proposition 3.2 for $\tilde{\beta}_0$. Therefore, γ' is essential and satisfying the conditions in the statement. Similar to the argument in Lemma 3.8, if C is 2-sided, then C and γ' are not homotopic. □

Lemma 3.12. *Let G be a polyhedral quadrangulation of a closed surface F^2 having a 2-cell region D with $\partial D = v_0v_1v_2v_3v_0v_4v_5v_6$ containing a unique vertex u as shown in the left-hand side of R_6 in* Figure 1*, and let G' denote a quadrangulation obtained from G by an R_6 of D. If G' is not polyhedral, then there exists an essential simple closed curve γ' such that*

(i) *γ' intersects at most two vertices of G',*

(ii) *γ' passes through at least one vertex of $[v_1v_6], [v_2v_5]$ and $[v_3v_4]$, and*

(iii) *γ' does not pass through v_0.*

In particular, if $C = v_0[v_1v_6][v_2v_5][v_3v_4]$ is 2-sided, then γ' is not homotopic to C.

Proof. The most part is same as the argument in Lemma 3.11, and hence we implicitly omit the argument which had already done before. First, observe that there does not exist a face $f \notin D$ such that $v_0, v_2 \in \partial f$; otherwise, we can find a simple closed curve intersecting with G at exactly two vertices, which passes through the face $v_2v_3v_0v_4$ and f. Similarly, there is no face $f \notin D$ of G such that $v_4, v_6 \in \partial f$. Further, in the case when G' is not simple, a loop of a vertex $[v_2v_5]$ might exist, unlike the argument in Lemma 3.11, and then, it is essential by Proposition 3.2.

Thus, we assume that G' has a diagonal (or semi-diagonal) 2-curve γ' passing $\{x, y\}$, and we may assume that y is one of $[v_1v_6], [v_2v_5]$ and $[v_3v_4]$ such that γ' and $C = v_0[v_1v_6][v_2v_5][v_3v_4]$ cross at y transversally. If $x = v_0$, then y must be $[v_2v_5]$ by the same argument as in the previous lemma; recall the argument of $\tilde{\beta}_0$. However, under the condition, G would have a face $f \notin D$ such that $v_0, v_2 \in \partial f$, which is passed by a γ'-segment, a contradiction. Thus, γ' does not pass through v_0 in the following argument. If γ' is trivial, then $\{x, y\} = \{[v_1v_6], [v_3v_4]\}$, and γ' crosses C exactly twice by the former argument. Similarly, there exists a face $f \notin D$ such that $v_4, v_6 \in \partial f$ and f is passed by a γ'-segment, a contradiction. Therefore, γ' is essential. Further, it is not difficult to see that γ' is not homotopic to C when C is 2-sided. □

Lemma 3.13. *Let G be a polyhedral quadrangulation of a closed surface F^2 having an annular region A formed by three faces $v_0v_1v_4v_3, v_1v_2v_5v_4$ and $v_2v_0v_3v_5$ as shown in the left-hand side of R_7 in* Figure 1*, and let G' be a quadrangulation obtained from G by an R_7 of A. If G' is not polyhedral, then there exists an essential simple closed curve γ' such that*

(i) *γ' intersects exactly two vertices of G',*

(ii) *γ' passes through exactly one vertex of $[v_0v_3]$, $[v_1v_4]$ and $[v_2v_5]$, and*

(iii) *$C = [v_0v_3][v_1v_4][v_2v_5]$ and γ' are not homotopic.*

Proof. Almost the same argument as in the proofs of Lemmas 3.11 and 3.12 holds, and hence we omit the proof. (This is easier than those proofs.) Since any two homotopic 2-sided simple closed curves on a closed surface cross even times, (iii) immediately holds from (ii). □

Lemma 3.14. *Let G be a polyhedral quadrangulation of a closed surface F^2 having an annular region A formed by four faces $v_0v_1v_6v_5, v_1v_2v_7v_6, v_2v_3v_0v_7$ and $v_0v_5v_4v_7$ as shown in the left-hand side of R_8 in* Figure 1*, and let G' be a quadrangulation obtained from G by an R_8 of A. If G' is not polyhedral, then there exists an essential simple closed curve γ' such that*

(i) *γ' intersects exactly two vertices of G',*

(ii) *γ' passes through at least one vertex of $[v_0v_4]$, $[v_1v_5]$, $[v_2v_6]$ and $[v_3v_7]$, and*

(iii) *$C = [v_0v_4][v_1v_5][v_2v_6][v_3v_7]$ and γ' are not homotopic.*

Proof. Note that there does not exist a face $f \notin A$ (resp., $f' \notin A$) such that $v_0, v_2 \in \partial f$ (resp., $v_5, v_7 \in \partial f'$), similar to the argument in the proof of Lemma 3.12. Furthermore, for example, there might be an edge v_2v_5 in G such that 2-cycle $C' = [v_1v_5][v_2v_6]$ formed by a pair of multiple edges is essential in G'; this is different from the previous lemma. The argument is almost same, and hence we omit it as well. □

4 Main result

First, we refer to the following lemma, which plays an important role in the proof of our main result.

Lemma 4.1 (Juvan, Malnič and Mohar [8])**.** *For any closed surface F^2 and any non-negative integer k, there exists a constant $f(k, F^2)$ such that if $\mathcal{L}$ is a set of pairwise non-homotopic simple closed curves on F^2 such that any two elements of $\mathcal{L}$ cross at most k times, then $|\mathcal{L}| \leq f(k, F^2)$.*

In the next lemmas, we show that there is an upper bound of the maximum degree (resp., the diameter) of $\{R_1, \ldots, R_6\}$-irreducible (resp., $\{R_1, \ldots, R_8\}$-irreducible) polyhedral quadrangulations of a non-spherical closed surface F^2.

