ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 23 (2023) #P3.08
https://doi.org/10.26493/1855-3974.2694.56a
(Also available at http://amc-journal.eu)

The A-Möbius function of a finite group

Francesca Dalla Volta
Dipartimento di Matematica e Applicazioni, University of Milano-Bicocca, Via Cozzi 55, 20126 Milano, Italy

Andrea Lucchini *
Università degli Studi di Padova, Dipartimento di Matematica "Tullio Levi-Civita", Via Trieste 63, 35121 Padova, Italy

Received 11 September 2021, accepted 30 September 2022, published online 27 January 2023

Abstract

The Möbius function of the subgroup lattice of a finite group G has been introduced by Hall and applied to investigate several different questions. We propose the following generalization. Let A be a subgroup of the automorphism group $\mathrm{Aut}(G)$ of a finite group G and denote by $\mathcal{C}_A(G)$ the set of A-conjugacy classes of subgroups of G. For $H \leq G$ let $[H]_A = \{ H^a \mid a \in A\}$ be the element of $\mathcal{C}_A(G)$ containing H. We may define an ordering in $\mathcal{C}_A(G)$ in the following way: $[H]_A \leq [K]_A$ if $H^a \leq K$ for some $a \in A$. We consider the Möbius function μ_A of the corresponding poset and analyse its properties and possible applications.

Keywords: Groups, subgroup lattice, Möbius function.

Math. Subj. Class. (2020): 20D30, 05E16

1 Introduction

The Möbius function of a finite partially ordered set (poset) P is the map $\mu_P\colon\ P \times P \to \mathbb{Z}$ satisfying $\mu_P(x, y) = 0$ unless $x \leq y$, in which case it is defined inductively by the equations $\mu_P(x, x) = 1$ and $\sum_{x\leq z\leq y} \mu_P(x, z) = 0$ for $x < y$.

In a celebrated paper [5], P. Hall used for the first time the Möbius function μ of the subgroup lattice of a finite group G to investigate some properties of G, in particular to compute the number of generating t-tuples of G. A detailed investigation of the properties of the function μ associated to a finite group G is given by T. Hawkes, I. M. Isaacs and

*Corresponding author.
E-mail addresses: francesca.dallavolta@unimib.it (Francesca Dalla Volta), lucchini@math.unipd.it (Andrea Lucchini)

M. Özaydin in [6]. In that paper, the authors also consider the Möbius function λ of the poset of conjugacy classes of subgroups of G, where $[H] \leq [K]$ if $H \leq K^g$ for some $g \in G$ (see [6, Section 7]). In particular, they propose the interesting and intriguing question of comparing the values of μ and λ.

In this paper we aim to generalize the definitions and main properties of the functions μ and λ to a more general context. Let G and A be a finite group and a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G, respectively. Denote by $\mathcal{C}_A(G)$ the set of A-conjugacy classes of subgroups of G. For $H \leq G$ let $[H]_A = \{ H^a \mid a \in A\}$ be the element of $\mathcal{C}_A(G)$ containing H. We may define an ordering in $\mathcal{C}_A(G)$ in the following way: $[H]_A \leq [K]_A$ if $H^a \leq K$ for some $a \in A$; we consider the Möbius function μ_A of the corresponding poset. We will write $\mu_A(H, K)$ in place of $\mu_A([H]_A, [K]_A)$. When $A = \mathrm{Inn}(G)$, we write $\mathcal{C}(G)$ and $[H]$, in place of $\mathcal{C}_{\mathrm{Inn}(G)}(G)$ and $[H]_{\mathrm{Inn}(G)}$. When $A = 1$, $\mu_A = \mu$ is the Möbius function in the subgroup lattice of G, introduced by P. Hall. In the case when $A = \mathrm{Inn}(G)$ is the group of the inner automorphism, $\mu_{\mathrm{Inn}(G)}$ coincides the Möbius function λ of the poset of conjugacy classes of subgroups of G, defined above. Note that for any subgroup A of $\mathrm{Aut}(G)$, we get $[G]_A = \{G\}$.

In Section 2, we prove some general properties of μ_A. In particular we prove the following result:

Proposition 1.1. *Let G be a finite solvable group. If $G' \leq K \leq G$ and A is the subgroup of $\mathrm{Inn}(G)$ obtained by considering the conjugation with the elements of K, then $\mu_A(H, G) = \lambda(H, G)$ for every $H \leq G$.*

To illustrate the meaning of the previous proposition, consider the following example. Let $G = A_4$ be the alternating group of degree 4 and A the subgroup of $\mathrm{Inn}(G)$ induced by conjugation with the elements of $G' \cong C_2 \times C_2$. The posets $\mathcal{C}(G)$ and $\mathcal{C}_A(G)$ are different. For example there are three subgroups of G of order 2, which are conjugated in G, but not A-conjugated. However $\lambda(H, G) = \mu_A(H, G)$ for any $H \leq G$.

In Section 3, we generalize some result given by Hall in [5], about the cardinality $\phi(G, t)$ of the set $\Phi(G, t)$ of t-tuples $(g_1, \dots, g_t)$ of group elements g_i such that $G = \langle g_1, \dots, g_t\rangle$. As observed by P. Hall, using the Möbius inversion formula, it can be proved that

$$\phi(G, t) = \sum_{H \leq G} \mu(H, G)|H|^t. \tag{1.1}$$

We generalize this formula, showing that $\phi(G, t)$ can be computed with a formula involving μ_A for any possible choice of A.

