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An extension of the Erdős-Ko-Rado theorem to uniform set partitions

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Abstract

A (k, ℓ)-partition is a set partition which has ℓ blocks each of size k. Two (k, ℓ)-partitions P and Q are said to be *partially t-intersecting* if there exist blocks P_i in P and Q_j in Q such that $|P_i \cap Q_j| \geq t$. In this paper we prove a version of the Erdős-Ko-Rado theorem for partially 2-intersecting (k, ℓ)-partitions. In particular, we show for ℓ sufficiently large, the set of all (k, ℓ)-partitions in which a block contains a fixed pair is the largest set of 2-partially intersecting (k, ℓ)-partitions. For $k = 3$, we show this result holds for all ℓ.

Keywords: Erdős-Ko-Rado Theorem, uniform set partitions, ratio bound, clique, coclique, quotient graphs.

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1 Introduction

In 1961, Erdős, Ko, and Rado proved that if $\mathcal{F}$ is a t-intersecting family of k-subsets of $\{1, 2, \ldots, n\}$, then $\binom{n-t}{k-t}$ is a tight upper bound on the size of $\mathcal{F}$, provided that n is sufficiently large [7]. This result has motivated consideration of "intersecting" families of many

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other combinatorial objects using diverse proof techniques and has developed into an active and broad area of research. There are many recent results giving analogs of the EKR theorem; see, for example, [9, 13, 16, 20, 26] or [12] and the references within. In this work, we prove an extension of the EKR theorem to systems of uniform set partitions.

A (k,ℓ)*-partition* is a set partition of $\{1,2,\dots,k\ell\}$ with exactly ℓ blocks each of size k. These are also called *uniform set partitions*. We use $\mathcal{U}_{k,\ell}$ to denote the set of all (k,ℓ)-partitions, and $u_{k,\ell}=|\mathcal{U}_{k,\ell}|$. It is easy to see that

$$u_{k,\ell}=\frac{1}{\ell!}\binom{k\ell}{k}\binom{k\ell-k}{k}\binom{k\ell-2k}{k}\cdots\binom{k}{k}=\frac{(k\ell)!}{(k!)^{\ell}\ell!}. \tag{1.1}$$

In [8], Erdős and Székely considered different types of intersection for partitions. In one of these types, and the one we consider here, two partitions P and Q are *intersecting in a pair* if there exist blocks P_i in P, and Q_j in Q such that $|P_i\cap Q_j|\geq 2$. In [20], Meagher and Moura generalized this definition: two partitions P and Q are *partially t-intersecting* if there exist P_i in P, and Q_j in Q such that $|P_i\cap Q_j|\geq t$. The work of Meagher and Moura is different than that of Erdős and Székely since only uniform partitions are considered in [20].

A set of partitions is *partially t-intersecting* if the partitions in the set are pairwise partially t-intersecting. Meagher and Moura [20] conjectured if $\mathcal{P}\subset\mathcal{U}_{k,\ell}$ is a set of partially t-intersecting partitions, with $t\leq k$, then $|\mathcal{P}|\leq\binom{k\ell-t}{k-t}u_{k,\ell-1}$. A set of this size can be formed by fixing a t-subset T and taking all (k,ℓ)-partitions with a block that contains T; such a set is called *canonically t-intersecting*. Meagher and Moura further conjectured that only the canonically t-intersecting (k,ℓ)-partitions attain this maximum size. As pointed out by Brunk in [2], this conjecture additionally requires that $k\leq\ell(t-1)$, since if $k>\ell(t-1)$, then any two (k,ℓ)-partitions are t-partially intersecting.

If $k=t=2$, then the $(2,\ell)$-partitions are perfect matchings in the complete graph on 2ℓ vertices, and partially 2-intersecting is equivalent to intersecting (as sets of edges). The Meagher-Moura conjecture has been proven in this case in [13], so in this paper we only consider $k\geq 3$. In particular, we prove the Meagher-Moura conjecture for $t=2$ with $k=3$ and all values of ℓ, and for all $k\geq 4$, provided that ℓ is sufficiently large. Our approach is to define a graph in which the cocliques (also known as independent sets) are equivalent to partially 2-intersecting (k,ℓ)-partitions from $\mathcal{U}_{k,\ell}$. Then we use a version of the algebraic method from [13] to find the size of a maximum coclique in the graph. This is an approach that has been very effective in proving many EKR-type results, indeed it is the main topic of the book [12]. This method is particularly effective when considering intersecting permutations in groups and it has been applied to many families of groups, see for example [5, 6, 17, 21, 22, 23, 25].

2 Overview of method

In a graph X a *clique* is a set of vertices which induce a complete subgraph; and a *coclique* is a set of vertices which induce an empty subgraph. The size of a largest clique and a largest coclique are denoted by $\omega(X)$ and $\alpha(X)$, respectively. The *adjacency matrix* $A(X)$ of X is a matrix in which rows and columns are indexed by the vertices in X and the (i,j)-entry is 1 if i and j are adjacent, and 0 otherwise. The *eigenvalues* of X refer to the eigenvalues of its adjacency matrix. We use $\mathbf{1}$ to denote the all-ones vector; for any d-regular graph, the all-ones vector is an eigenvector with eigenvalue d.

In general, finding the largest coclique of a graph X is known to be NP-hard, but the *Delsarte-Hoffman (ratio)* bound gives an upper bound on $\alpha(X)$. This bound is based on the ratio between the largest and the smallest eigenvalue of the adjacency matrix of the graph. A proof of this result can be found in [4] or in [12, Section 2.4], we also recommend Haemers' paper [15] on the history of this bound.

Theorem 2.1 (Delsarte-Hoffman bound [4])**.** *Let A be the adjacency matrix for a d-regular graph X on vertex set $V(X)$. If the least eigenvalue of A is τ, then*

$$\alpha(X) \leq \frac{|V(X)|}{1-\frac{d}{\tau}}.$$

If equality holds for some coclique S with characteristic vector ν_S, then

$$\nu_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.

Define $X_{k,\ell}$ to be the graph with $\mathcal{U}_{k,\ell}$ as its vertex set, in which two partitions P and Q are adjacent if every pair of blocks, one from P and one from Q, have at most 1 element in common. The group $\mathrm{Sym}(k\ell)$ acts transitively on the vertices of $X_{k,\ell}$ and preserves the edges. This means the $X_{k,\ell}$ is vertex transitive and regular. We will denote the degree by $d_{k,\ell}$, or simply d when the context is clear.

A resolvable packing design on $k\ell$ points with block size k and index $\lambda = 1$ is equivalent to a clique in this graph. Further, a resolvable balanced incomplete block design on $k\ell$ points with block size k and index $\lambda = 1$, if it exists, gives a maximum clique.

For any distinct $i, j \in \{1, \ldots, k\ell\}$, let $S_{i,j}$ be the subset of partitions in $\mathcal{U}_{k,\ell}$ for which the elements i and j are in the same block. Then $S_{i,j}$ is a coclique in the graph $X_{k,\ell}$ and the size of $S_{i,j}$ is

$$\frac{1}{(\ell-1)!}\binom{k\ell-2}{k-2}\binom{k\ell-k}{k}\cdots\binom{k}{k}.$$

The main goal in this paper is to prove, using the ratio bound, that $S_{i,j}$ is a maximum coclique in $X_{k,\ell}$. For the ratio bound to hold with equality, we need to prove if τ is the least eigenvalue of $X_{k,\ell}$, then

$$1 - \frac{d_{k,\ell}}{\tau} = \frac{u_{k,\ell}}{|S_{i,j}|} = \frac{k\ell-1}{k-1}.$$

Thus we need to prove two facts: first that $\tau = -\frac{d_{k,\ell}(k-1)}{k(\ell-1)}$ is an eigenvalue of $X_{k,\ell}$; and second that τ is the least eigenvalue of $X_{k,\ell}$.

