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Almost simple groups as flag-transitive automorphism groups of symmetric designs with λ prime*

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Abstract

In this article, we study symmetric designs with λ prime admitting a flag-transitive and point-primitive automorphism group G of almost simple type with socle X. We prove that either $\mathcal{D}$ is one of the six well-known examples of biplanes and triplanes, or $\mathcal{D}$ is the point-hyperplane design of $\mathrm{PG}(n-1, q)$ with $\lambda = (q^{n-2}-1)/(q-1)$ prime and $X = \mathrm{PSL}_n(q)$.

Keywords: Almost simple group, automorphism group, flag-transitive, point-primitive, symmetric design.

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1 Introduction

Symmetric designs admitting flag-transitive automorphism groups are of most interest. Kantor [14] classified flag-transitive symmetric $(v, k, 1)$ designs known as projective planes of order n, and showed that either $\mathcal{D}$ is a Desarguesian projective plane and $\mathrm{PSL}_3(n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, where n is even and $n^2 + n + 1$ is prime. Regueiro gave a classification of nontrivial symmetric designs with $\lambda = 2$ (biplanes) admitting flag-transitive automorphism groups apart from those groups contained in a 1-dimensional affine group [17, 18, 19, 20, 21]. Dong, Fang and Zhou studied flag-transitive automorphism groups G of nontrivial symmetric $(v, k, 3)$ designs (triplanes), and in conclusion, excluding the case where $G \leqslant \mathrm{A\Gamma L}_1(q)$ where $q = p^m$ with $p \geqslant 5$ prime, they determined all such possible symmetric designs [13, 25, 26, 27, 28].

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Table 1: Some symmetric designs with λ prime.

Line	v	k	λ	X	H	G	Designs	References*
1	7	4	2	$\mathrm{PSL}_2(7)$	S_4	$\mathrm{PSL}_2(7)$	Complement of the Fano plane	[2, 8]
2	11	5	2	$\mathrm{PSL}_2(11)$	A_5	$\mathrm{PSL}_2(11)$	Hadamard	[2, 8]
3	11	6	3	$\mathrm{PSL}_2(11)$	A_5	$\mathrm{PSL}_2(11)$	Complement of line 2	[2, 8]
4	15	7	3	A_7	$\mathrm{PSL}_3(2)$	A_7	$\mathrm{PG}_2(3,2)$	[8, 26, 29]
5	45	12	3	$\mathrm{PSU}_4(2)$	$2.(\mathrm{A}_4{\times}\mathrm{A}_4).2$	$\mathrm{PSU}_4(2)$	-	[7, 11, 22]
6	45	12	3	$\mathrm{PSU}_4(2)$	$2.(\mathrm{A}_4{\times}\mathrm{A}_4).2{:}2$	$\mathrm{PSU}_4(2){:}2$	-	[7, 11, 22]

Note: The last column addresses to references in which a design with the parameters in the line has been constructed.

Recently, Z. Zhang, Y. Zhang and Zhou in [24] proved that if $\mathcal{D}$ is a nontrivial symmetric (v,k,λ) design with λ prime and G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$, then G must be of affine or almost simple type. In this paper, we study symmetric designs with λ prime admitting a flag-transitive and point-primitive automorphism group of almost simple type. Indeed, we have already shown in [4] that almost simple exceptional groups of Lie type give rise to no possible symmetric designs with λ prime. In addition, we investigated the case where G is an almost simple group with socle X being a finite simple classical group of Lie type, and proved that $\mathcal{D}$ is either the point-hyperplane design of a projective space $\mathrm{PG}(n-1,q)$, or it is of parameters $(7,4,2)$, $(11,5,2)$, $(11,6,3)$ or $(45,12,3)$. Here, we focus on the case where G is an almost simple group with socle X an alternating group:

Theorem 1.1. *Let $\mathcal{D}$ be a nontrivial symmetric (v,k,λ) design with λ prime and α a point of $\mathcal{D}$, and let G be a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ of almost simple type. If the socle of G is an alternating group A_c with $c \geqslant 5$, then $\mathcal{D} = \mathrm{PG}_2(3,2)$ with parameters $(15,7,3)$ and $G = \mathrm{A}_7$ with the point-stabiliser $G_\alpha = \mathrm{PSL}_3(2)$.*