Theorem 1.2. *For any finite group G and any subgroup A of $\mathrm{Aut}(G)$,*

$$\phi(G, t) = \sum_{[H]_A \in \mathcal{C}_A(G)} \mu_A(H, G)| \cup_{a \in A} (H^a)^t|.$$

If G is not cyclic, then $\phi(G, 1) = 0$, so we obtain the following equality, involving the values of μ_A.

Corollary 1.3. *If G is not cyclic, then*

$$0 = \sum_{[H]_A \in \mathcal{C}_A(G)} \mu_A(H, G)| \cup_{a \in A} H^a|.$$

Further generalizations are given in Section 4, where we consider the function $\phi^*(G,t)$, which is an analogue of $\phi(G,t)$: actually, $\phi^*(G,t)$ denotes the cardinality of the set of of t-tuples $(H_1,\dots,H_t)$ of subgroups of G such that $G=\langle H_1,\dots,H_t\rangle$. As a corollary of our formula for computing $\phi^*(G,t)$, we obtain we following unexpected result.

Proposition 1.4. *Let $\sigma(X)$ denote the number of subgroups of a finite group X. For any finite group G, the following equality holds:*

$$1=\sum_{H\leq G}\mu(H,G)\sigma(H).$$

Finally, in Section 5, we consider one question originated from a result given by Hawkes, Isaacs and Özaydin in [6]: they proved that the equality

$$\mu(1,G)=|G'|\lambda(1,G)$$

holds for any finite solvable group G; later Pahlings [7] generalized the result proving that

$$\mu(H,G)=|N_{G'}(H):G'\cap H|\cdot\lambda(H,G) \tag{1.2}$$

holds for any $H\leq G$ whenever G is finite and solvable. Following [3], we say that G satisfies the (μ,λ)-property if (1.2) holds for any $H\leq G$. Several classes of non-solvable groups satisfy the (μ,λ)-property, for example all the minimal non-solvable groups (see [3]). However it is known that the (μ,λ)-property does not hold for every finite group. For instance, it does not hold for the following finite almost simple groups: A_9, S_9, A_{10}, S_{10}, A_{11}, S_{11}, A_{12}, S_{12}, A_{13}, S_{13}, J_2, $PSU(3,3)$, $PSU(4,3)$, $PSU(5,2)$, M_{12}, M_{23}, M_{24}, $PSL(3,11)$, HS, $\mathrm{Aut}(HS)$, He $\mathrm{Aut}(H)$, McL, $PSL(5,2)$, $G_2(4)$, Co_3, $P\Omega^-(8,2)$, $P\Omega^+(8,2)$. It is somehow intriguing to notice that although the (μ,λ)-property fails for the sporadic groups M_{12}, J_2, McL, it holds for their automorphism groups.

We prove the following generalization of Pahlings's result.

Theorem 1.5. *Let N be a solvable normal subgroup of a finite group G. If G/N satisfies the (μ,λ)-property, then G also satisfies the (μ,λ)-property.*

An almost immediate consequence of the previous theorem is the following.

Corollary 1.6. *$PSU(3,3)$ is the smallest group which does not satisfy the (μ,λ) property.*

In the last part of Section 5, we use Theorem 1.2 to deduce some consequences of the (μ,λ)-property. In particular we prove the following theorem.

Theorem 1.7. *Suppose that a finite group G satisfies the (μ,λ)-property. Then, for every positive integer t, the following equality is satisfied:*

$$\sum_{[H]\in\mathcal{C}(G)}\lambda(H,G)\left(\frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|}-|\cup_{a\in A}(H^a)^t|\right)=0.$$

Some open questions are proposed along the paper.

2 Applying some general properties of the Möbius function

Given a poset P, a closure on P is a function $^{-}\colon P \to P$ satisfying the following three conditions:

(a) $x \leq \bar{x}$ for all $x \in P$;

(b) if $x, y \in P$ with $x \leq y$, then $\bar{x} \leq \bar{y}$;

(c) $\bar{\bar{x}} = \bar{x}$ for all $x \in P$.

If $^{-}$ is a closure map on P, then $\overline{P} = \{x \in P \mid \bar{x} = x\}$ is a poset with order induced by the order on P. We have:

Theorem 2.1 (The closure theorem of Crapo [2]). *Let P be a finite poset and let $^{-}\colon P \to P$ be a closure map. Fix $x, y \in P$ such that $y \in \overline{P}$. Then*

$$\sum_{x \leq z \leq y, \bar{z} = y} \mu_P(x, z) = \begin{cases} \mu_{\bar{P}}(x, y) \text{ if } x = \bar{x} \\ 0 \text{ otherwise.} \end{cases}$$

In [5], P. Hall proved that if $H < G$, then $\mu(H, G) \neq 0$ only if H is an intersection of maximal subgroups of G. Using the previous theorem, the following more general statement can be obtained.

Proposition 2.2. *If $H < G$ and $\mu_A(H, G) \neq 0$, then H can be obtained as intersection of maximal subgroups of G.*

Proof. Let H be a proper subgroup of G and let $\overline{H}$ be the intersection of the maximal subgroups of G containing H. Moreover let $\overline{G} = G$. The map $[H]_A \mapsto [\overline{H}]_A$ is a well defined closure map on $\mathcal{C}_A(G)$. Apply Theorem 2.1, with $x = [H]_A$ and $y = [G]_A$. Since $\overline{K} = G$ if and only if $K = G$, we have that $\mu_A(H, G) = 0$ if $H \neq \overline{H}$. □

An element a of a poset $\mathcal{P}$ is called conjunctive if the pair $\{a, x\}$ has a least upper bound, written $a \vee x$, for each $x \in \mathcal{P}$.