In Section 3, we show how the eigenvalues of $X_{k,\ell}$ are related to the irreducible characters of $\mathrm{Sym}(k\ell)$, and we prove some bounds on the degrees of these irreducible characters. Next, in Section 4, we calculate three of the eigenvalues of $X_{k,\ell}$; one of these eigenvalues is the τ above. Next we prove if there is an eigenvalue of $X_{k,\ell}$ that is strictly smaller than τ, there is a function that is an upper bound on the eigenvalue's multiplicity. In Section 6 we prove that this function is bounded by $\binom{k\ell}{3} - \binom{k\ell}{2}$ for ℓ sufficiently large. This uses the result from Section 5, that the limit of ratio $u_{k,\ell}/d_{k,\ell}$ is finite as ℓ goes to ∞. The bounds from Section 3 then prove that no such eigenvalues exist and this proves the

Meagher-Moura Conjecture with $t = 2$, for all values of k, provided that ℓ is sufficiently large. Finally, in Section 7, we prove a weaker bound on $u_{k,\ell}/d_{k,\ell}$ when $k = 3$ that holds for all ℓ. Thus we prove the Meagher-Moura Conjecture for $t = 2$, $k = 3$ for all values of ℓ.

3 Representations of the symmetric group

In this section we will explain the connection between the eigenvalues of the graph $X_{k,\ell}$ and the irreducible characters of the symmetric group. We also recall some results on the degree of the irreducible characters that are involved in the eigenvalues.

For any character χ of $\mathrm{Sym}(n)$, we can consider its restriction to $H \leq \mathrm{Sym}(n)$ which is denoted by $\mathrm{res}\,(\chi)_H$. Similarly if χ is a character of $H \leq \mathrm{Sym}(n)$, then its induced character on $\mathrm{Sym}(n)$ is denoted by $\mathrm{ind}\,(\chi)^{\mathrm{Sym}(n)}$. The trivial character on a group H is denoted by 1_H.

The stabilizer of a partition in $\mathcal{U}_{k,\ell}$ is the group $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ (this is called the *wreath product* of $\mathrm{Sym}(k)$ and $\mathrm{Sym}(\ell)$). The cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$ are in one-to-one correspondence with the partitions of $\mathcal{U}_{k,\ell}$. The action of $\mathrm{Sym}(k\ell)$ on the partitions is equivalent to the action of $\mathrm{Sym}(k\ell)$ on the cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell))$ and this action is clearly transitive. The permutation representation of this action is

$$\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}.$$

The module for this representation can be thought of as the vector space of length-$u_{k,\ell}$ vectors with the characteristic vectors of $P \in \mathcal{U}_{k,\ell}$, denoted by v_P, as its basis. The group $\mathrm{Sym}(k\ell)$ acts on this vector space by the action on the partitions, for any $\sigma \in \mathrm{Sym}(k\ell)$ the action is $\sigma(v_P) = v_{P^\sigma}$.

This representation can be decomposed as the sum of irreducible representations of $\mathrm{Sym}(k\ell)$. If the multiplicity of each irreducible representation in the decomposition is equal to 1, then the representation is called *multiplicity-free*. In general, the group $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ is not multiplicity free in $\mathrm{Sym}(k\ell)$. In fact it is not multiplicity free unless $k = 2$, $\ell = 2$, or (k, ℓ) is one of $(3, 3)$, $(4, 3)$, $(5, 3)$ or $(3, 4)$ [11].

3.1 Orbital association scheme

The set of *orbitals* of the action of a group G on a set Ω is the set of orbits of the action of G on $\Omega \times \Omega$. Each orbital of $\mathrm{Sym}(k\ell)$ on $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$ can be represented by an object called a *meet table*. The meet table for two (k, ℓ)-partitions is a $\ell \times \ell$ array in which the (i, j)-entry is $|P_i \cap Q_j|$. Two meet tables are *isomorphic* if one can be obtained from the other by permuting the rows and the columns. In [12, Section 15.4] it is shown that the set of non-isomorphic meet tables corresponds to the set of orbitals. For each orbital $\mathcal{O}$ there is a corresponding meet table M; this means for $P, Q \in \mathcal{U}_{k,\ell}$ the meet table of P and Q is M if and only if $(P, Q) \in \mathcal{O}$. Further, each orbital can be represented as a $u_{k,\ell} \times u_{k,\ell}$ matrix, with the (P, Q)-entry equal to 1 if and only if the meet table of P and Q is isomorphic to the table representing the orbital. The set of these $u_{k,\ell} \times u_{k,\ell}$-matrices of the orbitals forms an *association scheme* if and only if $\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$ is multiplicity-free. In general, these matrices form a *homogeneous coherent configuration*.

The graph $X_{k,\ell}$ is the union of the orbitals from the action of $\mathrm{Sym}(k\ell)$ on the cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$ that are represented by a meet table that has no entry greater than 1. This means for every permutation $\sigma \in \mathrm{Sym}(k\ell)$ its action on the partitions is an automorphism of $X_{k,\ell}$. In particular, if M_σ is the permutation representation of σ, then

$$M_{\sigma^{-1}}\, A(X_{k,\ell})\, M_\sigma = A(X_{k,\ell}).$$

Further, if v is any θ-eigenvector of $X_{k,\ell}$, then $M_\sigma v$ is also a θ-eigenvector. This implies the eigenspaces of $X_{k,\ell}$ are invariant under the action of $\mathrm{Sym}(k\ell)$ and thus a union of irreducible modules in the decomposition of

$$\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}.$$

We say that an eigenvalue θ *belongs* to a module if the module is a subspace of the θ-eigenspace.

3.2 Degree of the irreducible characters of $\mathrm{Sym}(k\ell)$

In this section we will give some results on the irreducible representations of $\mathrm{Sym}(n)$. We refer the reader to [24], or any similar reference on this topic, for details and background. It is well-known that the irreducible representations of $\mathrm{Sym}(n)$ correspond to integer partitions on n. We will use $\lambda \vdash n$ to indicate that λ is an integer partition of n, this means that $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_j]$, each λ_i is an integer with $\sum_{i=1}^{j} \lambda_i = n$. We will use χ_λ to represent the irreducible character of $\mathrm{Sym}(n)$ corresponding to the partition λ.

From [13] we have a list of irreducible representations of the symmetric group with small degree.

Lemma 3.1. *For $n \geq 9$, let χ be a character of $\mathrm{Sym}(n)$ with degree less than $(n^2-n)/2$. If χ_λ is a constituent of χ, then λ is one of the following partitions of n:*

$$[n],\ [1^n],\ [n-1,1],\ [2,1^{n-2}],\ [n-2,2],\ [2,2,1^{n-4}],\ [n-2,1,1],\ [3,1^{n-3}].$$

This proof uses the branching rule, which we state here. For a proof of this rule see [3, Corollary 3.3.11].

Lemma 3.2. *Let $\lambda \vdash n$, then*

$$\mathrm{res}\,(\chi_\lambda)_{\mathrm{Sym}(n-1)} = \sum \chi_{\lambda^-},$$

where the sum is taken over all partitions λ^- of $n-1$ that have a Young diagram which can be obtained by the deletion of a single box from the Young diagram of λ. Further,

$$\mathrm{ind}\,(\chi_\lambda)^{\mathrm{Sym}(n+1)} = \sum \chi_{\lambda^+},$$

where the sum is taken over partitions λ^+ of $n+1$ that have a Young diagram which can be obtained by the addition of a single box to Young diagram of λ. □

Using the same approach as the proof for Lemma 3.1 we can get a second family of irreducible characters with slightly larger, but still small degree.

Lemma 3.3. *For $n \geq 13$, let χ be an irreducible character of $\mathrm{Sym}(n)$ with degree less than $\binom{n}{3} - \binom{n}{2}$. If χ_λ is a constituent of χ, then λ is one of the following partitions of n:*

$$[n], \quad [1^n], \quad [n-1,1], \quad [2,1^{n-2}], \quad [n-2,2], \quad [2,2,1^{n-4}],$$
$$[n-2,1,1], \quad [3,1^{n-3}], \quad [n-3,3], \quad [2,2,2,1^{n-6}].$$

Proof. The hook length formula confirms that each of the 10 characters above have degree less than or equal to $\binom{n}{3} - \binom{n}{2}$.

We prove this result by induction. For $n = 13$ and 14 this can be calculated directly using the GAP character table library [10]. We assume for $n \geq 14$ that the lemma holds for n and $n-1$, and we will prove that the lemma holds for $n+1$.

Assume that χ is an irreducible character of $\mathrm{Sym}(n+1)$ that has dimension less than

$$\binom{n+1}{3} - \binom{n+1}{2} = \frac{(n+1)n(n-4)}{6},$$

but is not one of the ten irreducible characters listed in the statement of the lemma. We will show that such a χ cannot exist.