By Theorem 1.1 and the main results of [3, 23], we consequently obtain all symmetric designs with λ prime admitting flag-transitive and point-primitive automorphism groups of almost simple type:

Corollary 1.2. *Let $\mathcal{D}$ be a nontrivial symmetric (v,k,λ) design with λ prime. Suppose that G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ of almost simple type with socle X. Then $\lambda = 2,3$ and $(\mathcal{D},G)$ is as in one of the lines of* Table 1*, or $\mathcal{D}$ is the point-hyperplane design of* $\mathrm{PG}(n-1,q)$ *with $\lambda = (q^{n-2}-1)/(q-1)$ prime and $X = \mathrm{PSL}_n(q)$.*

In order to prove Theorem 1.1, for the case where $\lambda \leqslant 100$ or $\gcd(k,\lambda) = 1$, by [29, 30], we obtain the designs in the statement. Then we assume that $\lambda > 100$ and λ divides k (as λ is prime). In this case, we show that there is no symmetric deign with λ prime and flag-transitive and point-primitive automorphism group G. Here, we first observe that the point-stabiliser H of G has to be large, that is to say, $|G| \leqslant |H|^3$, see Corollary 2.2. The possibilities for H can be read off from [5]. In Section 3, we examine these possibilities and achieve our desired result. In Section 4, we give a detailed proof of Corollary 1.2 which follows immediately from Theorem 1.1 and the main results of [3, 23].

1.1 Definitions and notation

All groups and incidence structures in this paper are finite. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$, where X is a nonabelian simple group. Symmetric and alternating groups on c letters are denoted by S_c and A_c, respectively. We write "n" for group of order n. A symmetric (v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of v blocks such that each block is a k-subset of $\mathcal{P}$ and each pair of distinct points is contained in exactly λ blocks. We say that $\mathcal{D}$ is nontrivial if $2 < k < v - 1$. A *flag* of $\mathcal{D}$ is a point-block pair (α, B) such that $\alpha \in B$. An *automorphism* of $\mathcal{D}$ is a permutation on $\mathcal{P}$ which maps blocks to blocks and preserving the incidence. The *full automorphism* group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$. For a given positive integer n and a prime divisor p of n, we denote the p-part of n by n_p, that is to say, $n_p = p^t$ with $p^t \mid n$ but $p^{t+1} \nmid n$. Further notation and definitions in both design theory and group theory are standard and can be found, for example in [6, 9, 12, 15, 16].

2 Preliminaries

In this section, we state some useful facts in both design theory and group theory. If a group G acts on a set $\mathcal{P}$ and $\alpha \in \mathcal{P}$, the *subdegrees* of G are the length of orbits of the action of the point-stabiliser G_α on $\mathcal{P}$.

Lemma 2.1 ([2, Lemma 2.1])**.** *Let $\mathcal{D}$ be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If α is a point of $\mathcal{D}$, then*

(i) $k(k-1) = \lambda(v-1)$*;*

(ii) *k divides $|G_\alpha|$, and $\lambda v < k^2$;*

(iii) *$k \mid \lambda d$, for all nontrivial subdegrees d of G.*

For a point-stabiliser H of a flag-transitive automorphism group G of a design $\mathcal{D}$, by Lemma 2.1(ii), we conclude that $\lambda|G| \leqslant |H|^3$, and so we have that

Corollary 2.2. *Let $\mathcal{D}$ be a symmetric (v, k, λ) design with a flag-transitive automorphism group G and α a point of $\mathcal{D}$. Then $|G| \leqslant |G_\alpha|^3$.*

Lemma 2.3. *Suppose that s and t are positive integers. Then*

(i) *if $t > s \geqslant 9$, then $\binom{s+t}{s} > s^2t^3$;*

(ii) *if $s \geqslant 4$ and there exists $t_0 \geqslant 7$ such that $\binom{s+t_0}{s} > s^2t_0^3$, then $\binom{s+t}{s} > s^2t^3$ for all $t \geqslant t_0$.*

Proof. (i) If $t > s = 9$, then we observe that the inequality $\binom{s+t}{s} = \binom{t+9}{9} > 81t^3 = s^2t^3$ holds. If $t > s \geqslant 10$, then $10 \leqslant s \leqslant \frac{s+t}{2}$, and so that $\binom{s+t}{s} \geqslant \binom{10+t}{10} > t^5$. Since $t > s$, we have that $t^5 > s^2t^3$, and hence $\binom{s+t}{s} > t^5 > s^2t^3$.