Lemma 2.3 ([6, Lemma 2.7]). *Let $\mathcal{P}$ be a poset with a least element 0, and let $a > 0$ be a conjunctive element of $\mathcal{P}$. Then, for each $b > a$, we have*

$$\sum_{a \vee x = b} \mu_{\mathcal{P}}(0, x) = 0.$$

From the above 2.3, the following Lemma 2.4 follows easily. Together with Lemma 2.5 and Lemma 2.7, this allows us to prove Proposition 1.1.

Lemma 2.4. *Let N be an A-invariant normal subgroup of G and $H \leq G$. If $H < HN < G$, then*

$$\mu_A(H, G) = -\sum_{[Y]_A \in \mathcal{S}_A(H,N)} \mu_A(H, Y),$$

with $\mathcal{S}_A(H, N) = \{[Y]_A \in \mathcal{C}_A(G) \mid [H]_A \leq [Y]_A < [G]_A \text{ and } YN = G\}$.

Proof. Let $\mathcal{P}$ be the interval $\{[K]_A \in \mathcal{C}_G(A) \mid [H]_A \leq [K]_A \leq [G]_A]\}$. Notice that $[HN]_A$ is a conjunctive element of $\mathcal{P}$. Indeed $[HN]_A \vee [K]_A = [KN]_A$ for every $[K]_A \in \mathcal{P}$. So the conclusion follows immediately from Lemma 2.3. □

Lemma 2.5. *Let K and A be a subgroup of G and the subgroup of* $\mathrm{Inn}(G)$ *induced by the conjugation with the elements of K, respectively. Assume that N is an abelian minimal normal subgroup of G contained in K and $H < HN \leq G$. Then*

$$\mu_A(H,G) = -\mu_A(HN,G)\gamma_A(N,H),$$

where $\gamma_A(N,H)$ is the number of A-conjugacy classes of complements of N in G containing H.

Proof. If $HN = G$, then H is a maximal subgroup of G, hence $\mu_A(H,G) = -1$, while $\mu_A(HN,G) = \mu_A(G,G) = 1$ and $\gamma_A(N,H) = 1$, so the statement is true. So we may assume $HN < G$ and apply Lemma 2.4. Suppose $[Y]_A \in \mathcal{S}_A(H,N)$. Notice that, since $YN = G$ and N is abelian, $Y \cap N$ is normal in G. Moreover $N \not\leq G$, since $Y < G = YN$. By the minimality of N as normal subgroup, we conclude $Y \cap N = 1$. Let

$$\mathcal{C} = \{J \leq G \mid H \leq J \leq Y\}, \quad \mathcal{D} = \{L \leq G \mid HN \leq L\}$$
$$\mathcal{C}_A = \{[J]_A \in \mathcal{C}_A(G) \mid [H]_A \leq [J]_A \leq [Y]_A\}, \mathcal{D}_A = \{[L]_A \in \mathcal{C}_A(G) \mid [HN]_A \leq [L]_A\}.$$

The map $\eta\colon \mathcal{C} \to \mathcal{D}$ sending J to JN is an order preserving bijection. Clearly, if $J_2 = J_1^x$ for some $x \in K$, then $\eta(J_2) = NJ_2 = NJ_1^x = (NJ_1)^x = (\eta(J_1))^x$. Conversely assume $\eta(J_2) = (\eta(J_1))^x$ with $x \in K$. Since $YN = G$, $x = yn$ with $n \in N$ and $y \in Y \cap K$. Thus $J_2N = (J_1N)^x = (J_1N)^y$ and consequently, applying the Dedekind law, $J_2 = J_2(Y \cap N) = J_2N \cap Y = (J_1N)^y \cap Y = (J_1N \cap Y)^y = J_1^y$. It follows that η induces an order preserving bijection from $\mathcal{C}_A$ to $\mathcal{D}_A$, but then $\mu_A(H,Y) = \mu_A(HN,YN) = \mu_A(HN,G)$. □

The statement of the previous lemma leads to the following open question.

Question 2.6. Let G be a finite group, $A \leq \mathrm{Aut}(G)$ and N an A-invariant normal subgroup of G. Does $\mu_A(HN,G)$ divide $\mu_A(H,G)$ for every $H \leq G$?

The following lemma is straightforward.

Lemma 2.7. *Let A be a subgroup of* $\mathrm{Aut}(G)$ *and N an A-invariant normal subgroup of G. Every $a \in A$ induces an automorphism $\overline{a}$ of G/N. Let $\overline{A} = \{\overline{a} \mid a \in A\}$. Then, for any $H \leq G$, $\mu_A(HN,G) = \mu_{\overline{A}}(HN/N,G/N)$.*

Proof of Proposition 1.1. We work by induction on $|G| \cdot |G:H|$. The statement is true if G is abelian. Assume that G' contains a minimal normal subgroup, say N, of G. If $N \leq H$, then, by Lemma 2.7

$$\lambda(H,G) = \lambda(H/N,G/N) = \mu_{\overline{A}}(H/N,G/N) = \mu_A(H,G).$$

So we may assume $N \not\leq H$. If H is not an intersection of maximal subgroups of G, then $\lambda(H,G) = \mu_A(H,G) = 0$. Suppose $H = M_1 \cap \cdots \cap M_t$ where $M_1, \ldots, M_t$ are maximal subgroups of G. In particular N is not contained in M_i for some i, so M_i is a complement of N in G containing H and $N \cap H = 1$. By Lemma 2.5, we have

$$\lambda(H,G) = -\lambda(HN,G)\gamma(N,H), \quad \mu_A(H,G) = -\mu_A(HN,G)\gamma_A(N,H),$$

where $\gamma(N,H)$ is the number of conjugacy classes of complements of N in G containing H and $\gamma_A(N,H)$ is the number of A-conjugacy classes of these complements. Suppose that K_1, K_2 are two conjugated complements of N in G containing H. Then $K_2 = K_1^n$ for some $n \in N_N(H)$. Since $N \leq G' \leq K$, it follows $\gamma(N,H) = \gamma_A(N,H)$. Moreover, by induction, $\lambda(HN,G) = \mu_A(HN,G)$, hence we conclude $\lambda(H,G) = \mu_A(H,G)$. □