Lemma 4.2. *Let G be an $\{R_1, \ldots, R_6\}$-irreducible polyhedral quadrangulation of a non-spherical closed surface F^2. Then the maximum degree of G is bounded by a constant depending only on F^2.*

Proof. We prove that $\Delta(G) \leq 640f(5, F^2) + 79$, where $f(\cdot, F^2)$ is the function in Lemma 4.1. Suppose, for a contradiction, that G has a vertex v with $\deg(v) \geq$

$640f(5, F^2) + 80$. Let L_v be the link walk of v in G. Give a direction to L_v and denote the directed cycle by $\overrightarrow{L}_v$. Let

$$a_1^1, \ldots, a_{16}^1, b_1^1, \ldots, b_7^1, c_1^1, \ldots, c_{17}^1, a_1^2, \ldots, a_{16}^2, b_1^2, \ldots, b_7^2, c_1^2, \ldots, c_{17}^2, \quad \ldots, \quad a_1^l, \ldots, a_{16}^l, b_1^l, \ldots, b_7^l, c_1^l, \ldots, c_{17}^l$$

be $40l$ consecutive vertices of L_v taken along $\overrightarrow{L}_v$, where $l \geq 16f(5, F^2)+2$. Then, we may assume that $vb_1^1 b_2^1 b_3^1$ is a face of G; note that $vb_1^i b_2^i b_3^i$ is also a face for each $i \in \{2, \ldots, l\}$ under the assumption. Let $P(a, b)$ denote the path in L_v starting at $a \in V(L_v)$ and ending at $b \in V(L_v)$ along $\overrightarrow{L}_v$.

In the former half of the proof, we show the following fact: For each $i \in \{1, \ldots, l\}$, there exists either (A) a cycle of length at most 6 containing a path $b_s^i v b_t^i$ $(1 \leq s < t \leq 6)$, or (B) a cycle of length at most 4 containing a path $b_s^i vu$ $(1 \leq s \leq 6)$ where $u \in V(L_v)$. We call the cycle having the above property (A) (resp., (B)) a *type-A cycle* (resp., *type-B cycle*). Note that there might be a cycle having both properties (A) and (B); in that case, we can classify it into either.

In the following argument, we discuss several cases around vertices $b_1^i, \ldots, b_6^i$ and b_7^i. To simplify notation, we put $b_j^i = b_j$ for each $j \in \{1, \ldots, 7\}$ by omitting the upper subscript "i". First of all, assume that $\deg(b_2) \geq 4$. In this case, we apply an R_1 of $vb_1b_2b_3$ at $\{b_1, b_3\}$, i.e., identifying b_1 and b_3. By Lemma 3.4, we can easily find our desired cycle containing a path b_1vb_3; take such a path using edges of faces passed by the diagonal 3-curve or the semi-diagonal 3-curve. The same fact holds for b_4 and b_6, and hence we assume that $\deg(b_h) = 3$ for each $h \in \{2, 4, 6\}$ below.

Next, assume $\deg(b_3) = 3$. Then, there exist faces $b_1b_2xy, b_2b_3b_4x$ and b_4b_5zx for $x, y, z \in V(G)$. If $\deg(x) \geq 4$, then we can find our desired cycle containing a path b_1vb_5 by Lemma 3.6 as a type-A cycle. On the other hand, if $\deg(x) = 3$, i.e., $y = z$ in this case, then $b_2b_3b_4x$ is an attached 4-cycle. In this case, there exists either a type-A cycle or a type B cycle, both of which contain vb_1, by Lemma 3.7. Thus, we assume that $\deg(b_3) \geq 4$ and $\deg(b_5) \geq 4$ in the following argument.

For the face $vb_3b_4b_5$, there is

(i) an essential 4-cycle vb_3b_4x for $x \notin \{v, b_3, b_4, b_5\}$,

(ii) an essential diagonal 3-curve γ passing through v and b_4, or

(iii) an essential semi-diagonal 3-curve γ passing through v and b_4, by Lemma 3.4.

First, we discuss (i). In this case, x is a vetex of L_v such that $xv \in E(G)$, and hence there exists our desired type-B cycle. Secondly, assume (ii), and let $f_1 = vb_3b_4b_5, f_2 = b_4pqr$ and $f_3 = vsqt$ be faces passed by γ where $q, s, t \in V(L_v)$ (see the left-hand side of Figure 2). Since $\deg(v_4) = 3$, we have $|\{b_3, b_5\} \cap \{p, r\}| = 1$. Without loss of generality, we may assume that $p = b_3$, and we find our desired type-B 4-cycle vb_3qs.

Thirdly, we discuss (iii). We further divide this case into the following two subcases:

(1) γ passes through $f_1 = vb_3b_4b_5, f_2 = b_4pqr$ and $f_3 = vqst$ where $q, s, t \in V(L_v)$, and

(2) γ passes through $f_1 = vb_3b_4b_5, f_2 = b_4pqr$ and $f_3 = vsrt$ where $s, r, t \in V(L_v)$.

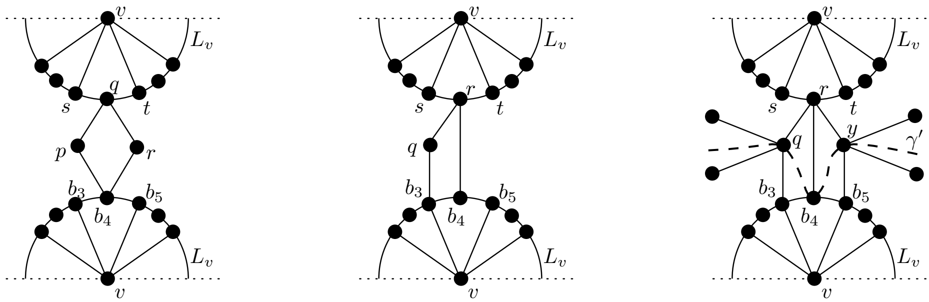

Figure 2: Configurations around L_v.

First, assume the former case (iii)(1). Similar to the above argument, we have $|\{b_3, b_5\} \cap \{p, r\}| = 1$ since $\deg(v_4) = 3$, and we may assume that $p = b_3$ here. In this case, we find a type-B cycle vb_3q of length 3.

Next, suppose the latter case (iii)(2). Similarly, we have $\deg(v_4) = 3$, and hence we may assume that $p = b_3$ (see the center of Figure 2). Furthermore, if $\deg(r) = 3$, then q must be either s or t, and hence we find our desired type-B cycle vb_3q of length 3. Thus, we assume $\deg(r) \geq 4$ in the following argument. By applying Lemma 3.4 to $f_2 = b_3b_4rq$ since $\deg(b_3) \geq 4$ and $\deg(r) \geq 4$, we find either a 2-path P joining q and b_4 such that the cycle b_4b_3qP is essential, or an essential simple closed curve γ' passing $\{q, b_4, x\}$ for $x \in V(G)$. If the former holds, then $P = qb_5b_4$ since $\deg(b_4) = 3$. In this case, there exists our desired type-A cycle vb_3qb_5 of length 4. Next, we assume the latter, and suppose that γ' is an essential diagonal 3-curve. If γ' passes through rb_4b_5y for $y \in V(G)$, then there exists a 2-path P' joining y and q along γ' (see the right-hand side of Figure 2). That is, there exists a type-A cycle $vb_3qP'yb_5$ of length 6. If γ' passes through $b_3b_4b_5v$, then $q \in V(L_v)$ and γ' passes $\{v, b_4, q\}$. In this case, there exists a type-B cycle vb_3qq' of length 4 where $qq' \in E(L_v)$. When γ' is an essential semi-diagonal 3-curve, similar argument holds, and we have either a type-A cycle of length 5 or a type-B cycle of length 3.