(ii) It suffices to show that $\binom{s+t_0+1}{s} > s^2(t_0+1)^3$. Note that

$$\binom{s+t_0+1}{s} = \binom{s+t_0}{s}\frac{(s+t_0+1)}{(t_0+1)} > s^2t_0^3\frac{(s+t_0+1)}{(t_0+1)}$$
$$= s^2(t_0+1)^3\frac{(s+t_0+1)t_0^3}{(t_0+1)^4}.$$

Since $t_0 \geqslant 7$ and $s \geqslant 4$, it follows that $(s+t_0+1)t_0^3 \geqslant (t_0+5)t_0^3 > (t_0+1)^4$. Therefore, $\binom{s+t_0+1}{s} > s^2(t_0+1)^3$. □

3 Proof of Theorem 1.1

Suppose that G is a flag-transitive and point-primitive automorphism group of a symmetric (v,k,λ) design $\mathcal{D}$ with λ prime. Suppose that X is the alternating group A_c of degree $c \geqslant 5$ on $\Omega = \{1,\dots,c\}$ and that $H := G_\alpha$ with α a point of $\mathcal{D}$. Then H is maximal in G by [12, Corollary 1.5A], and since $G = HX$, we conclude that

$$v = \frac{|X|}{|H \cap X|}. \tag{3.1}$$

If $\lambda \leqslant 100$ or $\gcd(k,\lambda) = 1$, then by [29, 30], we conclude that $\mathcal{D}$ is $\mathrm{PG}_2(3,2)$ with parameters $(15,7,3)$, and $G = \mathrm{A}_7$ with the point-stabiliser $H = \mathrm{PSL}_3(2)$. Therefore, we can assume that $\lambda > 100$ and $\gcd(k,\lambda) \neq 1$. Since $k(k-1) = \lambda(v-1)$ and $\gcd(k,\lambda) \neq 1$, we conclude that $\lambda \mid k$, and so by Lemma 2.1(ii), the parameter λ divides $|H|$. In what follows, assuming that $\lambda > 100$ divides k and $\gcd(k,\lambda) \neq 1$, we show that there is no flag-transitive and point-primitive automorphism group of a symmetric (v,k,λ) design $\mathcal{D}$ with λ prime.

Let $H_0 := H \cap X$. Then by [5, Theorem 2 and Proposition 6.1], one of the following holds:

(i) H_0 is intransitive on $\Omega = \{1,\dots,c\}$;

(ii) H_0 is transitive and imprimitive on $\Omega = \{1,\dots,c\}$;

(iii) $G = \mathrm{S}_c$ and (c,H) is one of the following:

$$(5,\mathrm{AGL}_1(5)),\quad (6,\mathrm{PGL}_2(5)),\quad (7,\mathrm{AGL}_1(7)),\quad (8,\mathrm{PGL}_2(7)),$$
$$(9,\mathrm{AGL}_2(3)),\quad (10,\mathrm{A}_6{\cdot}2^2),\quad (12,\mathrm{PGL}_2(11));$$

(iv) $G = \mathrm{A}_6{\cdot}2 = \mathrm{PGL}_2(9)$ and H is D_{20} or a Sylow 2-subgroup P of G of order 16;

(v) $G = \mathrm{A}_6{\cdot}2 = \mathrm{M}_{10}$ and H is $\mathrm{AGL}_1(5)$ or a Sylow 2-subgroup P of G of order 16;

(vi) $G = \mathrm{A}_6{\cdot}2^2 = \mathrm{P\Gamma L}_2(9)$ and H is $\mathrm{AGL}_1(5){\times}2$ or a Sylow 2-subgroup P of G of order 32;