3 Generalizing a formula of Philip Hall

We begin with introducing the functions $\Psi_A(H,t)$ and $\psi_A(H,t)$, analogue of $\Phi(H,t)$ and $\phi(H,t)$ in the general case of any possible subgroup A of $\mathrm{Aut}(G)$.

For any $H \in \mathcal{C}_A(G)$ and any positive integer t, let

1. $\Omega_A(H,t) = \bigcup_{a\in A}(H^a)^t$;
2. $\omega_A(H,t) = |\Omega_A(H,t)|$;
3. $\Psi_A(H,t) = \{(g_1,\dots,g_t) \in G^t \mid \langle g_1,\dots,g_t\rangle = H^a \text{ for some } a \in A\}$;
4. $\psi_A(H,t) = |\Psi_A(H,t)|$.

If $(x_1,\dots,x_t) \in \Omega_A(H,t)$, then $\langle x_1,\dots,x_t\rangle \leq H^a$ for some $a \in A$, hence $\langle x_1,\dots,x_t\rangle = K$ for some $K \leq G$ with $[K]_A \leq [H]_A$. Thus

$$\sum_{[K]\leq_A[H]} \psi_A(K,t) = \omega_A(H,t)$$

and therefore, by the Möbius inversion formula,

$$\sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,t) = \psi_A(G,t).$$

On the other hand $\psi_A(G,t) = \phi(G,t)$ so we have proved the following formula.

Theorem 3.1. *For any finite group G and any subgroup A of* $\mathrm{Aut}(G)$,

$$\phi(G,t) = \sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,t).$$

Notice that if $A = 1$, then $\omega_A(H,t) = |H^t|$, so that the result by Hall given in (1.1) is a particular case of the previous theorem.

Corollary 3.2. *If G is not cyclic, then*

$$0 = \phi(G,1) = \sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\omega_A(H,1).$$

Taking $A = \mathrm{Inn}(G)$, we deduce in particular that if G is not cyclic, then

$$\sum_{H\in\mathcal{C}(H)} \lambda(H,G)\omega_{\mathrm{Inn}(G)}(H,1) = \sum_{H\in\mathcal{C}(H)} \lambda(H,G)|\cup_g H^g| = 0.$$

For example, if $G = S_4$, then the values of $\lambda(H,G)$ and $|\cup_g H^g|$ are as in the following table and $24 - 12 - 16 - 15 + 4 + 9 + 7 - 1 = 0$.

	$\lambda(H,G)$	$\lvert \cup_g H^g \rvert$
S_4	1	24
A_4	-1	12
D_4	-1	16
S_3	-1	15
K	1	4
$\langle(1,2,3,4)\rangle$	0	10
$\langle(1,2,3)\rangle$	1	9
$\langle(1,2\rangle$	1	7
$\langle(1,2)(3,4)\rangle$	0	4
1	-1	1

If $G = A_5$, then the values of $\lambda(H,G)$, $\omega_{\mathrm{Inn}(G)}(H,1) = \lvert \cup_g H^g \rvert$, $\omega_{\mathrm{Inn}(G)}(H,2) = \lvert \cup_g (H^g)^2 \rvert$ (taking only the subgroups H with $\lambda(H,G) \neq 0$) are as in the following table and $60 - 36 - 36 - 40 + 21 + 32 - 1 = 0$.

	$\lambda(H,G)$	$\lvert \cup_g H^g \rvert$	$\lvert \cup_g (H^g)^2 \rvert$
A_5	1	60	3600
A_4	-1	36	636
S_3	-1	36	306
D_5	-1	40	550
$\langle(1,2,3)\rangle$	1	21	81
$\langle(1,2)(3,4)\rangle$	2	16	46
1	-1	1	1

Moreover

$$3600 - 636 - 306 - 550 + 81 + 2 \cdot 46 - 1 = 2280 = \frac{19}{30} \cdot 3600 = \phi(A_5, 2).$$

If $G = D_p = \langle a, b \mid a^p = 1, b^2 = 1, a^b = a^{-1} \rangle$ and p is an odd prime, then the behaviour of the subgroups in $\mathcal{C}(G)$ is described by the following table.

	$\lambda(H,G)$	$\lvert \cup_g H^g \rvert$
D_p	1	$2p$
$\langle a \rangle$	-1	p
$\langle b \rangle$	-1	$p+1$
1	-1	1

Another interesting example is given by considering $G = C_p^n$ and $A = \mathrm{Aut}(G)$. Let $H \cong C_p^{n-1}$ be a maximal subgroup of G. Then, for $K \leq G$, $\mu_A(K,G) \neq 0$ if and only if either $[K]_A = [G]_A$ or $[K]_A = [H]_A$. Clearly $\cup_{\alpha \in \mathrm{Aut}(G)} H^\alpha = G$ so $\mu_A(G,G)\omega_A(G,1) - \mu_A(H,G)\omega_A(H,1) = |G| - |G| = 0$. More generally, $\Omega_A(H,t)$ is the set of t-tuples $(x_1, \ldots, x_t)$ such that $(x_1, \ldots, x_t) \in K^t$ for some maximal subgroup K of G, so $\mu_A(G,G)\omega_A(G,t) - \mu_A(H,G)\omega_A(H,t) = |G|^t - \omega_A(H,t)$ is the number of generating t-tuples of G.