In the latter half of the proof, we lead to a contradiction. For our purpose, let C_A^l denote a type-A cycle containing $b_s^l v b_t^l$ where $1 \leq s < t \leq 6$, and let $C_B^{i,j}$ denote a type-B cycle containing a 2-path $b_s^i vu$ where $1 \leq s \leq 6$ such that $u \in \{a_1^j, \dots, a_{16}^j, b_1^j, \dots, b_7^j, c_1^j, \dots, c_{17}^j\}$; i.e., $C_B^{i,j}$ was obtained by the argument above when discussing vertices $b_1^i, \dots, b_7^i$. (Note that $C_B^{i,i}$ might exist for some i.) Then, any two type-A cycles cross at most 5 times, since they cannot cross at a vertex v. Clearly, the number of crossing points of a type-B cycle and another type-A or type-B cycle is at most 4.

First, assume that there exist at least $2f(5, F^2) + 1$ type-A cycles. By the definition of the function, F^2 admits at most $f(5, F^2)$ simple closed curves which are pairwise non-homotopic and cross at most 5 times, and hence there exist three such homotopic cycles C_A^i, C_A^j and C_A^k $(i < j < k)$ by the Pigeonhole Principle. Let $\tilde{D}$ denote the configuration which is the union of the closed disk $\bar{D}$ bounded by L_v and the three cycles C_A^i, C_A^j and C_A^k. First, suppose that $\tilde{D}$ is an embedding on F^2 such that C_A^i, C_A^j and C_A^k are 2-sided. Moreover, assume that C_A^i (resp., C_A^j) contains $b_s^i v b_t^i$ with $1 \leq s < t \leq 6$ (resp., $b_{s'}^j v b_{t'}^j$ with $1 \leq s' < t' \leq 6$).

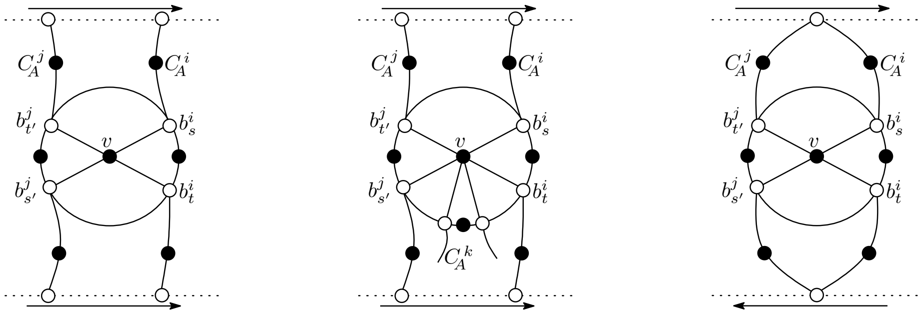

Figure 3: Type-A cycles around v.

Observe that in $\tilde{D}$, C_A^i and C_A^j bound a pinched annulus A (i.e., an annulus where the two boundary components might touch several times) having a pinched point v (see the left-hand side of Figure 3). If C_A^i and C_A^j have a common vertex other than v, then there exists a 2-cell region R in A bounded by a cycle of length either 4 or 6 such that $\bar{R}$ contains $P(b_t^i, b_{s'}^j)$ or $P(b_{t'}^j, b_s^i)$. However, this contradicts Lemma 3.10 since $P(b_t^i, b_{s'}^j)$ (resp., $P(b_{t'}^j, b_s^i)$) contains vertices $c_1^i, \dots, c_{17}^i, a_1^j, \dots, a_{15}^j$, and a_{16}^j (resp., $c_1^j, \dots, c_{17}^j, a_1^i, \dots, a_{15}^i$, and a_{16}^i). In the following argument, we call a region like the above R a *dense quadrangle* or a *dense hexagon*, which contains at least 5 or 17 inner vertices, respectively. Thus, we conclude that C_A^i and C_A^j have the unique common vertex v. However, under the situation, the third type-A cycle C_A^k must cross transversally either C_A^i or C_A^j (see the center of Figure 3), contradicting the same argument as above. In the case when each of C_A^i, C_A^j and C_A^k is 1-sided, any two of them must cross, and hence there exists a dense quadrangle or a dense hexagon, as well as the previous case (see the right-hand side of Figure 3).

Next, we discuss type-B cycles. Under our definition, for some $i \neq j$, $C_B^{i,j}$ and $C_B^{j,i}$ might exist; as an extreme example, $C_B^{i,j}$ might coincide with $C_B^{j,i}$. If so, i.e., there exist $C_B^{i,j}$ and $C_B^{j,i}$, then we choose one from them. By the above argument, we may assume that there exist at most $2f(5, F^2)$ type-A cycles. That is, there exist at least $7f(5, F^2) + 1$, which is the half of $14f(5, F^2) + 2$, distinct type-B cycles around v, such that the set of those cycles contains no pair of two cycles $C_B^{i,j}$ and $C_B^{j,i}$ for $1 \leq i \leq j \leq l$.

Similar to the argument for type-A cycles, there exist eight such homotopic cycles simply denoted by $\Gamma_1, \Gamma_2, \dots, \Gamma_8$ having a common vertex v such that they are placed on F^2 as shown in the left-hand side of Figure 4. Note that the lengths of those cycles are same, which is either 3 or 4, by Proposition 3.1. Furthermore, note that if Γ_i and Γ_{i+1} have a common vertex other than v for some $i \in \{1, \dots, 7\}$, then we can easily find a dense quadrangle or a dense hexagon, contradicting Lemma 3.10; only Γ_1 and Γ_8 might have a common vertex other than v. Therefore, $\Gamma_i \cup \Gamma_{i+1}$ bounds an octagonal (resp., a hexagonal) 2-cell region for each $i \in \{1, \dots, 7\}$ if $|\Gamma_i| = 4$ (resp., if $|\Gamma_i| = 3$).

