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A compact presentation for the alternating central extension of the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

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Abstract

This paper concerns the positive part U_q^+ of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. The algebra U_q^+ has a presentation involving two generators that satisfy the cubic q-Serre relations. We recently introduced an algebra $\mathcal{U}_q^+$ called the alternating central extension of U_q^+. We presented $\mathcal{U}_q^+$ by generators and relations. The presentation is attractive, but the multitude of generators and relations makes the presentation unwieldy. In this paper we obtain a presentation of $\mathcal{U}_q^+$ that involves a small subset of the original set of generators and a very manageable set of relations. We call this presentation the compact presentation of $\mathcal{U}_q^+$.

Keywords: q-Onsager algebra, q-Serre relations, q-shuffle algebra, tridiagonal pair.

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1 Introduction

The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is well known in representation theory [15] and statistical mechanics [20]. This algebra has a subalgebra U_q^+ called the positive part. The algebra U_q^+ has a presentation involving two generators (said to be standard) and two relations, called the q-Serre relations. The presentation is given in Definition 2.1 below.

Our interest in U_q^+ is motivated by some applications to linear algebra and combinatorics; these will be described shortly. Before going into detail, we have a comment about q. In the applications, either q is not a root of unity, or q is a root of unity with exponent large enough to not interfere with the rest of the application. To keep things simple, throughout the paper we will assume that q is not a root of unity.

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Our first application has to do with tridiagonal pairs [17]. A tridiagonal pair is roughly described as an ordered pair of diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act on the eigenspaces of the other one in a block-tridiagonal fashion [17, Definition 1.1]. There is a type of tridiagonal pair said to be q-geometric [18, Definition 2.6]; for this type of tridiagonal pair the eigenvalues of each map form a q^2-geometric progression. A finite-dimensional irreducible U_q^+-module on which the standard generators are not nilpotent, is essentially the same thing as a tridiagonal pair of q-geometric type [18, Theorem 2.7]; these U_q^+-modules are described in [18, Section 1]. See [13, 24] for more background on tridiagonal pairs.

Our next application has to do with distance-regular graphs [1, 14, 32]. Consider a distance-regular graph Γ that has diameter $d \geq 3$ and classical parameters (d, b, α, β) [14, p. 193] with $b = q^2$ and $\alpha = q^2 - 1$. The condition on α implies that Γ is formally self-dual in the sense of [14, p. 49]. Let A denote the adjacency matrix of Γ, and let A^* denote the dual adjacency matrix with respect to any vertex of Γ [19, Section 7]. Then by [19, Lemma 9.4], there exist complex numbers r, s, r^*, s^* with r, r^* nonzero such that $rA + sI$, $r^*A^* + s^*I$ satisfy the q-Serre relations. As mentioned in [19, Example 8.4], the above parameter restriction is satisfied by the bilinear forms graph [14, p. 280], the alternating forms graph [14, p. 282], the Hermitean forms graph [14, p. 285], the quadratic forms graph [14, p. 290], the affine E_6 graph [14, p. 340], and the extended ternary Golay code graph [14, p. 359].

Our next application has to do with uniform posets [23, 27]. Let $\mathrm{GF}(b)$ denote a finite field with b elements, and let N, M denote positive integers. Let H denote a vector space over $\mathrm{GF}(b)$ that has dimension $N + M$. Let h denote a subspace of H with dimension M. Let P denote the set of subspaces of H that have zero intersection with h. For $x, y \in P$ define $x \leq y$ whenever $x \subseteq y$. The relation $\leq$ is a partial order on P, and the poset P is ranked with rank N. The poset P is called an attenuated space poset, and denoted by $A_b(N, M)$ [21], [27, Example 3.1]. By [27, Theorem 3.2] the poset $A_b(N, M)$ is uniform in the sense of [27, Definition 2.2]. It is shown in [21, Lemma 3.3] that for $A_b(N, M)$ the raising matrix R and the lowering matrix L satisfy the q-Serre relations, provided that $b = q^2$.

Our last application has to do with q-shuffle algebras. Let $\mathbb{F}$ denote a field, and let x, y denote noncommuting indeterminates. Let V denote the free associative $\mathbb{F}$-algebra with generators x, y. By a letter in V we mean x or y. For an integer $n \geq 0$, by a word of length n in V we mean a product of letters $v_1 v_2 \cdots v_n$. The words in V form a basis for the vector space V. In [25, 26] M. Rosso introduced an algebra structure on V, called the q-shuffle algebra. For letters u, v their q-shuffle product is $u \star v = uv + q^{\langle u,v \rangle} vu$, where $\langle u, v\rangle = 2$ (resp. $\langle u, v\rangle = -2$) if $u = v$ (resp. $u \neq v$). By [25, Theorem 13], in the q-shuffle algebra V the elements x, y satisfy the q-Serre relations. Consequently there exists an algebra homomorphism $\natural$ from U_q^+ into the q-shuffle algebra V, that sends the standard generators of U_q^+ to x, y. By [26, Theorem 15] the map $\natural$ is injective.