Another generalization of (1.1), essentially due to Gaschütz, has been described by Brown in [1, Section 2.2]. Let N be a normal subgroup of G and suppose that G/N admits t generators for some integer t. Let $y = (y_1, \dots, y_t)$ be a generating t-tuple of G/N and denote by $P(G, N, t)$ the probability that a random lift of y to a t-tuple of G generates G. Then $P(G, N, t) = \phi(G, N, t)/|N|^t$, where $\phi(G, N, t)$ is the number of generating t-tuples of G lying over y. As is showed in [1, Section 2.2], using again the Möbius inversion formula it can be proved:

$$\phi(G, N, t) = \sum_{H \leq G, HN = G} \mu(H, G)|H \cap N|^t. \tag{3.1}$$

This formula can be generalized in our contest in the following way:

Theorem 3.3. *Let N be an A-invariant normal subgroup of G and fix $g_1, \dots, g_t \in G$ with the property that $G = \langle g_1, \dots, g_t \rangle N$. Define*

- $\Omega_A(H, N, t) = \{(n_1, \dots, n_t) \mid \langle g_1 n_1, \dots, g_t n_t \rangle \leq H^a$ *for some* $a \in A\}$;
- $\omega_A(H, N) = |\Omega_A(H, N, t)|$

and let $\mathcal{C}_A(G, N) = \{[H]_A \in \mathcal{C}_A(G) \mid HN = G\}$. Then

$$\phi(G, N, t) = \sum_{[H]_A \in \mathcal{C}_A(G,N)} \mu_A(H, G)\omega_A(H, N, t).$$

Proof. Fix $g_1, \dots, g_t \in G$ with the property that $G = \langle g_1, \dots, g_t \rangle N$. Then $\phi(G, N, t)$ is the cardinality of the set

$$\Phi(G, N, g_1, \dots, g_t) = \{(n_1, \dots, n_t) \in N^t \mid \langle g_1 n_1, \dots, g_t n_t \rangle = G\}.$$

Set:

$$\Psi_A(H, N, g_1, \dots, g_t) = \{(n_1, \dots, n_t) \in N^t \mid \langle g_1 n_1, \dots, g_t n_t \rangle = H^a \text{ for some } a \in A\};$$
$$\psi_A(H, N, t) = |\Psi_A(H, N, g_1, \dots, g_t)|.$$

Notice that $\omega_A(H, N, t) \neq 0$ if and only if $[H]_A \in \mathcal{C}_A(G, N)$. If $(n_1, \dots, n_t) \in \Omega_A(H, N, t)$, then $\langle g_1 n_1, \dots, g_t n_t \rangle \leq H^a$ for some $a \in A$, and $\langle g_1 n_1, \dots, g_t n_t \rangle = K$ for some $K \leq G$ with $[K]_A \leq [H]_A$. Thus

$$\sum_{[K]_A \leq [H]_A} \psi_A(K, N, t) = \omega_A(H, N, t)$$

and therefore, by the Möbius inversion formula

$$\sum_{[H] \in \mathcal{C}_A(G,N)} \mu_A(H, G)\omega_A(H, N, t) = \psi_A(G, N, t) = \phi(G, N, t) \qquad \square$$

4 Another application of Möbius inversion formula

Denote by $\Phi^*(G,t)$ the set of t-tuples $(H_1,\dots,H_t)$ of subgroups of G such that $G = \langle H_1,\dots,H_t\rangle$ and by $\phi^*(G,t)$ the cardinality of this set. For any $H \in \mathcal{C}_A(G)$ and any positive integer t, let

1. $\Sigma_A(H,t) = \{(H_1,\dots,H_t) \mid \langle H_1,\dots,H_t\rangle \le H^a$ for some $a \in A\}$;
2. $\sigma_A(H,t) = |\Sigma_A(H,t)|$;
3. $\Gamma_A(H,t) = \{(H_1,\dots,H_t) \mid \langle H_1,\dots,H_t\rangle = H^a$ for some $a \in A\}$;
4. $\gamma_A(H,t) = |\Gamma_A(H,t)|$.

Theorem 4.1.

$$\phi^*(G,t) = \sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\sigma_A(H,t).$$

Proof. If $(H_1,\dots,H_t) \in \Sigma_A(H,t)$, then $\langle H_1,\dots,H_t\rangle = K$ for some $K \le G$ with $[K]_A \le [H]_A$. Thus

$$\sum_{[K]\le_A[H]} \gamma_A(K,t) = \sigma_A(H,t)$$

and therefore, by the Möbius inversion formula,

$$\sum_{[H]\in\mathcal{C}_A(G)} \mu_A(H,G)\sigma_A(H,t) = \gamma_A(G,t) = \phi^*(G,t). \qquad \square$$

In the particular case when $A = 1$, $\sigma_A(H,t) = \sigma(H)^t$, denoting with $\sigma(H)$ the number of subgroups of H. So we obtain the following corollary:

Corollary 4.2.

$$\phi^*(G,t) = \sum_{H\le G} \mu(H,G)\sigma(H)^t.$$

Clearly $\Sigma^*(G,t) = \{G\}$, so $\phi^*(G,1) = 1$ and therefore it follows:

Corollary 4.3.

$$1 = \sum_{H\in H_A} \mu_A(H,G)\sigma_A(H,1).$$

In particular:

Corollary 4.4.

$$1 = \sum_{H\le G} \mu(H,G)\sigma(H).$$

For example, if $G = A_5$ then the subgroups of G with $\mu(H,G) \ne 0$ are listed in the following table (where $\kappa(H,G)$ denote the numbers of conjugate of H in G).