If one of the ten irreducible characters of $\mathrm{Sym}(n)$ with degree less than $\binom{n}{3} - \binom{n}{2}$ is a constituent of $\mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}$, then we can determine the possible constituents of χ with the branching rule.

Constituent of $\mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}$	Constituents of χ
$[n]$	$[n+1]$, $[n,1]$
$[n-1,1]$	$[n,1]$, $[n-1,2]$, $[n-1,1,1]$
$[n-2,2]$	$[n-1,2]$, $[n-2,3]$, $[n-2,2,1]$
$[n-2,1,1]$	$[n-1,1,1]$, $[n-2,2,1]$, $[n-2,1,1,1]$
$[n-3,3]$	$[n-2,3]$, $[n-3,4]$, $[n-3,3,1]$
$[1^n]$	$[2,1^{n-1}]$, $[1^{n+1}]$
$[2,1^{n-2}]$	$[3,1^{n-2}]$, $[2,2,1^{n-3}]$, $[2,1^{n-1}]$
$[2,2,1^{n-4}]$	$[3,2,1^{n-4}]$, $[2,2,2,1^{n-5}]$, $[2,2,1^{n-3}]$
$[3,1^{n-3}]$	$[4,1^{n-3}]$, $[3,2,1^{n-4}]$, $[3,1^{n-2}]$
$[2,2,2,1^{n-6}]$	$[3,2,2,1^{n-6}]$, $[2,2,2,2,1^{n-7}]$, $[2,2,2,1^{n-5}]$

Table 1: Constituents of χ, if $\mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}$ has a constituent with degree less than $\binom{n}{3} - \binom{n}{2}$.

By Frobenius reciprocity, for any character ϕ of $\mathrm{Sym}(n)$

$$\langle \mathrm{res}\,(\chi)_{\mathrm{Sym}(n)}, \phi \rangle_{\mathrm{Sym}(n)} = \langle \chi, \mathrm{ind}\,(\phi)^{\mathrm{Sym}(n+1)} \rangle_{\mathrm{Sym}(n+1)}.$$

Character	Degree
$[n-3,4]$	$(n+1)n(n-1)(n-7)/24$
$[n-3,3,1]$	$(n+1)n(n-2)(n-5)/8$
$[n-2,2,1]$	$(n+1)(n-1)(n-3)/3$
$[n-2,1,1,1]$	$n(n-1)(n-2)/6$
$[2,2,2,2,1^{n-8}]$	$(n+1)n(n-1)(n-7)/24$
$[3,2,2,1^{n-6}]$	$(n+1)n(n-2)(n-5)/8$
$[3,2,1^{n-4}]$	$(n+1)(n-1)(n-3)/3$
$[4,1^{n-3}]$	$n(n-1)(n-2)/6$

Table 2: Degrees of the characters from Table 1 that are larger than $\frac{(n+1)n(n-4)}{6}$ for $n \geq 13$.

This means if ϕ is a constituent of $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$, then χ is one of the constituents of $\operatorname{ind}(\phi)^{\operatorname{Sym}(n+1)}$. The possible constituents of $\operatorname{ind}(\phi)^{\operatorname{Sym}(n+1)}$ are recorded in Table 1; the second column lists the irreducible characters that, according to the branching rule, are constituents of representation of $\operatorname{Sym}(n+1)$ induced by the character in the first column.

From these lists, and the degrees of the characters given in Table 2, we see that either χ is one of the ten listed in the theorem, or the degree of χ is larger than $\binom{n}{3} - \binom{n}{2}$ (again, the degrees are calculated using the hook length formula). Thus $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ does not contain any of the ten irreducible characters of $\operatorname{Sym}(n)$ in the statement of the theorem.

Next consider the case where the decomposition of $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ contains at least two irreducible characters of $\operatorname{Sym}(n)$ which are not in the list of the ten irreducible characters with dimension less $\binom{n}{3} - \binom{n}{2} = n(n-1)(n-5)/6$. In this case, the degree of χ must be at least $n(n-1)(n-5)/3$. But since $n > 7$, this is strictly larger than $(n+1)n(n-4)/6$.

Finally we need to consider the case where $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ contains exactly one irreducible character of $\operatorname{Sym}(n)$, which is not one of the ten listed in the theorem. By the branching rule the only irreducible characters of $\operatorname{Sym}(n+1)$ for which $\operatorname{res}(\chi)_{\operatorname{Sym}(n)}$ contains only one irreducible character have a rectangular Young diagram, so $\chi = \chi_{[s^t]}$ for some s and t.

Next consider $\operatorname{res}(\chi)_{\operatorname{Sym}(n-1)}$, this is the restriction of $\chi = \chi_{[s^t]}$ to $\operatorname{Sym}(n-1)$. By the branching rule, this can contain only the irreducible characters of $n-1$ that correspond to the partitions $\lambda' = [s^{t-1}, s-2]$ and $\lambda'' = [s^{t-2}, s-1, s-1]$.

If λ' is one of the ten partitions that correspond to irreducible characters of $\operatorname{Sym}(n-1)$ with degree less than $\binom{n-1}{3} - \binom{n-1}{2}$, then one of the following cases must hold:

- $t = 1$ and $\lambda' = [n-1]$ and $s = n+1$,
- $t = 2$ and $\lambda' = [n-1,1]$, $[n-2,2]$ or $[n-3,3]$, and $s \leq 5$, or
- $2 < t < 4$ and and $s \leq 2$.

The first of these cases implies $\chi = [n+1]$, which contradicts the degree of χ, and none of the other cases can happen, since $n = st$ and n is assumed to be at least 13.

Similarly, assume $\lambda'' = [s^{t-2}, s-1, s-1]$ is one of the partitions corresponding to the ten characters of $\operatorname{Sym}(n-1)$ that have degree less than $\binom{n-1}{3} - \binom{n-1}{2}$. Then one of the following cases must hold:

- $t = 2$ and $\lambda'' = [s-1, s-1]$ and $s \leq 4$,
- $2 < t \leq 5$ and $\lambda'' = [s^{t-1}, 1, 1]$ and $s \leq 2$, or
- $s = 1$.

The first two cases imply that $n \leq 10$ and the final case implies that $\chi = [1^{(n+1)}]$ which has degree 1.

Thus $\operatorname{res}(\chi)_{\mathrm{Sym}(n-1)}$ has two characters with degree at least $\binom{n-1}{3} - \binom{n-1}{2}$, so the degree of χ is at least $(n-1)(n-2)(n-6)/3$, which is strictly greater than $(n+1)n(n-4)/6$ for $n \geq 13$. This is a contradiction, so no such χ exists. □

Next we will show that there are only three irreducible characters in the decomposition of $\operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$ that have degree no more than $\binom{k\ell}{3} - \binom{k\ell}{2}$. To do this we will consider the action of different Young subgroups on $\mathcal{U}_{k,\ell}$. For any integer partition $\lambda \vdash n$ we will denote the Young subgroup by

$$\mathrm{Sym}(\lambda) = \mathrm{Sym}(\lambda_1) \times \mathrm{Sym}(\lambda_2) \times \cdots \times \mathrm{Sym}(\lambda_k).$$

Theorem 3.4. *Assume $k\ell \geq 13$ and $k \geq 3$. Then the only partitions in the decomposition of $\operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$ with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ are*

$$\chi_{[k\ell]}, \quad \chi_{[k\ell-2,2]}, \quad \chi_{[k\ell-3,3]}.$$

Proof. Lemma 3.3 lists the 10 irreducible representations of $\mathrm{Sym}(k\ell)$ with dimension no more than $\binom{k\ell}{3} - \binom{k\ell}{2}$. We only need to show which of these representations are in the decomposition. The tool we use is Frobenius reciprocity along with the action of different Young subgroups on $\mathcal{U}_{k,\ell}$.

By Frobenius reciprocity

$$\left\langle \operatorname{ind}(1_{\mathrm{Sym}(\lambda)})^{\mathrm{Sym}(k\ell)}, \quad \operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)} \right\rangle_{\mathrm{Sym}(k\ell)} = \left\langle 1, \quad \operatorname{res}(\operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)})_{\mathrm{Sym}(\lambda)} \right\rangle_{\mathrm{Sym}(\lambda)}.$$

The second inner product above gives the number of orbits of the action of $\mathrm{Sym}(\lambda)$ on the cosets $\mathrm{Sym}(k\ell)/(\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell))$; or, equivalently, the number of orbits of $\mathrm{Sym}(\lambda)$ on the partitions in $\mathcal{U}_{k,\ell}$. Using this fact with different Young subgroups will allow us to determine that many of the representations with small degree do not occur in the decomposition of $\operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$.