(vii) $G = \mathrm{A}_c$ and (c,H) is one of the following:

$$(5,\mathrm{D}_{10}),\quad (6,\mathrm{PSL}_2(5)),\quad (7,\mathrm{PSL}_2(7)),\quad (8,\mathrm{AGL}_3(2)),$$
$$(9,3^2{\cdot}\mathrm{SL}_2(3)),\quad (9,\mathrm{P\Gamma L}_2(8)),\quad (10,\mathrm{M}_{10}),\quad (11,\mathrm{M}_{11}),$$
$$(12,\mathrm{M}_{12}),\quad (13,\mathrm{PSL}_3(3)),\quad (15,\mathrm{A}_8),\quad (16,\mathrm{AGL}_4(2)),$$
$$(24,\mathrm{M}_{24}).$$

Since λ is a prime divisor of k, it follows from Lemma 2.1(ii) that λ is a prime divisor of $|H|$. For the possibilities recorded in (iii) – (vii), we then have $\lambda \in \{2,3,5,7,11,13,23\}$, and this violates our assumption that $\lambda > 100$, see Table 2. Therefore, H_0 is either intransitive, or imprimitive.

Table 2: The possibilities for λ in cases (iii) – (vii) in Section 3.

Line	H	$\|H\|$	λ
1	$\mathrm{AGL}_1(5)$	$2^2 \cdot 5$	$2, 5$
2	$\mathrm{PGL}_2(5)$	$2^3 \cdot 3 \cdot 5$	$2, 3, 5$
3	$\mathrm{AGL}_1(7)$	$2 \cdot 3 \cdot 7$	$2, 3, 7$
4	$\mathrm{PGL}_2(7)$	$2^4 \cdot 3 \cdot 7$	$2, 3, 7$
5	$\mathrm{AGL}_2(3)$	$2^4 \cdot 3^3$	$2, 3$
6	$\mathrm{A}_6 \cdot 2^2$	$2^5 \cdot 3^2 \cdot 5$	$2, 3, 5$
7	$\mathrm{PGL}_2(11)$	$2^3 \cdot 3 \cdot 5 \cdot 11$	$2, 3, 5, 11$
8	D_{10}	$2 \cdot 5$	$2, 5$
9	$\mathrm{PSL}_2(5)$	$2^2 \cdot 3 \cdot 5$	$2, 3, 5$
10	$\mathrm{PSL}_2(7)$	$2^3 \cdot 3 \cdot 7$	$2, 3, 7$
11	$\mathrm{AGL}_3(2)$	$2^6 \cdot 3 \cdot 7$	$2, 3, 7$
12	$3^2 \cdot \mathrm{SL}_2(3)$	$2^3 \cdot 3^3$	$2, 3$
13	$\mathrm{P\Gamma L}_2(8)$	$2^3 \cdot 3^3 \cdot 7$	$2, 3, 7$
14	M_{10}	$2^4 \cdot 3^2 \cdot 5$	$2, 3, 5$
15	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$2, 3, 5, 11$
16	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$2, 3, 5, 11$
17	$\mathrm{PSL}_3(3)$	$2^4 \cdot 3^3 \cdot 13$	$2, 3, 13$
18	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$2, 3, 5, 7$
19	$\mathrm{AGL}_4(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	$2, 3, 5, 7$
20	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2, 3, 5, 7, 11, 23$
21	D_{20}	$2^2 \cdot 5$	$2, 5$
22	$\mathrm{AGL}_1(5)$	$2^2 \cdot 5$	$2, 5$
23	P	2^4	2
24	$\mathrm{AGL}_1(5) \times 2$	$2^3 \cdot 5$	$2, 5$
25	P	2^5	2

Note: In line 23, P is a Sylow 2-subgroup of $G = \mathrm{A}_6 \cdot 2$ of order 16.
In line 25, P is a Sylow 2-subgroup of $G = \mathrm{A}_6 \cdot 2^2$ of order 32.