Let $D_{i,j}$ denote an octagonal (or a hexagonal) region bounded by $\Gamma_i \cup \Gamma_j$ for $1 \leq i < j \leq 8$. By Euler's formula, $\Gamma_{4,5}$ contains a vertex u of degree 3; e.g., see Lemma 4.1 in [23]. By Lemma 3.3, u belongs to a connected component of $\langle V_3 \rangle_G$ which is

(i) a 4-cycle,

(ii) a 2-path,

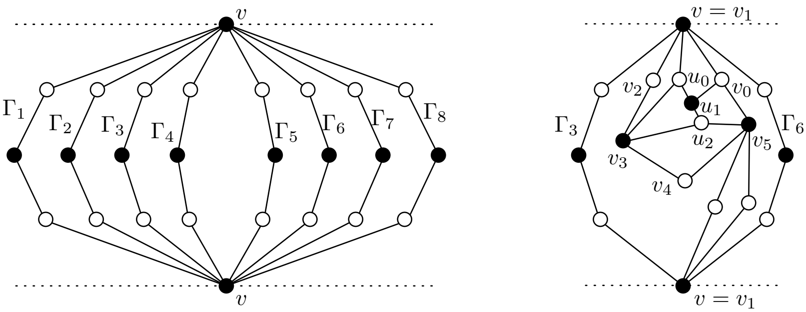

Figure 4: Type-B cycles around v.

(iii) a K_2 or

(iv) an isolated vertex.

First, we assume that $|\Gamma_i| = 4$, and discuss the above four cases in order.

Case (i): In this case, an attached cube H with $\partial(H) = v_0v_1v_2v_3$ containing u as an vertex of the attached 4-cycle is in $\bar{D}_{3,6}$. (Observe that faces incident to u are in $D_{4,5}$, and the other two faces in the 2-cell region bounded by $\partial(H)$ are at least in $D_{3,6}$.) Then, by Lemma 3.7 and the existence of Γ_2 and Γ_7, one of v_0, v_1, v_2 and v_3, say v_0 without loss of generality, must be v; we call the above Γ_2 and Γ_7 *obstructions* throughout the proof. However, Lemma 3.8 requires one more essential simple closed curve which does not pass through $v = v_0$, a contradiction; by the existence of obstructions again.

Case (ii): We assume that u belongs to a 2-path $P = u_0u_1u_2$ and the configuration around P is given by the left-hand side of R_3 in Figure 1. Similarly, the hexagon bounded by $v_0v_1v_2v_3v_4v_5$ is contained in $\bar{D}_{3,6}$, and hence the obstructions, which are Γ_2 and Γ_7, play the same role in this argument. By Lemma 3.6, one of v_1 and v_5, say v_1 without loss of generality, must be v (see the right-hand side of Figure 4). Since $\deg(v_3) \geq 4$ and $\deg(v_5) \geq 4$, there is an essential diagonal 3-curve passing $\{v_4, u_2, v_0\}$ or $\{v_4, u_2, u_0\}$ by Lemma 3.4. However, in each case, such three vertices are inner vertices of $D_{2,7}$, a contradiction.

Case (iii): We assume that $u_0u_1 \in E(G)$ is a connected component of $\langle V_3 \rangle_G$, and there are four faces $v_0v_1u_1v_4, v_1v_2u_2u_1$ and $u_1u_2v_3v_4$ and $u_2v_2v_0'v_3$ contained in $D_{3,6}$. Here, we locally color vertices in $\bar{D}_{3,6}$ by two colors black and white; we assume that v is colored by black. Further, we may assume that v_0' is colored by black without loss of generality; note that v_0, v_2 and v_3 are white vertices. When considering a face $v_0v_1u_1v_4$, there is an essential diagonal 3-curve passing either $\{v_0, u_1, v_2\}$ or $\{v_0, u_1, v_3\}$ by Lemma 3.4, since we have $\deg(v_1) \geq 4$ and $\deg(v_4) \geq 4$. By the existence of obstructions, one of v_0, v_2 and v_3 must be v under the situation. However, it contradicts the above bipartition.

Case (iv): Assume that u is incident to three faces $v_0v_1uv_6, v_1v_2v_4u$ and $uv_4v_5v_6$, which are in $D_{4,5}$, and note that $\deg(v_i) \geq 4$ for each $i \in \{1, 4, 6\}$. As well as the previous case, we locally color vertices in $\bar{D}_{3,6}$; assume that v is colored by black. If u is a white vertex, then it contradicts Lemma 3.4 by the existence of obstructions; note that there should be a diagonal 3-curve passing three white vertices including u. Therefore, u is a black vertex below. By Lemma 3.4 again, exactly one of v_0, v_2 and v_5, say v_0 without loss

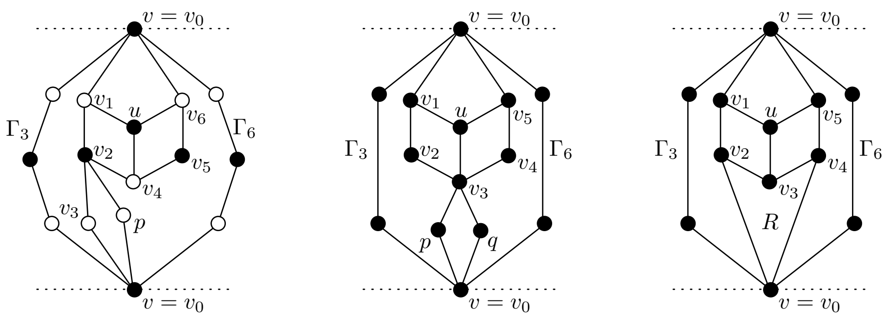

Figure 5: Configurations in the 2-cell region bounded by Type-B cycles.

of generality, coincides with v, and there exists a diagonal 3-curve passing through three faces $v_0v_1uv_6, v_1v_2v_4u$ and $v_2v_3v_0p$ for $v_3, p \in V(G)$, up to symmetry (see the left-hand side of Figure 5). If $\deg(v_2) \geq 4$, then Lemma 3.4 works for $v_2v_3v_0p$, and it contradicts the existence of the obstructions. Thus, we conclude that $|\{v_1, v_4\} \cap \{v_3, p\}| = 1$, and we may suppose $v_4 = p$ since $\{v_1, v_4\} \cap \{v_3, p\} \neq \{v_1\}$; otherwise, G would have multiple edges. Then, G has an octagonal region bounded by $v_0v_1v_2v_3v_0v_4v_5v_6$ satisfying the condition of a reduction R_6. However, it contradicts Lemma 3.12 by the existence of the obstructions.

Next, we assume that $|\Gamma_i| = 3$. We implicitly omit the same argument as in the case assuming $|\Gamma_i| = 4$. (That is, we give only the different and important points below.)