To start, it is clear that $\mathrm{Sym}(k\ell)$ has one orbit on the (k,ℓ)-partitions, so $\chi_{[k\ell]}$ has multiplicity 1 in the decomposition. Next consider the group $\mathrm{Sym}([k\ell-1,1])$, it is also straight-forward that this group only has one orbit on the partitions. Using the definition of the Specht modules and the labelling of the irreducible characters of the symmetric group it is straight forward to see that

$$\operatorname{ind}(1_{\mathrm{Sym}([k\ell-1,1])})^{\mathrm{Sym}(k\ell)} = \chi_{[k\ell]} + \chi_{[k\ell-1,1]},$$

so we have that

$$\left\langle \chi_{[k\ell]} + \chi_{[k\ell-1,1]}, \operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)} \right\rangle = 1.$$

Since we know that $\chi_{[k\ell]}$ occurs in this decomposition with multiplicity 1, this implies that $\chi_{[k\ell-1,1]}$ does not occur in the decomposition of $\operatorname{ind}(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$.

Next we consider the group $\mathrm{Sym}([k\ell-2,2])$. This group has two orbits on the partitions of $\mathcal{U}_{k,\ell}$. Again, from the definition of the Specht modules and the labelling of the irreducible characters, we have that

$$\left\langle \operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)}, \quad \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)} \right\rangle =$$
$$\left\langle \chi_{[k\ell]}+\chi_{[k\ell-1,1]}+\chi_{[k\ell-2,2]}, \quad \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)} \right\rangle = 2.$$

This implies that $\chi_{[k\ell-2,2]}$ occurs in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ with multiplicity 1.

We continue this process with the group $\mathrm{Sym}([k\ell-2,1,1])$. It has two orbits on the partitions of $\mathcal{U}_{k,\ell}$. Since

$$\operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-2,1,1])}\right)^{\mathrm{Sym}(k\ell)} = \chi_{[k\ell]}+\chi_{[k\ell-1,1]}+\chi_{[k\ell-2,2]}+\chi_{[k\ell-2,1,1]},$$

we conclude that $\chi_{[k\ell-2,1,1]}$ does not occur in the decomposition.

Next, we consider the group $\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,2])$. This group has two orbits on the partitions of $\mathcal{U}_{k,\ell}$. Again the decomposition of $\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)}$ is well-known (a proof can be found in [11, Proposition 1.4]) and we have

$$\begin{aligned}\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,2])}\right)^{\mathrm{Sym}(k\ell)} &= \chi_{[k\ell]}+\chi_{[k\ell-1,1]}+\chi_{[k\ell-2,2]}+\chi_{[k\ell-2,1,1]}\\ &\quad+\chi_{[1^{k\ell}]}+\chi_{[2,1^{k\ell-2}]}+\chi_{[2,2,1^{k\ell-4}]}+\chi_{[3,1^{k\ell-3}]}.\end{aligned}$$

This implies that none of $\chi_{[1^{k\ell}]}$, $\chi_{[2,1^{k\ell-2}]}$, $\chi_{[2,2,1^{k\ell-4}]}$ and $\chi_{[3,1^{k\ell-3}]}$ occur in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$.

Next we consider the group $\mathrm{Sym}([k\ell-3,3])$. This group has three orbits on the partitions of $\mathcal{U}_{k,\ell}$ and from the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-3,3])}\right)^{\mathrm{Sym}(k\ell)}$ we have that

$$\langle\operatorname{ind}\left(1_{\mathrm{Sym}([k\ell-3,3])}\right)^{\mathrm{Sym}(k\ell)}, \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}\rangle =$$
$$\langle\chi_{[k\ell]}+\chi_{[k\ell-1,1]}+\chi_{[k\ell-2,2]}+\chi_{[k\ell-3,3]}, \operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}\rangle = 3.$$

This implies that $\chi_{[k\ell-3,3]}$ occurs in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ with multiplicity 1.

Next we consider the group $\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,1,1])$. This group has 2 orbits on the partitions of $\mathcal{U}_{k,\ell}$ and

$$\begin{aligned}\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-2,1,1])}\right)^{\mathrm{Sym}(k\ell)} &= \chi_{[k\ell]}+\chi_{[k\ell-1,1]}+\chi_{[k\ell-2,2]}+\chi_{[k\ell-2,1,1]}\\ &\quad\chi_{[1^{k\ell}]}+\chi_{[2,1^{k\ell-2}]}+\chi_{[2,2,1^{k\ell-4}]}+\chi_{[3,1^{k\ell-3}]}.\end{aligned}$$

Since $\chi_{[k\ell]}$, and $\chi_{[k\ell-2,2]}$ are in the decomposition, none of the irreducible representations $\chi_{[1^{k\ell}]}$, $\chi_{[2,1^{k\ell-2}]}$, $\chi_{[2,2,1^{k\ell-4}]}$, or $\chi_{[3,1^{k\ell-3}]}$ occur in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$.

Finally, we consider the group $\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-3,3])$. This group has three orbits on the partitions of $\mathcal{U}_{k,\ell}$ and

$$\begin{aligned}\operatorname{ind}\left(1_{\mathrm{Alt}(k\ell)\cap\mathrm{Sym}([k\ell-3,3])}\right)^{\mathrm{Sym}(k\ell)} &= \chi_{[k\ell]}+\chi_{[k\ell-1,1]}+\chi_{[k\ell-2,2]}+\chi_{[k\ell-3,3]}\\ &\quad+\chi_{[1^{k\ell}]}+\chi_{[2,1^{k\ell-2}]}+\chi_{[2^2,1^{k\ell-4}]}+\chi_{[2^3,1^{k\ell-6}]}.\end{aligned}$$

Which shows $\chi_{[2,2,2,1^{k\ell-6}]}$ is not in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$. $\square$

4 Eigenvalues of $X_{k,\ell}$ with $k \geq 3$

In this section we will find three of the eigenvalues of $X_{k,\ell}$. For ease of notation, we will denote the irreducible representation of χ_λ by the λ-module. Also, the number of vertices in $X_{k,\ell}$, which is equal to $u_{k,\ell}$, will be denoted simply by v and the degree of the graph $X_{k,\ell}$ will be simply written as d, rather than $d_{k,\ell}$.

Any subgroup $H \leq \mathrm{Sym}(k\ell)$ acts on the vertices of $X_{k,\ell}$ and the orbits of this action form an equitable partition. From any equitable partition, we can form a quotient graph and the eigenvalues of this quotient graph will be eigenvalues of the $X_{k,\ell}$ (details can be found in [13, Section 2.2]). The trivial case is $H = \mathrm{Sym}(k\ell)$, since this group is transitive, the equitable partition has all the vertices of $X_{k,\ell}$ in a single part. The quotient graph for this is simply the 1×1 matrix with the single entry d. The eigenvalue of this matrix is simply d, and the eigenvector is the all ones vector and the eigenspace is isomorphic to the trivial representation of $\mathrm{Sym}(k\ell)$. So d belongs to the $[k\ell]$-module.

Since the subgroup $\mathrm{Sym}([k\ell-1,1])$ has only one orbit on the vertices of $X_{k,\ell}$, the next subgroup we consider is the Young subgroup $\mathrm{Sym}([k\ell-2,2])$, considered as the stabilizer of the set $\{1,2\}$. The action of $\mathrm{Sym}([k\ell-2,2])$ on the partitions will give us another eigenvalue of the graph.