(I) Suppose that $H_0 = (\mathrm{S}_s \times \mathrm{S}_{c-s}) \cap \mathrm{A}_c$ is intransitive on $\Omega = \{1, \dots, c\}$ with $1 \leqslant s < c/2$. In this case, $H = (\mathrm{S}_s \times \mathrm{S}_{c-s}) \cap G$. Note that H is maximal in G as long as $s \neq c - s$. Since λ is an odd prime divisor of $|H|$, it follows that λ divides $s!$ or $(c-s)!$, and since $s < c - s$, we conclude that

$$\lambda \leqslant \max\{s, c-s\} = c - s. \tag{3.2}$$

Note that H_0 contains all the even permutations of H, and hence $H_0 = H$ if $G = \mathrm{A}_c$, or the index of H_0 in H is 2 if $G = \mathrm{S}_c$. Since G is flag-transitive, H is transitive on the set of blocks passing through α. Hence H fixes exactly one point in $\mathcal{P}$, and so stabilizes exactly one s-subset, say S, in Ω. Therefore, we can identify the point α of $\mathcal{P}$ with the unique s-subset S of Ω stabilized by H. Thus $v = \binom{c}{s}$. Since H_0 acting on Ω is intransitive, it has at least two orbits. According to [10, page 82], two points of $\mathcal{P}$ are in the same orbit under H_0 if and only if the corresponding s-subsets S_1 and S_2 of Ω intersect S in the same number of points. Thus G acting on $\mathcal{P}$ has rank $s+1$, and each H_0-orbit $\mathcal{O}_i$ on $\mathcal{P}$ corresponds to a possible size $i \in \{0, 1, \ldots, s\}$ and these are precisely the families of s-subsets of Ω that intersect S, see also [1, Proposition 2.5]. Then if d_i is the length of a G-orbit on $\mathcal{P}$, then $d_0 = 1$, and $d_j = \binom{s}{j-1}\binom{c-s}{s-j+1}$ when $G = \mathrm{A}_c$ or $d_j = \binom{s}{j-1}\binom{c-s}{s-j+1}/2$ when $G = \mathrm{S}_c$ for $j = 1, \ldots, s$.

By Lemma 2.1(iii), we have that k divides λd_j for all nontrivial subdegrees d_j of G. By taking $j = s$, we have that k divides $\lambda s(c-s)$, and so $k \leqslant \lambda s(c-s)$. As $\lambda v < k^2$ by Lemma 2.1(ii), it follows from (3.2) that

$$v = \binom{c}{s} < \lambda s^2(c-s)^2 \leqslant s^2(c-s)^3.$$

Set $t := c - s$. Thus

$$\binom{s+t}{s} < s^2 t^3. \tag{3.3}$$

Applying Lemma 2.3(i), we conclude that (3.3) holds only for $s \leqslant 8$.

If $s = 1$, then $v = c \geqslant 5$. Note that G is $(v-2)$-transitive on $\mathcal{P}$. Since $2 < k \leqslant v-2$, G acts k-transitively on $\mathcal{P}$. Then $\binom{c}{k} = |B^G| = |\mathcal{B}| = v = c$ for every block $B \in \mathcal{B}$. This implies that $k = 1$ or $k = c-1$. Since $k \geqslant \lambda > 100$, we conclude that $k = c-1$, that is to say, $\mathcal{D}$ is a trivial design, which is a contradiction.

If $s = 2$, then the subdegrees are 1, $\binom{c-2}{2}$, $2(c-2)$ if $G = \mathrm{A}_c$, or 1, $\binom{c-2}{2}/2$, $(c-2)$ if $G = \mathrm{S}_c$. Thus G is a primitive permutation group of rank 3. Therefore, the possibilities for $\mathcal{D}$ can be read off from [11], which gives no example with $\lambda > 100$ prime.

If $s = 3$, then by Lemma 2.1(iii), k divides $\lambda d_3 = 3\lambda(c-3)$, and so $k = 3\lambda(c-3)/u$ for some positive integer u. We apply Lemma 2.1(i) and since $v - 1 = \binom{c}{3} - 1 = (c-3)(c^2+2)/6$, we deduce that

$$\frac{3\lambda(c-3)}{u} \cdot \left(\frac{3\lambda(c-3)}{u} - 1\right) = \frac{\lambda(c-3)(c^2+2)}{6},$$