Case (i): The same argument as in the case of $|\Gamma_i| = 4$ works.

Case (ii): We may assume that $v_1 = v$, and there is an essential semi-diagonal 3-curve passing $\{v_4, u_2, v_0\}$, $\{v_4, u_2, u_1\}$ or $\{v_4, u_2, u_0\}$ by Lemma 3.4. However, in any case, such three vertices are inner vertices of $D_{2,7}$, a contradiction.

Case (iii): In this case, the similar argument (not using the bipartition) leads us to the conclusion that $v_0 = v_0' = v$ such that the 3-cycle $v_0v_1v_2$ is homotopic to Γ_i. However, it contradicts Lemma 3.11 by the existence of the obstructions.

Case (iv): Assume that u is incident to three faces $v_0v_1uv_5, uv_1v_2v_3$ and $uv_3v_4v_5$, which are in $D_{4,5}$, and note that $\deg(v_i) \geq 4$ for each $i \in \{1, 3, 5\}$. For a face $v_0v_1uv_5$, there exists a semi-diagonal 3-curve passing either $\{v_0, u, v_3\}$ or $\{v_0, u, v_4\}$, up to symmetry, by Lemma 3.4. Fist assume the former case. If $v = v_0$, then there is a face $f = v_3pvq$ for $p, q \in V(G)$ (see the center of Figure 5). For f, Lemma 3.4 works and we conclude a contradiction by the existence of the obstructions since $\deg(v_3) \geq 4$. On the other hand, if $v = v_3$, then there is a face vsv_0t for $s, t \in V(G)$. As well as the previous case, we can apply Lemma 3.4 for vsv_0t since $\deg(v_0) \geq 4$; if $\{v_1, v_5\} \cap \{s, t\} \neq \emptyset$, then G would not become 3-representative.

Next, we assume the latter case. In this case, v is either v_0 or v_4, say v_0, up to symmetry. By the assumption, there exists an edge v_4v_0 such that $v_0v_5v_4$ is homotopic to Γ_i. Furthermore, applying Lemma 3.4 for a face $v_1v_2v_3u$, there must be a semi-diagonal 3-curve passing $\{v_0, u, v_2\}$; note that v_2, u, v_4 and v_5 are vertices in $\bar{D}_{4,5}$, i.e., inner vertices of $D_{3,6}$. That is, we have $v_2v_0 \in E(G)$ such that $v_2v_0v_4v_3$ bounds a 2-cell region R inside $D_{4,5}$ (see the right-hand side of Figure 5). By the above argument of (i), we may assume that $D_{4,5}$ does not contain a vertex of degree 3 belonging to an attached 4-cycle, and hence

R is a face of G by Lemma 3.9. However, v_3 has degree 3, contrary to u being an isolated vertex of $\langle V_3 \rangle_G$. Therefore, we got our desired conclusion. □

Lemma 4.3. *Let G be an $\{R_1, R_2, R_3\}$-irreducible polyhedral quadrangulation of a non-spherical closed surface F^2. For any vertex $v \in V(G)$, there exists an essential cycle of length at most 6 either*

(i) *containing v, or*

(ii) *containing $u \in V(G)$ such that $uv \in E(G)$.*

Proof. First, assume that $\deg(v) = 3$, and let u_0, u_1 and u_2 be vertices adjacent to v. If two of u_0, u_1 and u_2, say u_0 and u_1 without loss of generality, have degree at least 4, then we can easily find our desired cycle by Lemma 3.5. Thus, by Lemma 3.3, we may assume that $\deg(u_0) = \deg(u_1) = 3$ and $\deg(u_2) \geq 4$ below. If v is contained in a 4-cycle of $\langle V_3 \rangle_G$, then there exists such a cycle by Lemma 3.7. On the other hand, if v is not contained in the above 4-cycle in $\langle V_3 \rangle_G$, that is, if a 2-path $u_0 v u_1$ is a connected component of $\langle V_3 \rangle_G$, then G also has our desired cycle by Lemma 3.6.

Next, we assume $\deg(v) \geq 4$, and let u_0 and u_1 be vertices adjacent to v such that $u_0 v u_1$ forms a corner of a face of G. If one of u_0 and u_1, say u_0 without loss of generality, has degree 3, then G has a cycle of length at most 6 passing through u_0 by the above argument, and hence it satisfies (ii) of the statement in the lemma. If $\deg(u_0) \geq 4$ and $\deg(u_1) \geq 4$, then there exists our desired cycle by Lemma 3.5 again. □

Lemma 4.4. *Let G be an $\{R_1, \dots, R_8\}$-irreducible polyhedral quadrangulation of a non-spherical closed surface F^2. Then the diameter of G is bounded by a constant depending only on F^2.*

Proof. In this proof, we prove that $\mathrm{diam}(G) \leq 50 f(0, F^2) - 1$ where $\mathrm{diam}(G)$ is a *diameter* of G and $f(\cdot, F^2)$ is the function in Lemma 4.1. Suppose, for a contradiction, that G has two vertices x and y with distance at least $50 f(0, F^2)$. Let P be a path from x to y attaining the distance, and let $x = v_1, v_2, \dots, v_k$ be the vertices on P lying in this order, where $k \geq 5 f(0, F^2) + 1$, so that the distance between v_i and v_{i+1} is exactly 10 on P, for each $i \in \{1, \dots, k-1\}$. Then, there exists a cycle C_i of length at most 6 passing through either v_i or a vertex u_i adjacent to v_i for each $i \in \{1, \dots, k\}$ by Lemma 4.3. Since the distance between v_i and v_j is at least 10 for any $i < j$, two cycles C_i and C_j are mutually disjoint. Since F^2 admits only $f(0, F^2)$ pairwise non-crossing non-homotopic essential cycles, and since we assumed $k \geq 5 f(0, F^2) + 1$, we can take six pairwise homotopic cycles from $\{C_1, \dots, C_k\}$ by the Pigeonhole Principle. Let $\Gamma_1, \dots, \Gamma_6$ be such six cycles of length at most 6, which are mutually homotopic. Note that those cycles are 2-sided since any two of them are disjoint, and that the parities of those cycles are pairwise same. We may assume that these $\Gamma_1, \dots, \Gamma_6$ lie on an annulus in this order.