Lemma 4.1. *For integers k and ℓ, with $k,\ell \geq 2$, $\tau = -\frac{(k-1)d}{k(\ell-1)}$ is an eigenvalue of $X_{k,\ell}$ with multiplicity at least $\binom{k\ell}{2} - \binom{k\ell}{1}$.*

Proof. The action of $\mathrm{Sym}([k\ell-2,2])$ on the (k,ℓ)-partitions has exactly 2 orbits: S_1, the set of all partitions that have 1 and 2 in the same block, and S_2, the set of all partitions in which 1 and 2 are in different blocks. The orbit S_1 is a coclique in $X_{k,\ell}$ so the quotient matrix for this partition has the form

$$\begin{pmatrix} 0 & d \\ -\tau & d+\tau \end{pmatrix}. \tag{4.1}$$

The eigenvalues of the quotient matrix (4.1) are d and τ. We can calculate the value of τ by counting edges between S_1 and S_2. Since S_1 is a coclique, each vertex in it is adjacent to exactly d vertices in S_2, and each vertex in S_2 is adjacent to $-\tau$ vertices in S_1. Using the sizes of S_1 and S_2, we have that the number of edges between S_1 and S_2 is equal to

$$|S_1|\, d = \binom{k\ell-2}{k-2} u_{k,\ell-1}\, d$$

and also to

$$|S_2|(-\tau) = \binom{k\ell-2}{k-1}\binom{k\ell-k-1}{k-1} u_{k,\ell-2}(-\tau).$$

Thus

$$\tau = -\frac{(k-1)d}{k(\ell-1)} \tag{4.2}$$

is a second eigenvalue for $X_{k,\ell}$. Since this eigenvalue arises from the action of $\mathrm{Sym}([k\ell-2,2])$, it belongs to a module that is common between the two representations $\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$ and $\mathrm{ind}\,(1_{\mathrm{Sym}([k\ell-2,2])})^{\mathrm{Sym}(k\ell)}$. Thus it belongs to the module $[k\ell-2,2]$, as this is the only common module, and must have dimension at least $\binom{k\ell}{2} - \binom{k\ell}{1}$. □

We denote this eigenvalue by τ since in Section 6 it will be shown that τ is the least eigenvalue of $X_{k,\ell}$, provided that ℓ is sufficiently large. We also note that a second irreducible module may also have τ as the eigenvalue belonging to it, so the multiplicity of τ could be higher than the degree of the $[k\ell - 2, 2]$-module.

Next we will consider the Young subgroup $\mathrm{Sym}([k\ell - 3, 3])$, thought of as the group that stabilizes the set $\{1, 2, 3\}$. The action of this subgroup on $\mathcal{U}_{k,\ell}$ has 3 orbits: T_1, the set of all partitions with $1, 2, 3$ in the same block; T_2 the set of all partitions in which $1, 2, 3$ are in exactly two different blocks; and T_3 the set of all partitions in which $1, 2, 3$ are in three different blocks. Any vertex in T_1 is adjacent only to vertices in T_3. Similarly, a vertex in T_2 can be adjacent to vertices in T_2 and T_3. The quotient graph for this equitable partition is

$$M = \begin{pmatrix} 0 & 0 & d \\ 0 & a & d-a \\ b & c & d-b-c \end{pmatrix},$$

where a, b, c are all non-negative.

The eigenvalues for this quotient graph will be the eigenvalues that belong to modules that are both in the decomposition of $\mathrm{ind}\,(1_{\mathrm{Sym}([k\ell-3,3])})^{\mathrm{Sym}(k\ell)}$ and the decomposition of $\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}$. Thus the eigenvalues will belong to the $[k\ell]$, $[k\ell - 2, 2]$ and $[k\ell - 3, 3]$ modules. We have already seen that the eigenvalue for $[k\ell]$ is d, and the eigenvalue for $[k\ell - 2, 2]$ is τ. We will denote the eigenvalue belonging to $[k\ell - 3, 3]$ by θ.

Since the trace of the matrix is the sum of the eigenvalues we have that

$$d + a - b - c = d + \tau + \theta. \tag{4.3}$$

The number of edges between T_1 and T_3 is equal to

$$d|T_1| = d\binom{k\ell - 3}{k-3} u_{k,\ell-1},$$

and also to

$$b|T_3| = b\binom{k\ell - 3}{k-1}\binom{k\ell - k - 2}{k-1}\binom{k\ell - 2k - 1}{k-1} u_{k,\ell-3}.$$

Setting these equations equal to each other, then expanding the binomial coefficients and rearranging yields

$$\frac{(k-1)(k-2)}{k^2(\ell-1)(\ell-2)} d = b.$$

Replacing $d = -\frac{k(\ell-1)}{k-1}\tau$ shows that

$$b = -\frac{(k-1)(k-2)}{k^2(\ell-1)(\ell-2)} \frac{k(\ell-1)}{(k-1)} \tau = -\frac{k-2}{k(\ell-2)} \tau. \tag{4.4}$$

Putting this into Equation 4.3 produces the following formula

$$\theta = a + \frac{k-2}{k(\ell-2)} \tau - c - \tau = a - c + \frac{(k-2) - k(\ell-2)}{k(\ell-2)} \tau. \tag{4.5}$$

Similarly, counting the number of edges between T_2 and T_3 yields

$$3\binom{k\ell-3}{k-2}\binom{k\ell-k-1}{k-1}u_{k,\ell-2}(d-a) =$$
$$\binom{k\ell-3}{k-1}\binom{k\ell-k-2}{k-1}\binom{k\ell-2k-1}{k-1}u_{k,\ell-3}(c).$$

Again, expanding the binomial coefficients and rearranging shows that

$$a = d - \frac{(\ell-2)k}{3(k-1)}c.$$

The characteristic polynomial of M is

$$x^3 + (-a+b+c-d)x^2 + (-ab+ad-bd-cd)x + abd.$$

Substituting in the values we have computed for b and c, and using the fact that τ is a root of the characteristic polynomial we get

$$a = \frac{2(k-1)}{k(\ell-1)}d. \tag{4.6}$$

From this we can compute that

$$c = \frac{3(k\ell-3k+2)(k-1)}{k^2(\ell-1)(\ell-2)}d. \tag{4.7}$$

Lemma 4.2. *For integers k and ℓ, with $k, \ell \geq 3$,*

$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}$$

is an eigenvalue of $X_{k,\ell}$ with multiplicity at least $\binom{k\ell}{3} - \binom{k\ell}{2}$.

Proof. By Equations (4.5), (4.6) and (4.7), we can calculate that

$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}. \tag{4.8}$$

From the comments above, $\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}$ is the eigenvalue belonging to the unique $[k\ell-3,3]$-module in $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$. Since the dimension of the irreducible representation of $[k\ell-3,3]$ is $\binom{k\ell}{3} - \binom{k\ell}{2}$, the multiplicity of θ is at least $\binom{k\ell}{3} - \binom{k\ell}{2}$. $\square$

5 Bound on degree of $X_{k,\ell}$

In this section we will find a lower bound on the degree of $X_{k,\ell}$ for all sufficiently large ℓ. If P and Q are two partitions that are adjacent in $X_{k,\ell}$, then the meet table of P and Q is an $\ell \times \ell$ matrix with entries either 0 or 1, and further, the entries in each row and column in the meet table sum to k. We define $\mathcal{M}_{k,\ell}$ to be the set of all such meet tables, so all $\ell \times \ell$ matrices with entries either 0 or 1, and row and columns sums equal to k. To find the degree of $X_{k,\ell}$, we first state a result on the number of such meet tables. Next, for a fixed partition P and a meet table $M \in \mathcal{M}_{k,\ell}$, we count the number of partitions Q for which the meet table of P and Q is M.

Bender [1] determined the asymptotic cardinality of $\mathcal{M}_{k,\ell}$. (In fact, Bender found a much more general result, but we only state the result that we need here.)

Theorem 5.1 (Bender [1]). *For positive integers k, ℓ*

$$\lim_{\ell \to \infty} \frac{(k!)^{2\ell}}{(k\ell)!} |\mathcal{M}_{k,\ell}| = e^{-\frac{(k-1)^2}{2}}.$$

To get a lower bound on $d_{k,\ell}$, we fix a partition P in $\mathcal{U}_{k,\ell}$, then for each $M \in \mathcal{M}_{k,\ell}$, we will count the number of Q so that the meet table of P and Q is M, then we use Theorem 5.1 to bound the size of $\mathcal{M}_{k,\ell}$.

Lemma 5.2. *For positive integers k, ℓ with $k \leq \ell$,*

$$d_{k,\ell} = \frac{k!^\ell}{\ell!} |\mathcal{M}_{k,\ell}|.$$

Proof. Fix a partition $P \in \mathcal{U}_{k,\ell}$. Define a bipartite multigraph with the vertices in one part the meet tables in $\mathcal{M}_{k,\ell}$, and the vertices in the other part the partitions in the neighbourhood of P in $X_{k,\ell}$. Two vertices M and Q are adjacent if the meet table of P and Q is M. By counting the number of edges in this graph in two ways, we will determine the size of the neighbourhood of P in terms of $|\mathcal{M}_{k,\ell}|$.