	$\mu(H,G)$	$\kappa(H,G)$	$\sigma(H)$
A_5	1	1	59
A_4	-1	5	10
S_3	-1	10	6
D_5	-1	6	8
$\langle(1,2,3)\rangle$	2	10	2
$\langle(1,2)(3,4)\rangle$	4	15	2
1	-60	1	1

According with Corollary 4.4, $1 = 59 - 5\cdot 10 - 10\cdot 6 - 6\cdot 8 + 2\cdot 10\cdot 2 + 4\cdot 15\cdot 2 - 60$.

For a finite group G, denote by $P(G,t)$ and $P^*(G,t)$ the probability of generating G with, respectively, t elements or t subgroups. It can be easily seen that $P(G,t) = P(G/\operatorname{Frat}(G),t)$, but in general $P^*(G,t) \neq P^*(G/\operatorname{Frat}(G),t)$. For example, if $G \cong C_{p^a}$, then G and $H \cong C_{p^{a-1}}$ are the unique subgroups of G with non trivial Möbius number and therefore

$$P(G,t) = \frac{|G|^t - |H|^t}{|G|^t} = 1 - \frac{1}{p^t},$$
$$P^*(G,t) = \frac{\sigma(G)^t - \sigma(H)^t}{\sigma(G)^t} = 1 - \frac{a^t}{(a+1)^t}.$$

So $P(G,t)$ is independent of a, while $P^*(G,t)$ tends to 0 when a tends to infinity.

5 The (μ,λ)-property

Proof of Thereom 1.5. Working by induction on the order of G, it suffices to prove the statement in the particular case when N is an abelian minimal normal subgroup of G. Let H be a subgroup of G. If $N \leq H$, then

$$\begin{aligned}\mu(H,G) &= \mu(H/N,G/N) = \lambda(H/N,G/N)|N_{G'N/N}(H/N) : H/N \cap G'N/N|\\ &= \lambda(H,G)|N_{G'N}(H) : H \cap G'N| = \lambda(H,G)|NN_{G'}(H) : N(H\cap G')|\\ &= \lambda(H,G)\frac{|N_{G'}(H) : H \cap G'|}{|N \cap N_{G'}(H) : N \cap H \cap G'|} = \lambda(H,G)\frac{|N_{G'}(H) : H\cap G'|}{|N\cap G' : N \cap G'|}\\ &= \lambda(H,G)|N_{G'}(H) : H \cap G'|.\end{aligned}$$

So we may assume $N \not\leq H$. If H is not an intersection of maximal subgroups of G, then $\mu(G,H) = \lambda(G,H) = 0$. So we may assume $H = M_1 \cap \dots \cap M_t$ where $M_1,\dots,M_t$ are maximal subgroups of G. Since N is not contained in H, then N is not contained in M_i for some i, but then M_i is a complement of N in G containing H and $N \cap H = 1$. If $g \in N_G(HN)$, then $g = xn$ with $x \in M_i$ and $n \in N$. In particular $H^x \leq HN \cap M_i = H(N \cap M_i) = H$, so $N_G(HN) = N_G(H)N$. By Lemma 2.5, we have

$$\frac{\mu(H,G)}{\lambda(H,G)} = \frac{\mu(HN,G)}{\lambda(HN,G)}\frac{\kappa}{\delta} = |N_{G'N}(HN) : HN\cap G'N|\frac{\kappa}{\delta} = |NN_{G'}(H) : HN\cap G'N|\frac{\kappa}{\delta}$$

where k is the number of complements of N in G containing H and δ is the number of conjugacy classes of these complements. First assume that $N \leq Z(G)$. Then $\kappa = \delta$,

$G' = M_i' \leq M_i$, $N \cap G' = 1$ and

$$\frac{\mu(H,G)}{\lambda(H,G)} = |NN_{G'}(H) : HN \cap G'N|\frac{\kappa}{\delta} = |NN_{G'}(H) : HN \cap G'N|$$
$$= |NN_{G'}(H) : N(H \cap G')| = |N_{G'}(H) : H \cap G'|.$$

Finally assume $N \not\leq Z(G)$. Then $N \leq G'$, $\kappa/\delta = |N_N(H)|$ and

$$\frac{\mu(H,G)}{\lambda(H,G)} = |NN_{G'}(H) : HN \cap G'N|\frac{\kappa}{\delta} = |NN_{G'}(H) : N(H \cap G')||N_N(H)|$$
$$= \frac{|N||N_{G'}(H)|}{|N_N(H)|}\frac{|N_N(H)|}{|N||H \cap G'|} = |N_{G'}(H) : H \cap G'|. \qquad \square$$

Proof of Corollary 1.6. Suppose that G has minimal order with respect to the property that G does not satisfy the (μ, λ) property. By the previous proposition, G contains no abelian minimal normal subgroup and therefore $\mathrm{soc}(G) = S_1 \times \cdots \times S_t$ is a direct product of nonabelian finite simple groups. If $|G| \leq |PSU(3,3)| = 6048$, then either $t = 1$ or $G = \mathrm{soc}(G) = A_5 \times A_5$. So it suffices to check that $A_5 \times A_5$ and any almost simple group of order at most 6048 satisfies the (μ, λ) property. Recall that the table of marks of a finite group G is a matrix whose rows and columns are labelled by the conjugacy classes of subgroups of G and where for two subgroups A and B the (A, B)-entry is the number of fixed points of B in the transitive action of G on the cosets of A in G. Since, for every $H \leq G$, $\lambda(H,G)$ and $\mu(H,G)$ can be computed from the table of marks of G (see [7, Proposition 1]), our proof can be easily completed using the library of table of marks available in GAP [4]. $\square$

We may use Theorem 3.1 to deduce some consequences of the (μ, λ)-property.