Let $A_{i,j}$ denote the annular region bounded by Γ_i and Γ_j for $1 \leq i < j \leq 6$; similarly, $\bar{A}_{i,j}$ contains its two boundaries Γ_i and Γ_j. Note that there is no edge joining vertices on Γ_i and Γ_{i+1} for each $i \in \{1, \dots, 5\}$; for otherwise, the distance between v_i and v_{i+1} would be at most 9, contradicting that P is a shortest path joining x and y in G. Similar to the argument in Lemma 4.2, we call Γ_1 and Γ_6 obstructions for our purpose.

First, we discuss the case when G has a vertex u of degree 3 in $\bar{A}_{3,4}$. By Lemma 3.3, u belongs to a connected component of $\langle V_3 \rangle_G$ which is

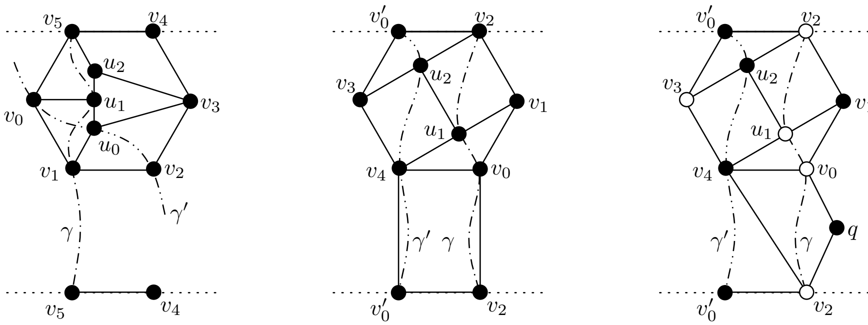

Figure 6: Configurations around connected components of $\langle V_3 \rangle_G$.

(i) a 4-cycle,

(ii) a 2-path,

(iii) a K_2 or

(iv) an isolated vertex.

We discuss the above four cases in order.

Case (i): Under the assumption, an attached cube containing u as an vertex of an attached 4-cycle is in $\bar{A}_{2,5}$. (For example, even if u is on Γ_3, then there is no face f such that ∂f contains both u and a vertex on Γ_2, since there is no edge joining vertices on Γ_2 and Γ_3, and since $\deg(u) = 3$.) Similar argument in Case (i) in the proof of Lemma 4.2 works, and we conclude that this is not the case; i.e, we cannot take two essential simple closed curves γ_1 and γ_2 in Lemma 3.8 by the existence of the obstructions.

Case (ii): We assume that u belongs to a 2-path $P = u_0u_1u_2$ and the configuration around P is given by the left-hand side of R_3 in Figure 1. Similarly, the hexagonal region R bounded by $v_0v_1v_2v_3v_4v_5$ is contained in $\bar{A}_{2,5}$. By Lemma 3.6, there exists an essential diagonal (or a semi-diagonal) 3-curve γ passing $\{v_1, u_1, v_5\}$ (see the left-hand side of Figure 6). On the other hand, since $\deg(v_1) \geq 4$ and $\deg(v_3) \geq 4$ hold, there exists an essential diagonal (or semi-diagonal) 3-curve γ' passing $\{v_0, u_0, v_2\}$ by Lemma 3.4. Observe that both γ and γ' are homotopic to Γ_i by the existence of obstructions. Under the situation, γ and γ' cross transversally in R, and it must cross transversally one more time since these two curves are 2-sided. This implies that there should be a face incident to four vertices v_0, v_1, v_2 and v_5, in which γ and γ' pass through. However, it contradicts that G is simple.

Case (iii): Assume that $u_1u_2 \in E(G)$ is a connected component of $\langle V_3 \rangle_G$, and there are four faces $v_0v_1u_1v_4, v_1v_2u_2u_1, u_1u_2v_3v_4$ and $u_2v_2v_0'v_3$ incident to u_1 and u_2. Note that $\deg(v_i) \geq 4$ for any $i \in \{1, 2, 3, 4\}$. When considering a face $v_0v_1u_1v_4$, there exists an essential diagonal (or semi-diagonal) 3-curve γ passing either $\{v_0, u_1, u_2\}$ or $\{v_0, u_1, v_2\}$ by Lemma 3.4, up to symmetry. Note that γ is homotopic to Γ_i. In the former case, we have $v_0 = v_0'$, and hence we discuss an R_5 to the hexagonal region containing u_1 and u_2. However, it immediately contradicts that G is $\{R_1, \ldots, R_8\}$-irreducible by the existence of obstructions and by Lemma 3.11.

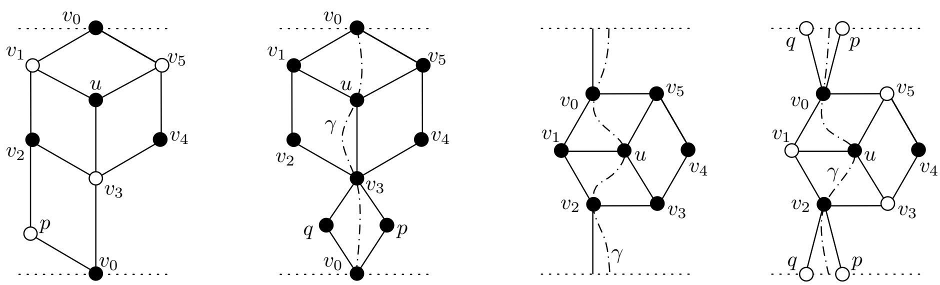

Figure 7: Configurations around connected components of $\langle V_3 \rangle_G$.

Therefore, we assume the latter case. In this case, we may assume that there exists an essential diagonal (or semi-diagonal) 3-curve γ' passing $\{v_0', u_2, v_4\}$ by the same argument as above. Note that γ' is homotopic to γ under the condition. If γ and γ' are both essential semi-diagonal 3-curves (by Proposition 3.1) then, there exists a face $v_0 v_4 v_0' v_2$ by Lemma 3.9 and our former argument (see the center of Figure 6). However, since $\deg(v_2) \geq 4$ and $\deg(v_4) \geq 4$, we apply Lemma 3.4, and conclude a contradiction.

Thus, we suppose that γ is an essential diagonal 3-curve, and there is a face $f = v_0 p v_2 q$ for $p, q \in V(G)$ which is passed by γ. Here, observe that $v_1 \notin \{p, q\}$ by the simplicity of G, and hence we have $\deg(v_2) \geq 4$. For f, if $\deg(v_0) \geq 4$, then it is contrary to G being $\{R_1, \ldots, R_8\}$-irreducible by the existence of obstructions and by Lemma 3.4. Therefore, we assume that $\deg(v_0) = 3$ below. Without loss of generality, we may assume that $p = v_4$ (see the right-hand side of Figure 6). Under the situation, we can apply Lemma 3.12 to the octagonal region bounded by $v_2 v_1 v_0 q v_2 v_4 v_3 u_2$, and obtain a contradiction.