For any $M \in \mathcal{M}_{k,\ell}$, with $M = [m_{i,j}]$ assume that row i corresponds to the block $P_i \in P$. Construct a partition $Q = \{Q_1, Q_2, \ldots, Q_\ell\}$ so that the block Q_j corresponds to column j of M and $|P_i \cap Q_j| = m_{i,j}$. Since the entries of a row in M are either 0 or 1, and sum to k, there are $k!$ ways to select how the elements from P_i will be distributed to the blocks of Q. So for each meet table M, there are $k!^\ell$ partitions Q that can be constructed this way. It is possible that some of these partitions are equal, once the blocks are reordered, so this is a multigraph.

For every Q in the neighbourhood of P, there are $\ell!$ ways to order the blocks of Q, once the blocks are ordered the meet table for P and Q is uniquely defined. In the bipartite graph, Q is adjacent to each of these tables in the graph (again, these tables may not be distinct, so the graph is a multigraph). The degree of every vertex Q is $\ell!$.

Thus we have that the number of edges in the multigraph is

$$\ell! d_{k,\ell} = \sum_{M \in \mathcal{M}_{k,\ell}} k!^\ell,$$

and the result follows. □

Using Theorem 5.1 we have the asymptotic size of $d_{k,\ell}$.

Corollary 5.3. *For a fixed integer k with $k \geq 2$,*

$$\lim_{\ell \to \infty} \frac{u_{k,\ell}}{d_{k,\ell}} = e^{\frac{(k-1)^2}{2}}.$$

Proof. From Equation (1.1) and Lemma 5.2,

$$\frac{u_{k,\ell}}{d_{k,\ell}} = \frac{(k\ell)!}{(k!)^{\ell}\ell!}\frac{\ell!}{k!^{\ell}|\mathcal{M}_{k,\ell}|} = \frac{(k\ell)!}{(k!)^{2\ell}|\mathcal{M}_{k,\ell}|}.$$

The result then follows from Theorem 5.1. □

Thus for every $\epsilon > 0$, there exists an ℓ' such that for all $\ell \geq \ell'$,

$$\frac{u_{k,\ell}}{d_{k,\ell}} \leq e^{\frac{(k-1)^2}{2}} + \epsilon.$$

6 A bound on the multiplicity of eigenvalues with large absolute value

In Section 4 we found three eigenvalues, d, τ, and θ of $X_{k,\ell}$, and the ratio between d and τ is $\frac{d}{\tau} = \frac{k(1-\ell)}{k-1}$. If τ is the least eigenvalue of the graph, then by the ratio bound any coclique will have size no more than

$$\frac{|V(X_{k,\ell})|}{1-\frac{d}{\tau}} = \frac{|V(X_{k,\ell})|}{1-\frac{k(1-\ell)}{k-1}} = u_{k,\ell-1}.$$

This is exactly the size of a set of canonically 2-intersecting (k,ℓ)-partitions. Thus our goal in this section is to show that τ is the least eigenvalue of $X_{k,\ell}$. To this end, we first show if $X_{k,\ell}$ has an eigenvalue λ with $\lambda^2 > \tau^2$, then there is a bound on the multiplicity of λ.

Let

$$\{d^{(1)}, \tau^{(m_\tau)}, \theta^{(m_\theta)}, \lambda_2^{(m_2)}, \dots, \lambda_j^{(m_j)}\}$$

be the spectrum of the matrix $X_{k,\ell}$, where the values m_i represent the multiplicities of the eigenvalues. By squaring A and taking the trace, we have

$$vd = d^2 + m_\tau\tau^2 + m_\theta\theta^2 + \sum_{i=2}^{j} m_i\lambda_i^2.$$

Hence for every $2 \leq i \leq j$ we have

$$vd - d^2 - m_\tau\tau^2 - m_\theta\theta^2 \geq m_i\lambda_i^2.$$

Assume λ_i is an eigenvalue of $X_{k,\ell}$ with $\lambda_i^2 > \tau^2$, and also that λ_i is not the eigenvalue belonging to the $[k\ell]$, $[k\ell-2,2]$ or $[k\ell-3,3]$ modules, then

$$\frac{vd - d^2 - m_\tau\tau^2 - m_\theta\theta^2}{\tau^2} \geq m_i.$$

Expanding θ using Equation (4.8) in the above equation produces the following equation

$$\left(\frac{v}{d}-1\right)\frac{k^2(\ell-1)^2}{(k-1)^2} - m_\theta\frac{4(k-2)^2}{k^2(\ell-2)^2} - m_\tau \geq m_i.$$

Further, by Lemmas 4.1 and 4.2, it is known that $m_\tau \geq \binom{k\ell}{2} - \binom{k\ell}{1}$ and $m_\theta \geq \binom{k\ell}{3} - \binom{k\ell}{2}$, so this bound becomes

$$\left(\frac{v}{d}-1\right)\frac{k^2(\ell-1)^2}{(k-1)^2} - \frac{(k\ell)(k\ell-1)(k\ell-5)}{6}\frac{4(k-2)^2}{k^2(\ell-2)^2} - \frac{(k\ell)(k\ell-3)}{2} \geq m_i.$$

Our next step is to show that this upper bound on m_i is smaller than $\binom{k\ell}{3} - \binom{k\ell}{2}$. This will be a contradiction with Theorem 3.4 since we have assumed that λ does not belong to any of the $[k\ell]$, $[k\ell-2,2]$, and $[k\ell-3,3]$ modules. In other words, we need to prove that

$$\frac{v}{d} - 1 < \frac{\ell(k-1)^2\left(k^2(\ell-2)^2(k\ell-4)(k\ell+1) + 4(k-2)^2(k\ell-1)(k\ell-5)\right)}{6k^3(\ell-1)^2(\ell-2)^2}. \quad (6.1)$$

In the next result, we will see that this will follow from Corollary 5.3.

Theorem 6.1. *Fix an integer $k \geq 3$. For ℓ sufficiently large, the largest set of partially 2-intersecting uniform (k,ℓ)-partitions has size*

$$\binom{k\ell-2}{k-2} u_{k,\ell-1}.$$

Proof. For any distinct $i, j \in \{1, \dots, k\ell\}$, the set $S_{i,j}$ of all (k,ℓ)-partitions with i and j in the same block form a set of partially 2-intersecting (k,ℓ)-partitions of the size given in the theorem.

Corollary 5.3 shows that $\frac{v}{d}$ approaches a fixed constant, namely $e^{\frac{(k-1)^2}{2}}$, as ℓ goes to infinity. Since the right hand side of Equation (6.1) grows linearly in ℓ, we have that Equation (6.1) holds for ℓ sufficiently large. This implies if there is an eigenvalue λ of $X_{k,\ell}$ with $\lambda \leq \tau$, then the multiplicity of λ is less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$.

By Theorem 3.4, eigenspaces with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ can only include the $[k\ell]$, $[k\ell-2,2]$ or the $[k\ell-3,3]$-modules. The degree, d, is the eigenvalue belonging to the $[k\ell]$-module, and Lemma 4.1 and Lemma 4.2 shows that τ is the eigenvalue belonging to the $[k\ell-2,2]$-module and θ belongs to the $[k\ell-3,3]$-module. So we can conclude that $\tau = -\frac{(k-1)d}{k(\ell-1)}$ is the least eigenvalue of $X_{k,\ell}$ and that τ belongs only to the $[k\ell-2,2]$-module.

By the ratio bound, Theorem 2.1, the maximum size of coclique in $X_{k,\ell}$ is

$$\frac{|V(X_{k,\ell})|}{1-\frac{d}{\tau}} = \frac{v}{1-\frac{d}{-\frac{(k-1)d}{k(\ell-1)}}} = \frac{v}{1+\frac{k(\ell-1)}{k-1}} = \frac{v(k-1)}{k\ell-1} = \binom{k\ell-2}{k-2} u_{k,\ell-1}. \qquad \square$$

The previous result shows that the sets $S_{i,j}$ are the largest intersecting sets. We further conjecture that these sets are the only maximum intersecting sets.

Conjecture 6.2. *For $k \geq 3$ and ℓ sufficiently large, the only sets of partially 2-intersecting (k,ℓ)-partitions with size $\binom{k\ell-2}{k-2}u_{k,\ell-1}$ are the sets $S_{i,j}$.*

We can make a step towards this conjecture with the following weaker characterization of the maximum intersecting sets. Denote the characteristic vectors of the sets $S_{i,j}$ by $v_{i,j}$.