Next we recall the alternating elements in U_q^+ [30]. Let $v_1 v_2 \cdots v_n$ denote a word in V. This word is called alternating whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. Thus the alternating words have the form $\cdots xyxy \cdots$. The alternating words are displayed below:

$$
\begin{array}{lllll}
x, & xyx, & xyxyx, & xyxyxyx, & \ldots \\
y, & yxy, & yxyxy, & yxyxyxy, & \ldots \\
yx, & yxyx, & yxyxyx, & yxyxyxyx, & \ldots
\end{array}
$$

$$xy, \qquad xyxy, \qquad xyxyxy, \qquad xyxyxyxy, \qquad \ldots$$

By [30, Theorem 8.3] each alternating word is contained in the image of $\natural$. An element of U_q^+ is called alternating whenever it is the $\natural$-preimage of an alternating word. For example, the standard generators of U_q^+ are alternating because they are the $\natural$-preimages of the alternating words x, y. It is shown in [30, Lemma 5.12] that for each row in the above display, the corresponding alternating elements mutually commute. A naming scheme for alternating elements is introduced in [30, Definition 5.2].

Next we recall the alternating central extension of U_q^+ [29]. In [30] we displayed two types of relations among the alternating elements of U_q^+; the first type is [30, Propositions 5.7, 5.10, 5.11] and the second type is [30, Propositions 6.3, 8.1]. The relations in [30, Proposition 5.11] are redundant; they follow from the relations in [30, Propositions 5.7, 5.10] as pointed out in [2, Propositions 3.1, 3.2] and [5, Remark 2.5]; see also Corollary 6.3 below. The relations in [30, Proposition 6.3] are also redundant; they follow from the relations in [30, Propositions 5.7, 5.10] as shown in the proof of [30, Proposition 6.3]. By [30, Lemma 8.4] and the previous comments, the algebra U_q^+ is presented by its alternating elements and the relations in [30, Propositions 5.7, 5.10, 8.1]. For this presentation it is natural to ask what happens if the relations in [30, Proposition 8.1] are removed. To answer this question, in [29, Definition 3.1] we defined an algebra $\mathcal{U}_q^+$ by generators and relations in the following way. The generators, said to be alternating, are in bijection with the alternating elements of U_q^+. The relations are the ones in [30, Propositions 5.7, 5.10]. By construction there exists a surjective algebra homomorphism $\mathcal{U}_q^+ \to U_q^+$ that sends each alternating generator of $\mathcal{U}_q^+$ to the corresponding alternating element of U_q^+. In [29, Lemma 3.6, Theorem 5.17] we adjusted this homomorphism to get an algebra isomorphism $\mathcal{U}_q^+ \to U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$, where $\{z_n\}_{n=1}^{\infty}$ are mutually commuting indeterminates. By [29, Theorem 10.2] the alternating generators form a PBW basis for $\mathcal{U}_q^+$. The algebra $\mathcal{U}_q^+$ is called the alternating central extension of U_q^+.

We mentioned above that the algebra $\mathcal{U}_q^+$ is presented by its alternating generators and the relations in [30, Propositions 5.7, 5.10]. This presentation is attractive, but the multitude of generators and relations makes the presentation unwieldy. In this paper we obtain a presentation of $\mathcal{U}_q^+$ that involves a small subset of the original set of generators and a very manageable set of relations. This presentation is given in Definition 3.1 below; we call it the compact presentation of $\mathcal{U}_q^+$. At first glance, it is not clear that the algebra presented in Definition 3.1 is equal to $\mathcal{U}_q^+$. So we denote by $\mathcal{U}$ the algebra presented in Definition 3.1, and eventually prove that $\mathcal{U} = \mathcal{U}_q^+$. After this result is established, we describe some features of $\mathcal{U}_q^+$ that are illuminated by the presentation in Definition 3.1.