Case (iv): Assume that u is incident to three faces $v_0 v_1 u v_5$, $v_1 v_2 v_3 u$ and $u v_3 v_4 v_5$. Note that $\deg(v_i) \geq 4$ for any $i \in \{1, 3, 5\}$. Hence, for a face $v_0 v_1 u v_5$, we have

(a) an essential 4-cycle $v_0 v_1 u v_3$, or

(b) an essential diagonal 3-curve or semi-diagonal 3-curve γ passing

  (1) $\{v_0, u, v_3\}$ or

  (2) $\{v_0, u, v_2\}$

  by Lemma 3.4, up to symmetry.

First, assume (a). In this case, for a face $v_1 v_2 v_3 u$, there must be an essential diagonal 3-curve passing $\{v_0, u, v_2\}$ by Lemma 3.4; it is not difficult to check that this is the unique case by Proposition 3.1 and the existence of obstructions. Furthermore, by Lemma 3.9, there exists a face $v_2 p v_0 v_3$ for $p \in V(G)$, and it contradicts Lemma 3.12 for an octagonal region bounded by $v_0 v_1 v_2 p v_0 v_3 v_4 v_5$ by the similar argument as above (see the first figure of Figure 7).

Secondly, we assume (b)(1). In this case, γ is an essential semi-diagonal 3-curve, and hence there exists a face $v_0 p v_3 q$ for $p, q \in V(G)$ which γ passes through (see the second figure of Figure 7). Then, we have $\deg(v_0) \geq 4$ since $\{p, q\} \cap \{v_1, v_5\} = \emptyset$; otherwise, G would become representativity at most 2. Therefore, for $v_0 p v_3 q$, we apply Lemma 3.4, and obtain a contradiction as well as former cases.

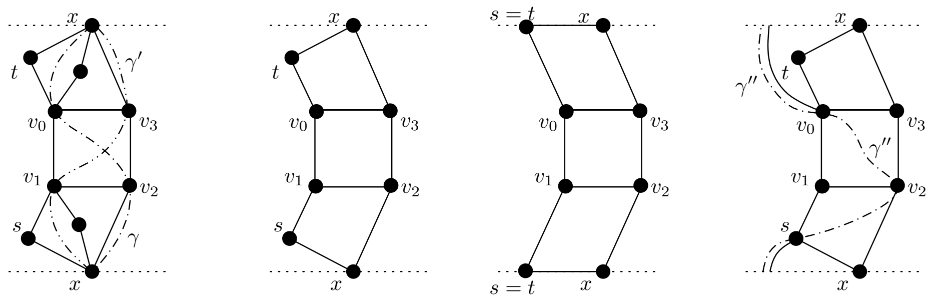

Figure 8: Configurations of Case (I) in Lemma 4.4.

Thirdly, we discuss the case (b)(2). First, assume that γ is an essential semi-diagonal 3-curve; i.e., $v_0v_2 \in E(G)$ which is along γ (see the third figure of Figure 7). Then, for a face $uv_3v_4v_5$, there exists either v_4v_0 or v_4v_2, say v_4v_0 without loss of generality, as an edge of G such that $v_0v_5v_4$ is homotopic to Γ_i. Under the situation, there exists a 2-cell region R bounded by $v_0v_4v_3v_2$, which is a face of G by Lemma 3.9 and the former argument. However, we obtain a contradiction since $\deg(v_3) \geq 4$. Therefore, we suppose that γ is an essential diagonal 3-curve; i.e., there exists a face bounded by v_0pv_2q for $p, q \in V(G)$ (see the last figure of Figure 7). If $\{p, q\} \cap \{v_3, v_5\} \neq \emptyset$, then it gives rise to the above case (a), which had already discussed. On the other hand, if $v_1 \in \{p, q\}$, then G would have multiple edges, a contradiction. Thus, we have $\deg(v_0) \geq 4$ and $\deg(v_2) \geq 4$, and conclude a contradiction by Lemma 3.4, similar to the former cases.

Therefore, in the following argument, we discuss the case when $\deg(u) \geq 4$ for any vertex u in $\bar{A}_{3,4}$. In this case, we focus on a face $f = v_0v_1v_2v_3$ in $\bar{A}_{3,4}$ with $\deg(v_i) \geq 4$ for each $i \in \{0, 1, 2, 3\}$. By Propositions 3.1 and 3.2, Lemma 3.4 and the existence of obstructions, it suffices to discuss the following two cases (I) and (II), up to symmetry.

Case (I): There exist two essential semi-diagonal 3-curves γ and γ' passing $\{v_0, v_2, x\}$ and $\{v_1, v_3, x\}$, respectively, for $x \in V(G)$ such that γ and γ' are homotopic to Γ_i (see the first figure of Figure 8). Then, there are two faces $f = v_0v_3xt$ and $f' = v_1sxv_2$ for $s, t \in V(G)$ by Lemma 3.9 (see the second figure of Figure 8). Under the situation, if $s = t$, then there exists an annular region A bounded by two 3-cycles sv_0v_1 and xv_3v_2 which contains exactly three edges dividing it into three faces (see the third figure of Figure 8). Then, we apply Lemma 3.13 to A and obtain a contradiction by the existence of the obstructions.

Thus, we assume $s \neq t$ below, and hence s, t, v_2 and v_3 are distinct vertices; i.e., we have $\deg(x) \geq 4$. Then, we apply Lemma 3.4 to f' and find an essential semi-diagonal 3-curve γ'' passing $\{s, v_2, z\}$ for $z \in V(G)$. By the existence of the obstructions, γ' and γ'' should be homotopic. That is, γ' and γ'' cross even times (actually twice), and hence we have $z = v_0$ and $sv_0 \in E(G)$ (see the last figure of Figure 8). Then, there exists a 2-cell region bounded by sv_0v_3x, and it contradicts Lemma 3.9 since $s \neq t$.