Corollary 6.3. *For a fixed integer $k \geq 3$ and ℓ sufficiently large, let S be any maximum partially 2-intersecting set of (k,ℓ)-partitions. Then the characteristic vector of S is a linear combination of the vectors $v_{i,j}$.*

Proof. For $k \geq 3$ and ℓ sufficiently large, $S_{i,j}$ is a maximum coclique in $X_{k,\ell}$ and equality holds in the ratio bound. Let $v_{i,j}$ be the characteristic vector of $S_{i,j}$. Since we have equality in the ratio bound, this implies that

$$v_{i,j} - \frac{k-1}{k\ell-1}\mathbf{1}$$

is a τ-eigenvector (where $\mathbf{1}$ is the all ones vector). Since no other modules have have eigenvalue τ, these vectors are in the $[k\ell-2,2]$-module. Further, the set of vectors

$$\left\{ v_{i,j} - \frac{k-1}{k\ell-1}\mathbf{1} \mid i,j \in \{1,\dots,k\ell\} \right\}$$

is invariant under the action of $\mathrm{Sym}(k\ell)$, so they form a module. Since the $[k\ell-2,2]$-module is irreducible, these vectors span the entire $[k\ell-2,2]$-module; this also implies that the vectors $\{v_{i,j} \mid i,j \in \{1,\dots,k\ell\}\}$ span the $[k\ell]$ and $[k\ell-2,2]$-modules.

Let S be a partially 2-intersecting set of (k,ℓ)-partition of maximum size, and let v_S denote the characteristic vector of S. Then $v_S - \frac{k-1}{k\ell-1}\mathbf{1}$ is in the $[k\ell-2,2]$-module. Thus v_S is in the span of the $[k\ell]$ and $[k\ell-2,2]$-module, so v_S is a linear combination of the vectors $v_{i,j}$. □

7 Exact result for $k = 3$

In this section we will prove Theorem 6.1 holds for all $\ell \geq 3$, when $k = 3$. It is already known, see Corollary 7.5.6 in [19], that Theorem 6.1 holds in the case where $k = 3$ and ℓ is odd; this follows from the existence of resolvable packing designs of an appropriate size.

For $k = 3$, we observed experimentally that the ratio $u_{3,\ell}/d_{3,\ell}$ converges to $e^{\frac{(k-1)^2}{2}} = e^2$ surprisingly quickly. If the sequence of $u_{3,\ell}/d_{3,\ell}$ was non-increasing this would be sufficient, but we have no proof of this. Rather, in this section we show an upper bound on the ratio $u_{3,\ell}/d_{3,\ell}$ for all ℓ, or, equivalently, a lower bound on $d_{3,\ell}$. This bound holds for $\ell > 10$, and we simply directly check the theorem for the specific graphs with smaller values of ℓ.

Lemma 7.1. *For $\ell > 10$, the degree, $d_{3,\ell}$ is greater than $u_{3,\ell}/24$.*

Proof. We will use a truncated inclusion-exclusion argument to bound the degree. Since $X_{k,\ell}$ is vertex transitive, we obtain a bound on the degree by counting the neighbours of an arbitrary partition $P \in \mathcal{U}_{3,\ell}$.

Fix a partition $P \in \mathcal{U}_{3,\ell}$ and let $\mathcal{J}$ be the set of pairs $\{x,y\}$ of elements in $\{1,2,\dots,3\ell\}$ that are contained in the same block of P. Note that $|\mathcal{J}| = 3\ell$. For a pair $\{x,y\} \in \mathcal{J}$, we define $A_{\{x,y\}}$ to be the set of all partitions which contain x and y in the same block. Further, for a subset $J \subseteq \mathcal{J}$, define

$$N(J) = |\cap_{\{x,y\}\in J} A_{\{x,y\}}|$$

and for $0 \leq j \leq 3\ell$ let $N_j = \sum_{J,|J|=j} N(J)$. By inclusion-exclusion,

$$d_{3,\ell} = \sum_{j=0}^{3\ell} -1^j N_j. \tag{7.1}$$

Next we calculate N_j. First, we note that $N_0 = N(\emptyset) = u_{3,\ell}$.

For any subset $J \subseteq \mathcal{J}$, each block in P contains either $0,1,2$ or 3 of the pairs from J. For $i = 0,1,2,3$, let n_i be the number of blocks in P that have exactly i of their pairs in J. We call the 4-tuple (n_0,n_1,n_2,n_3) the *pair distribution of* J and note that $n_0+n_1+n_2+n_3 = \ell$. For each block of P its *block type relative to* J is the number of pairs in J that are in the block. With this terminology, we can find N_j.

First, fix a subset $J \subseteq \mathcal{J}$ with pair distribution (n_0, n_1, n_2, n_3) and count the number of partitions $Q \in \cap_{\{x,y\} \in J} A_{\{x,y\}}$. Each block of P with block type n_3 relative to J determines exactly which three elements are in a block of Q, as do the blocks of type n_2. Each of the blocks of type n_1 determines two of the three points in the block of Q. One more point must be chosen to complete each of these blocks, and this choice is ordered since each pair of type n_1 from J uniquely labels its corresponding block. Each of the blocks of type n_0 does not determine any points in Q. Thus the number of partitions Q which contain the pairs from J is given by the multinomial coefficient

$$\frac{1}{n_0!}\binom{3\ell - 3(n_3+n_2) - 2n_1}{1^{(n_1)}, 3^{(n_0)}}$$

(where the exponent in braces indicates the number is repeated that many times).

We now count the number of possible J which have pair distribution (n_0, n_1, n_2, n_3). The number of ways to select the type of each block in P is equal to the multinomial coefficient

$$\binom{\ell}{n_0, n_1, n_2, n_3},$$

since we are choosing the blocks from $\mathcal{P}$ that have either $0, 1, 2$ or 3 pairs in J. Each of the blocks of P with type n_3 has all of its three pairs in J, while for each block of type n_2 there are three ways to choose which two of the three possible pairs are in J. Similarly, for each block of type n_1 there is one pair in J, and there are three ways to chose this pair. Finally, each of the n_0 blocks does not contribute any pairs to J. Thus there are

$$3^{n_1+n_2}$$

different sets J in with pair distribution (n_0, n_1, n_2, n_3).

Finally, we sum the number of partitions $Q \in \cap_{\{x,y\} \in J} A_{\{x,y\}}$ over all possible pair distributions that J can have. Each pair distribution (n_0, n_1, n_2, n_3) is an ordered partition of ℓ into exactly four non-negative parts. The pair distribution (n_0, n_1, n_2, n_3) corresponds to a set J of size $n_1 + 2n_2 + 3n_3$. Define $\mathcal{C}(\ell, j)$ to be the set of compositions of ℓ into four parts with $n_1 + 2n_2 + 3n_3 = j$.

Then from our previous counting we have that

$$N_j = \sum_{(n_0,n_1,n_2,n_3) \in \mathcal{C}(\ell,j)} 3^{n_1+n_2} \binom{\ell}{n_0, n_1, n_2, n_3} \frac{1}{n_0!} \binom{3\ell - 3(n_3+n_2) - 2n_1}{1^{(n_1)}, 3^{(n_0)}}.$$

When we put this value in Equation 7.1 and truncate this sum after an odd j we will get a lower bound on $d_{3,\ell}$. Taking j up to 5 we sum over the following list of pair distributions:

$$\mathcal{C}(\ell,0) = \{(\ell,0,0,0)\}$$
$$\mathcal{C}(\ell,1) = \{(\ell-1,1,0,0)\}$$
$$\mathcal{C}(\ell,2) = \{(\ell-1,0,1,0), (\ell-2,2,0,0)\}$$
$$\mathcal{C}(\ell,3) = \{(\ell-1,0,0,1), (\ell-2,1,1,0), (\ell-3,3,0,0)\}$$
$$\mathcal{C}(\ell,4) = \{(\ell-2,1,0,1), (\ell-2,0,2,0), (\ell-3,2,1,0), (\ell-4,4,0,0)\}$$
$$\mathcal{C}(\ell,5) = \{(\ell-2,0,1,1), (\ell-3,2,0,1), (\ell-3,1,2,0), (\ell-4,3,1,0), (\ell-5,5,0,0)\}$$