Theorem 5.1. *Suppose that a finite group G satisfies the (μ, λ)-property. Then*

$$\sum_{[H]\in\mathcal{C}(G)} \lambda(H,G)\left(\frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|} - \omega(H,t)\right) = 0. \qquad (5.1)$$

Proof. By Theorem 3.1,

$$\sum_{H\in\mathcal{C}(G)} \lambda(H,G)\omega(H,t) = \phi(G,t) = \sum_{H\leq G} \mu(H,G)|H|^t$$
$$= \sum_{H\in\mathcal{C}(G)} \mu(H,G)|G : N_G(H)||H^t|$$
$$= \sum_{H\in\mathcal{C}(G)} \lambda(H,G)|N_{G'}(H) : G' \cap H||G : N_G(H)||H^t|$$
$$= \sum_{H\in\mathcal{C}(G)} \lambda(H,G)\frac{|H|^t|G||N_{G'}(H)|}{|G' \cap H||N_G(H)|}$$
$$= \sum_{H\in\mathcal{C}(G)} \lambda(H,G)\frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|}. \qquad \square$$

A natural question is whether (5.1) is also a sufficient condition for the (μ, λ)-property. For any $H \leq G$, set $\mu^*(H,G) = |N_{G'}(H) : G' \cap H|\lambda(H,G)$. The validity of (5.1) is equivalent to

$$\sum_{H\in\mathcal{C}(G)} \lambda(H,G)\omega(H,t) - \sum_{H\in\mathcal{C}(G)} \mu^*(H,G)|H|^t|G : N_G(H)| = 0.$$

In any case we must have

$$\sum_{H\in\mathcal{C}(G)} \lambda(H,G)\omega(H,t) - \sum_{H\in\mathcal{C}(G)} \mu(H,G)|H|^t|G : N_G(H)| = 0.$$

So (5.1) is equivalent to

$$\sum_{H\in\mathcal{C}(G)} \frac{(\mu(H,G) - \mu^*(H,G))|H|^t}{|N_G(H)|} = 0.$$

Let $\mathcal{T} = \{[H] \in \mathcal{C}(G) \mid \mu(H,G) \neq \mu^*(H,G)\}$. Then (5.1) is true if and only if

$$\sum_{[H]\in\mathcal{T}} \frac{(\mu(H,G) - \mu^*(H,G))|H|^t}{|N_G(H)|} = 0. \tag{5.2}$$

For example, if $G = PSU(3,3)$, then $\mathcal{T}$ consists of four conjugacy classes of subgroups and the corresponding values are given by the following table:

$\mu(H,G)$	$\mu^*(H,G)$	$\lvert H\rvert$	$\lvert N_G(H)\rvert$
-48	0	2	96
3	0	6	18
0	-4	8	32
1	2	24	24

In this case (5.2) is equivalent to

$$2^{t-1} - 6^{t-1} - 8^{t-1} + 24^{t-1} = 0$$

which is true only if $t = 1$.

For any positive integer n let

$$\tau(n) = \sum_{H\in\mathcal{T},|H|=n} \frac{\mu(H,g) - \mu^*(H,G)}{|N_G(H)|}.$$

Proposition 5.2. *A finite group G satisfies (5.1) for every positive integer t if and only if $\tau(n) = 0$ for any $\in \mathbb{N}$.*

Question 5.3. Does $\tau(n) = 0$ for all $n \in \mathbb{N}$ imply $\mu^*(H,G) = \mu(H,G)$ for all $H \leq G$?

For any $H \leq G$, consider

$$\alpha(H,t) = \frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|}, \quad \beta(H,t) = \alpha(H,t) - \omega(H,t).$$

Let $\mathcal{C}^*(G) = \{[H] \in \mathcal{C}(H) \mid [H] < [G] \text{ and } \lambda(H,G) \neq 0\}$. If G satisfies the (λ,μ)-property, then for any $t \in \mathbb{N}$, the vector

$$\beta_t(G) = (\beta(H,t))_{[H]\in\mathcal{C}^*(G)}$$

is an integer solution of the linear equation

$$\sum_{[H]\in C^*(G)} \lambda(H,G)x_H = 0. \tag{5.3}$$

One could investigate about the dimension of the vector space generated by the vectors $\beta_t(G)$, $t \in \mathbb{N}$. For example, if $G = A_5$, then we may order the elements of $\mathcal{C}^*(G)$ so that $H_1 = A_4$, $H_2 = S_3$, $H_3 = D_5$, $H_4 = \langle(1,2,3)\rangle$, $H_5 = \langle(1,2)(3,4)\rangle$, $H_6 = 1$. Then (5.3) can be written in the form

$$\sum_{[H]\in\mathcal{C}^*(G)} \lambda(H,G)x_H = -x_{H_1} - x_{H_2} - x_{H_3} + x_{H_4} + 2x_{H_5} - x_{H_6}$$

and

$$\begin{aligned}
\beta_1(G) &= (24, 24, 20, 39, 44, 59),\\
\beta_2(G) &= (84, 54, 50, 99, 74, 59),\\
\beta_3(G) &= (264, 114, 110, 279, 134, 59),\\
\beta_4(G) &= (804, 234, 230, 819, 254, 59),\\
\beta_5(G) &= (2424, 474, 470, 2439, 494, 59),\\
\beta_6(G) &= (7284, 954, 950, 7299, 974, 59).
\end{aligned}$$

The first three vectors $\beta_1(G), \beta_2(G), \beta_3(G)$ are linearly independent, while $\beta_4(G)$, $\beta_5(G)$ and $\beta_6(G)$ can be obtained as linear combinations of $\beta_1(G), \beta_2(G), \beta_3(G)$.