Case (II): There exists an essential diagonal 3-curve γ passing $\{v_1, v_3, x\}$ for $x \in V(G)$, and $v_0x, v_2x \in E(G)$ such that γ and the 4-cycle $v_0v_1v_2x$ are homotopic to Γ_i (see the left-hand side of Figure 9). Then, there are two faces $f = v_2v_1sx$ and $f' = v_0v_3tx$ for $s, t \in V(G)$ by Lemma 3.9 (see the center of Figure 9). By the simplicity of G, $s, t \notin \{v_0, v_1, v_2, v_3\}$, and hence $\deg(x) \geq 4$. Thus, for f, there exists an essential diagonal

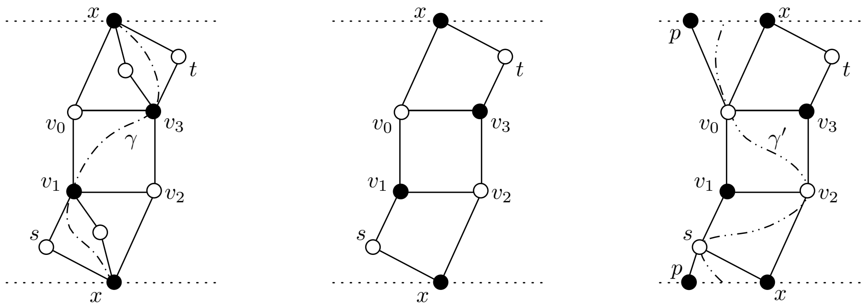

Figure 9: Configurations of Case (II) in Lemma 4.4.

3-curve γ' passing $\{v_0, v_2, s\}$ by Lemma 3.4; this is a unique case by the same argument as in Case (I). Then, by Lemma 3.9, there is a face $f'' = spv_0x$ for $p \in V(G)$ which γ' passes through (see the right-hand side of Figure 9). Apply Lemma 3.14 to the annular region bounded by two 4-cycles v_0v_1sp and v_3v_2xt, and obtain a contradiction. □

Now, we prove our main result as follows.

Proof of Theorem 1.1. Let G be a graph with maximum degree Δ and diameter d. Then, the following inequality holds.

$$|V(G)| \leq 1 + \sum_{k=1}^{d} \Delta(\Delta-1)^{k-1} = 1 + \frac{\Delta((\Delta-1)^d - 1)}{\Delta - 2}.$$

Therefore, every $\{R_1, \ldots, R_8\}$-irreducible quadrangulation G of F^2 has a finite number of vertices, since its maximum degree and diameter are bounded by Lemmas 4.2 and 4.4, respectively. Thus, F^2 admits only finitely many $\{R_1, \ldots, R_8\}$-irreducible quadrangulations, up to homeomorphism. □

5 Minimality of reductions

In the previous section, we proved that $\{R_1, \ldots, R_8\}$ is sufficient to finitize the number of minimal quadrangulations of any closed surface. However, one might think that the eight reductions are little too much. As mentioned in the introduction, Theorem 1.3 describes more accurate facts for the torus.

Proof of Theorem 1.3. See Figure 10. Each J_i represents an infinite series of $\{R_1, \ldots, R_8\} \setminus \{R_i\}$-irreducible quadrangulations of the torus. (To obtain the torus, identify two horizontal segments and two vertical segments of the rectangle, respectively.) In each gray colored quadrangular region in figures contains exactly four vertices which is of an attached 4-cycle. We can construct only J_6 and J_8 as bipartite quadrangulations since the others require essential cycles of length 3. Observe that we cannot apply R_8 to J_6, since the dual of J_6 has no essential cycle of length at most 4. Moreover, each of J_7 and J_8 is an infinite series of 4-regular quadrangulations of the torus. □

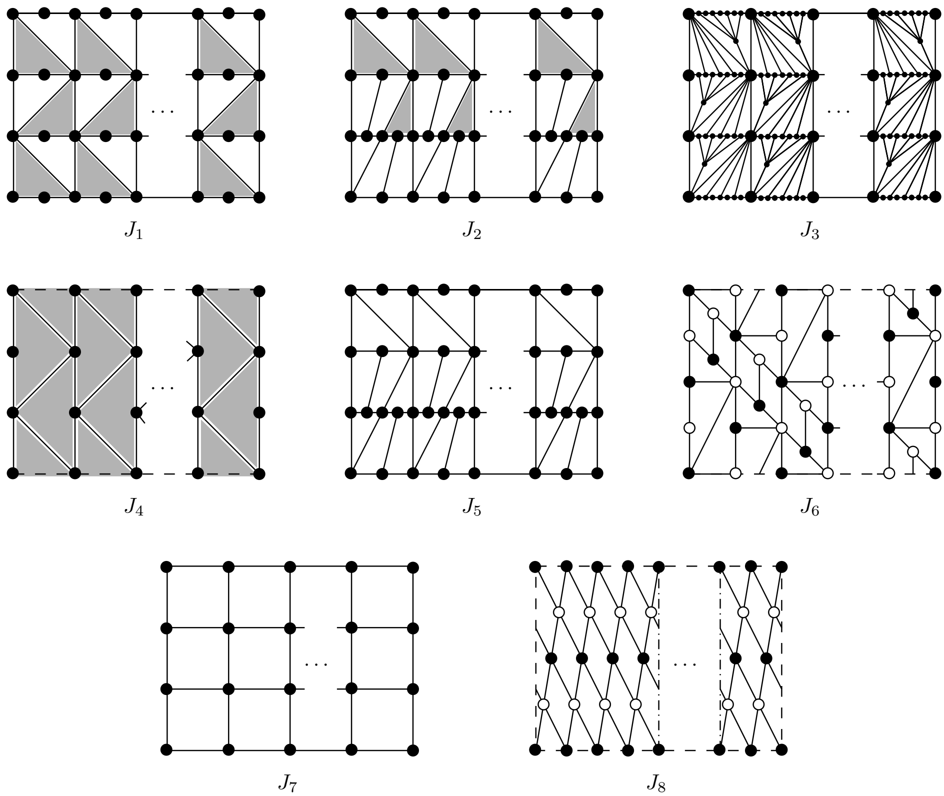

Figure 10: Infinite series of quadrangulations of the torus.

Proof of Theorem 1.4. As mentioned in the introduction, the projective plane does not admit 2-sided essential simple closed curves and hence $\{R_1, \dots, R_6\}$ is finitizable for polyhedral quadrangulations of the projective plane by Theorem 1.1. The infinite series of minimal graphs can be obtained in a similar way as those of torus; we leave it for readers. For example, an infinite series of polyhedral quadrangulations denoted by $I_{26}(2n+1)$ $(n \geq 2)$, which can be found in [23], is $\{R_1, \dots, R_5\}$-irreducible quadrangulations of the projective plane. □

In the end of the paper, we pose the following problem.

Problem 5.1. For any closed surface F^2 other than the sphere, the projective plane and the torus, is $\{R_1, \dots, R_8\}$ a minimal finitizable set of reductions for polyhedral quadrangulations of F^2?

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