and so

$$(c^2+2)u^2 + 18u - 54(c-3)\lambda = 0, \tag{3.4}$$

for some positive integer u. Define $f(x) := (c^2+2)x^2 + 18x - 54(c-3)\lambda$ with $x \geqslant 1$. Note here that $c > 2s = 6$ and $\lambda \leqslant c-3$ by (3.2). Then $f'(x) = 2(c^2+2)x + 18 > 0$ for $x \geqslant 1$. If $x \geqslant 8$, then since $\lambda \leqslant c-3$, we have that $f(x) = (c^2+2)x^2 + 18x - 54(c-3)\lambda \geqslant 10c^2 + 324c - 214 > 0$ for $c > 6$. Therefore, we cannot find any positive integer $u \geqslant 8$ satisfying (3.4), and hence $1 \leqslant u \leqslant 7$. Thus by (3.4), we have that $54\lambda = u^2(c+3) + (11u^2+18u)/(c-3)$, and so $c-3$ divides $11u^2+18u$, where $1 \leqslant u \leqslant 7$. Moreover, $\lambda > 100$ is a prime number less than 665 as $\lambda \leqslant c-3 \leqslant 11u^2+18u \leqslant 665$. For these values of u, c and λ, it is easy to check that (3.4) does not hold.

Table 3: An upper bound and a lower bound for t and c when $4 \leqslant s \leqslant 8$.

s	Bounds for t	Bounds for c
4	$5 \leqslant t \leqslant 373$	$9 \leqslant c \leqslant 377$
5	$6 \leqslant t \leqslant 46$	$11 \leqslant c \leqslant 51$
6	$7 \leqslant t \leqslant 22$	$13 \leqslant c \leqslant 28$
7	$8 \leqslant t \leqslant 14$	$15 \leqslant c \leqslant 21$
8	$9 \leqslant t \leqslant 11$	$17 \leqslant c \leqslant 19$

If $4 \leqslant s \leqslant 8$, then we apply (3.3) and Lemma 2.3(ii), and so we can find a lower bound and an upper bound for t as in the second column of Table 3. Since also $c = t - s$, we can find a lower bound and an upper bound for c as in the third column of Table 3. For example, if $s = 4$, then we take $t_0 = 374$ and observe that $\binom{s+t_0}{s} = \binom{378}{4} = 837222750 > 837017984 = 4^2 \cdot 374^3 = s^2 t_0^3$, then Lemma 2.3(ii) implies that if $t \geqslant 374$, then (3.3) does not hold, which is a contradiction. Thus $t \leqslant 373$. Moreover, it is easy to check that (3.3) holds for $t \geqslant 5$. Thus $5 \leqslant t \leqslant 373$. Note that $t = c - 4$, and hence $9 \leqslant c \leqslant 377$. This follows the first row of Table 3. For each t, s and c as in Table 3, we note by (3.2) that $\lambda \leqslant c - s = t$. Then we obtain $v = \binom{c}{s}$ and all the possibilities for prime $\lambda > 100$. But it is easy to check that for such parameters v and λ, we cannot find any possible parameters set (v, k, λ) satisfying Lemma 2.1.

(II) Suppose now that H_0 is transitive and imprimitive on $\Omega = \{1, \ldots, c\}$. In this case, $H = (\mathrm{S}_s \wr \mathrm{S}_{c/s}) \cap G$ is imprimitive, where s divides c and $2 \leqslant s \leqslant c/2$. Indeed, H_0 is transitive and imprimitive on $\Omega = \{1, \ldots, c\}$, H_0 acting on Ω preserves a partition Σ of Ω into t classes of size s with $t \geqslant 2$, $s \geqslant 2$ and $c = st$. Thus $H_0 \leqslant G_\Sigma < G$. Since G is isomorphic to S_c or A_c and since both natural actions of G and X on Ω are primitive, we conclude that H_0 contains all the even permutations of Ω preserving the partition Σ. By the same argument as in [10, Case 2], [17, (3.2)] and [29, pages 1489-1490], the imprimitive partition Σ is the only nontrivial partition of Ω preserved by H_0. Since X acts transitively on all the partitions of Ω into t classes of size s, we can identify the points of the $\mathcal{D}$ with the partitions of Ω into t classes of size s, and so $v = \binom{ts}{s}\binom{(t-1)s}{s} \cdots \binom{3s}{s}\binom{2s}{s}/(t!)$, that is to say,

$$v = \binom{ts-1}{s-1}\binom{(t-1)s-1}{s-1}\cdots\binom{3s-1}{s-1}\binom{2s-1}{s-1}. \tag{3.5}$$