The situation is completely different when $G = S_3$. We may order the elements of $C^*(G)$ so that $H_1 = \langle(1,2,3)\rangle$, $H_2 = \langle(1,2)\rangle$, $H_3 = 1$. The equation (5.3) has in this case the form $x_{H_1} + x_{H_2} - x_{H_3} = 0$ and $\beta_t(G) = (0,2,2)$ independently on the choice of t.

Some properties of the vectors $\beta_t(G)$ are described in the following propositions.

Proposition 5.4. *If $H \in \mathcal{C}^*(G)$, then $\beta(H,t) \geq 0$ with equality if and only if $G' \leq H$. In particular $\beta_t(G)$ is a non-negative vector and $\beta_t(G) = 0$ if and only if G is nilpotent.*

Proof. Notice that $\omega(H,t) \leq |G : N_G(H)|(|H|^t - 1) + 1$. So

$$\begin{aligned}
\beta(H,t) &\geq \frac{|H|^{t-1}|G||G'H|}{|G'N_G(H)|} - |G : N_G(H)|(|H|^t - 1) - 1\\
&= |H|^t|G : N_G(H)|\frac{|G' \cap N_G(H)|}{|G' \cap H|} - |G : N_G(H)|(|H|^t - 1) - 1 \geq 0
\end{aligned}$$

with equality if and only if $H \geq G'$. □

Proposition 5.5. *The vector $\beta_t(G)$ is independent on the choice of t if and only if G is a nilpotent group or a primitive Frobenius group, with cyclic Frobenius complement.*

Proof. By the previous proposition, if G is nilpotent then $\beta_t(G)$ is the zero vector for any $t \in \mathbb{N}$, so we may assume that G is not nilpotent. Assume that $\beta_t(G)$ is independent on the choice of t. Let H be a maximal non-normal subgroup of G. Then $\alpha(H,t) = |H|^t \cdot u$ with $u = |G : H|$. Let $H_1, \dots, H_u$ be the conjugates of H in G. For any $J \subseteq \{1, \dots, u\}$, let $\alpha_J = |\bigcap_{j \in J} H_j|$. Then

$$\beta(H,t) = \sum_{J \neq \{1,\dots,u\}} (-1)^{|J|+1} |\alpha_J|^t.$$

We must have $\alpha_J = 1$ for every choice of J, otherwise $\lim_{t\to\infty} \beta(H,t) = \infty$. Hence H is a Frobenius complement and, since H is a maximal subgroup, the Frobenius kernel V is an irreducible H-module. Since $\beta(V,t) = |V|^t(|H'| - 1)$ does not depends on t, H must be abelian, and consequently cyclic. So if $\beta_t(G)$ is independent of the choice of t, then G is a primitive Frobenius group with a cyclic Frobenius complement. Conversely assume $G = V \rtimes H$, where H is cyclic and V and irreducible H-module. If $X \in \mathcal{C}^*(G)$, then $\lambda(X, G) \neq 0$, so X is an intersection of maximal subgroups of G and therefore either $V = G' \leq X$, or X is conjugate to a subgroup of H. In the first case $\beta(H,t) = 0$. Assume $X = K^v$ for some $K \leq H$ and $v \in V$. Then $\beta(H,t) = |K|^t|V| - \omega(K,t) = |K|^t|V| - (|V|(|K|^t - 1) + 1) = |V| - 1$. □

ORCID iDs

Francesca Dalla Volta https://orcid.org/0000-0001-7368-4050
Andrea Lucchini https://orcid.org/0000-0002-2134-4991

References

[1] K. S. Brown, The coset poset and probabilistic zeta function of a finite group, *J. Algebra* **225** (2000), 989–1012, doi:10.1006/jabr.1999.8221, https://doi.org/10.1006/jabr.1999.8221.

[2] H. H. Crapo, Möbius inversion in lattices, *Arch. Math.* **19** (1969), 595–607, doi:10.1007/bf01899388, https://doi.org/10.1007/bf01899388.

[3] F. Dalla Volta and G. Zini, On two Möbius functions for a finite non-solvable group, *Commun. Algebra* **49** (2021), 4565–4576, doi:10.1080/00927872.2021.1924184, https://doi.org/10.1080/00927872.2021.1924184.

[4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021, https://www.gap-system.org.

[5] P. Hall, The Eulerian functions of a group, *Quart. J. Math. Oxford Ser.* **7** (1936), 134–151, doi:10.1093/qmath/os-7.1.134, https://doi.org/10.1093/qmath/os-7.1.134.

[6] T. Hawkes, I. M. Isaacs and M. Özaydin, On the Möbius function of a finite group, *Rocky Mt. J. Math.* **19** (1989), 1003–1034, doi:10.1216/rmj-1989-19-4-1003, https://doi.org/10.1216/rmj-1989-19-4-1003.

[7] H. Pahlings, On the Möbius function of a finite group, *Arch. Math. (Basel)* **60** (1993), 7–14, doi:10.1007/bf01194232, https://doi.org/10.1007/bf01194232.