Expanding this becomes

$$d_{3,\ell} \geq \sum_{j=0}^{5} -1^j \sum_{(n_0,n_1,n_2,n_3)\in\mathcal{C}(\ell,j)} 3^{n_1+n_2} \binom{\ell}{n_0,n_1,n_2,n_3} \frac{\binom{3\ell-3(n_3+n_2)-2n_1}{1^{(n_1)},3^{(n_0)}}}{n_0!}$$
$$= \frac{\binom{\ell}{\ell}\binom{3\ell}{3^{(\ell)}}}{(\ell)!} + \frac{-3\binom{\ell}{\ell-1,1}\binom{3\ell-2}{1,3^{(\ell-1)}}}{(\ell-1)!} + \frac{3\binom{\ell}{\ell-1,1}\binom{3\ell-3}{3^{(\ell-1)}}}{(\ell-1)!} + \frac{3^2\binom{\ell}{\ell-2,2}\binom{3\ell-4}{1^{(2)},3^{(\ell-2)}}}{(\ell-2)!}$$
$$+ \frac{-\binom{\ell}{\ell-1,1}\binom{3\ell-3}{3^{(\ell-1)}}}{(\ell-1)!} + \frac{-3^2\binom{\ell}{\ell-2,1^{(2)}}\binom{3\ell-5}{1,3^{(\ell-2)}}}{(\ell-2)!} + \frac{-3^3\binom{\ell}{\ell-3,3}\binom{3\ell-6}{1^{(3)},3^{(\ell-3)}}}{(\ell-3)!}$$
$$+ \frac{3\binom{\ell}{\ell-2,1^{(2)}}\binom{3\ell-5}{1,3^{(\ell-2)}}}{(\ell-2)!} + \frac{3^2\binom{\ell}{\ell-2,2}\binom{3\ell-6}{3^{(\ell-2)}}}{(\ell-2)!} + \frac{3^3\binom{\ell}{\ell-3,2,1}\binom{3\ell-7}{1^{(2)},3^{(\ell-3)}}}{(\ell-3)!}$$
$$+ \frac{3^4\binom{\ell}{\ell-4,4}\binom{3\ell-8}{1^{(4)},3^{(\ell-4)}}}{(\ell-4)!} + \frac{-3\binom{\ell}{\ell-2,1^{(2)}}\binom{3\ell-6}{3^{(\ell-2)}}}{(\ell-2)!} + \frac{-3^2\binom{\ell}{\ell-3,2,1}\binom{3\ell-7}{1^{(2)},3^{(\ell-3)}}}{(\ell-3)!}$$
$$+ \frac{-3^3\binom{\ell}{\ell-3,1,2}\binom{3\ell-8}{1,3^{(\ell-3)}}}{(\ell-3)!} + \frac{-3^4\binom{\ell}{\ell-4,3,1}\binom{3\ell-9}{1^{(3)},3^{(\ell-4)}}}{(\ell-4)!} + \frac{-3^5\binom{\ell}{\ell-5,5}\binom{3\ell-10}{1^{(5)},3^{(\ell-5)}}}{(\ell-5)!}$$
$$= \frac{(243\ell^6 - 2997\ell^5 + 13905\ell^4 - 32355\ell^3 + 42732\ell^2 - 32728\ell + 11200)(3\ell-10)!}{80(6^{\ell-4})(\ell-10)!(\ell^6 - 39\ell^5 + 625\ell^4 - 5265\ell^3 + 24574\ell^2 - 60216\ell + 60480)}.$$

Thus

$$\frac{u_{3,\ell}}{d_{3,\ell}} \leq \frac{5(729\ell^6 - 6561\ell^5 + 23085\ell^4 - 40095\ell^3 + 35586\ell^2 - 14904\ell + 2240)}{243\ell^6 - 2997\ell^5 + 13905\ell^4 - 32355\ell^3 + 4273\ell^2 - 32728\ell + 11200}.$$

For $\ell > 10$ this gives that $u_{3,\ell}/d_{3,\ell} < 24$. □

Theorem 7.2. *For $k = 3$ and all $\ell \geq 3$ the largest set of partially 2-intersecting uniform partitions has size*

$$(3\ell - 2)u_{3,\ell-1}.$$

Proof. For $\ell = 3$ all the eigenvalues of $X_{3,3}$ have long been known to be $\{36, 8, 2, -4, -12\}$ [18]. The ratio bound holds with equality, and the only irreducible representation that belongs to the least eigenvalue is $\chi_{[7,2]}$.

For $\ell = 4$, all the eigenvalues of $X_{3,4}$ are $\{1296, 96, 72, 48, 32, 0, -24, -48, -288\}$. These can be calculated by making a quotient graph of $X_{3,4}$ from the action of $\mathrm{Sym}(3) \wr \mathrm{Sym}(4)$ on the partitions. This equitable partition has a cell of size 1, so the eigenvalues of the quotient graph are exactly the eigenvalues of $X_{3,4}$. Further, the multiplicities of the eigenvalues can be calculated using the formulas in [14, Section 5.3] and the $[10, 2]$-module is the only module to which the eigenvalue -288 belongs.

For $\ell \in \{5, \ldots, 12\}$ the only irreducible representations with dimension less then $\binom{3\ell}{3} - \binom{3\ell}{2}$ in the decomposition of $\mathrm{ind}\left(1_{\mathrm{Sym}(3)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(3\ell)}$ are the three listed in Theorem 3.4—this can be checked using GAP [10]. Thus Theorem 3.4 holds for all $5 \leq \ell \leq 12$ when $k = 3$.

For all $\ell > 10$, Lemma 7.1 shows that $u_{k,\ell}/d_{k,\ell} - 1 < 23$. In this same range, the right hand side of Equation (6.1) is at least 26. Thus the inequality from Equation (6.1) holds for all $\ell > 10$.

For $5 \le \ell \le 10$ the degrees $d_{3,\ell}$ can be directly computed

$$\begin{aligned} d_{3,5} &= 132192, & d_{3,7} &= 3829057920, & d_{3,9} &= 333973115062272, \\ d_{3,6} &= 19258560, & d_{3,8} &= 1001695548672, & d_{3,10} &= 138348645213579264, \end{aligned}$$

and the inequality from Equation (6.1) directly checked. □

8 Further work

In this paper we only consider partially 2-intersecting partitions, but the conjecture in [20] is for partial t-intersection sets of partitions with $k \le \ell(t-1)$. It is possible that the approach in this paper could be applied for larger values of t, but there are some steps that we predict will be complicated.

It is straight-forward to generalize the definition of $X_{k,\ell}$ to partially t-intersecting partitions by defining the graph $X_{t,k,\ell}$. This graph will also have $\mathcal{U}_{k,\ell}$ as its vertex set, and two partitions P and Q are adjacent if and only if for all pairs of blocks $P_i \in P$ and $Q_j \in Q$ we have $|P_i \cap Q_j| < t$. A partially t-intersecting set of partitions is a coclique in $X_{t,k,\ell}$.

The conjecture is if $k < \ell(t-1)$, then the maximum cocliques in $X_{t,k,\ell}$ are exactly the canonical partially t-intersecting sets. The Young subgroup $\mathrm{Sym}([k\ell - t, t])$ is the stabilizer of a canonically partially t-intersecting set. The most significant complication is that for $t > 2$, there are more than two irreducible representations in both

$$\mathrm{ind}\left(1_{\mathrm{Sym}([k\ell-t,t])}\right)^{\mathrm{Sym}(k\ell)} \quad \text{and} \quad \mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}. \tag{8.1}$$

For the approach given in this paper to work, we believe the eigenvalues belonging to all the irreducible representations common to these two induced representations, except the trivial representation, should be the least eigenvalue of $X_{t,k,\ell}$. To make this happen we suspect that a weighted adjacency matrix of $X_{t,k,\ell}$ would be needed in the ratio bound, rather than just the adjacency matrix; the weighting would have to be chosen so that the common modules (except the trivial) in the representations in (8.1) all belong to the same eigenvalue. Another complication is that potentially more of the eigenvalues of $X_{t,k,\ell}$ would have to be calculated, at the very least all the eigenvalues belonging to the common representations would need to be known.

Bender's theorem is much more general than the version we stated here. We only state Bender's theorem for matrices with 01-entries, but the full theorem applies to matrices with entries less than t. Using the full theorem we would be able to approximate the degree of $X_{t,k,\ell}$ for $t \ge 2$.

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