We note that the suborbits of G on Ω can be described by the notion of j-cyclics introduced in [10, page 84]. Indeed, if a partition Σ_1 of Ω is a point of $\mathcal{P}$, then for $j = 2, \ldots, t$, the set Γ_j of j-cyclic partitions with respect to Σ_1 is a union of H-orbits on $\mathcal{P}$, see [10, Case 2] and [29, pages 1490-1491]. Therefore, by Lemma 2.1(iii), k divides λd_s, where

$$d_s = \begin{cases} s^2\binom{t}{2}, & \text{if } s \geqslant 3; \\ t(t-1), & \text{if } s = 2. \end{cases} \tag{3.6}$$

Therefore, by Lemma 2.1(iii), we have that k divides λd_s, where d_s is as in (3.6). Note that λ is a prime divisor of k, and so by Lemmas 2.1(ii), we conclude that $\lambda \leqslant c = st$.

If $s = 2$, then $t \geqslant 3$ as $c = st \geqslant 5$. By (3.5), we have that $v = \prod_{i=0}^{t-2}[2t-(2i+1)]$ and since k divides $\lambda d_2 = \lambda t(t-1)$ and $\lambda \leqslant c = 2t$, it follows from Lemma 2.1(ii) that

$$\prod_{i=0}^{t-2}[2t-(2i+1)] < \lambda d_2^2 \leqslant 2t^3(t-1)^2 < 2t^5.$$

This forces $t \leqslant 6$, and hence $\lambda \leqslant 2t = 12$, which is a contradiction as $\lambda > 100$.

If $s \geqslant 3$, then since,

$$\binom{is-1}{s-1} = \frac{is-1}{s-1} \cdot \frac{is-2}{s-2} \cdots \frac{is-(s-1)}{1} > i^{s-1}$$

with $2 \leqslant i \leqslant t$, by (3.5), we conclude that $v > t^{(s-1)(t-1)}$. Since also k divides $\lambda d_s = \lambda s^2\binom{t}{2}$ and $\lambda \leqslant st$, we deduce by Lemma 2.1(ii) that

$$t^{(s-1)(t-1)} < s^5 t\binom{t}{2}^2.$$

Thus $t^{(s-1)(t-1)-5} < s^5$. Note that $s \geqslant 3$. Then this inequality holds only for

$$\begin{aligned} &t = 2 \text{ and } 3 \leqslant s \leqslant 30;\\ &t = 3 \text{ and } 3 \leqslant s \leqslant 8;\\ &t = 4 \text{ and } s = 3, 4;\\ &t = 5 \text{ and } s = 3. \end{aligned}$$

However, for such pairs (t, s), we easily observe that $\lambda \leqslant st < 100$, which is a contradiction. This completes the proof.

4 Proof of Corollary 1.2

Let $\mathcal{D}$ be a nontrivial symmetric (v, k, λ) design with λ prime. Suppose that G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$ of almost simple type with socle X. Since λ is prime, by [23], the socle X cannot be a sporadic simple group. If the socle X is a simple group of Lie type, then by [3, Theorem 1], $\mathcal{D}$ is the point-hyperplane design of $\mathrm{PG}(n-1, q)$ with $\lambda = (q^{n-2}-1)/(q-1)$ prime and $X = \mathrm{PSL}_n(q)$, or $(\mathcal{D}, G)$ is as in one of the lines 1-3 and lines 5-6 of Table 1. If the socle X is an alternating group A_c of degree $c \geqslant 5$, then Theorem 1.1 implies that $\mathcal{D}$ is the unique design $\mathrm{PG}_2(3, 2)$ with parameters $(15, 7, 3)$, and $G = \mathrm{A}_7$ with the point-stabiliser $\mathrm{PSL}_3(2)$, and this follows line 4 of Table 1. This finishes the proof.

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