Our investigation of $\mathcal{U}_q^+$ is inspired by some recent developments in statistical mechanics, concerning the q-Onsager algebra O_q. In [9] Baseilhac and Koizumi introduce a current algebra $\mathcal{A}_q$ for O_q, in order to solve boundary integrable systems with hidden symmetries. In [12, Definition 3.1] Baseilhac and Shigechi give a presentation of $\mathcal{A}_q$ by generators and relations. This presentation and the discussion in [12, Section 4] suggest that $\mathcal{A}_q$ is related to O_q in roughly the same way that $\mathcal{U}_q^+$ is related to U_q^+. The relationship between $\mathcal{A}_q$ and O_q was conjectured in [7, Conjectures 1, 2] and [28, Conjectures 4.5, 4.6, 4.8], before being settled in [31, Theorems 9.14, 10.2, 10.3, 10.4]. The articles [3, 4, 6, 7, 8, 9, 10, 11, 12] contain background information on O_q and $\mathcal{A}_q$.

Earlier in this section, we indicated how U_q^+ has applications to tridiagonal pairs, distance-regular graphs, and uniform posets. Possibly $\mathcal{U}_q^+$ appears in these applications, and this possibility should be investigated in the future.

This paper is organized as follows. In Section 2 we review some facts about U_q^+. In Section 3, we introduce the algebra $\mathcal{U}$ and give an algebra homomorphism $U_q^+ \to \mathcal{U}$. In Section 4, we introduce the alternating generators for $\mathcal{U}$ and establish some formulas involving these generators. In Sections 5, 6 we use these formulas and generating functions to show that the alternating generators for $\mathcal{U}$ satisfy the relations in [30, Propositions 5.7, 5.10]. Using this result, we prove that $\mathcal{U} = \mathcal{U}_q^+$. Theorem 6.2 and Corollary 6.5 are the main results of the paper. In Section 7 we describe some features of $\mathcal{U}_q^+$ that are illuminated by the presentation in Definition 3.1. Appendix A contains a list of relations involving the generating functions from Section 5.

2 The algebra U_q^+

We now begin our formal argument. For the rest of the paper, the following notational conventions are in effect. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbb{F}$ denote a field. Every vector space and tensor product mentioned is over $\mathbb{F}$. Every algebra mentioned is associative, over $\mathbb{F}$, and has a multiplicative identity. Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad\qquad n \in \mathbb{N}.$$

For elements X, Y in any algebra, define their commutator and q-commutator by

$$[X, Y] = XY - YX, \qquad\qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3.$$

Definition 2.1 ([22, Corollary 3.2.6])**.** Define the algebra U_q^+ by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = 0, \qquad\qquad [W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = 0. \tag{2.1}$$

We call U_q^+ the *positive part of* $U_q(\widehat{\mathfrak{sl}}_2)$. The generators W_0, W_1 are called *standard*. The relations (2.1) are called the *q-Serre relations*.

We will use the following concept.

Definition 2.2 ([16, p. 299])**.** Let $\mathcal{A}$ denote an algebra. A *Poincaré-Birkhoff-Witt* (or *PBW*) basis for $\mathcal{A}$ consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω such that the following is a basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \dots, a_n \in \Omega, \qquad a_1 \le a_2 \le \cdots \le a_n.$$

We interpret the empty product as the multiplicative identity in $\mathcal{A}$.

In [16, p. 299] Damiani obtains a PBW basis for U_q^+ that involves some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad\qquad \{E_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad\qquad \{E_{n\delta}\}_{n=1}^{\infty}. \tag{2.2}$$

These elements are defined recursively as follows:

$$E_{\alpha_0} = W_0, \qquad E_{\alpha_1} = W_1, \qquad E_\delta = q^{-2}W_1W_0 - W_0W_1 \tag{2.3}$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q+q^{-1}}, \qquad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q+q^{-1}}, \tag{2.4}$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}W_0 - W_0E_{(n-1)\delta+\alpha_1}. \tag{2.5}$$

Proposition 2.3 ([16, p. 308]). *A PBW basis for U_q^+ is obtained by the elements (2.2) in the linear order*

$$E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots < E_\delta < E_{2\delta} < E_{3\delta} < \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}.$$

The elements (2.2) satisfy many relations [16]. We mention a few for later use.

Lemma 2.4 ([16, p. 300]). *For $i, j \in \mathbb{N}$ with $i > j$ the following hold in U_q^+.*

(i) *Assume that $i - j = 2r + 1$ is odd. Then*

$$E_{i\delta+\alpha_0}E_{j\delta+\alpha_0} = q^{-2}E_{j\delta+\alpha_0}E_{i\delta+\alpha_0} - (q^2 - q^{-2})\sum_{\ell=1}^{r} q^{-2\ell}E_{(j+\ell)\delta+\alpha_0}E_{(i-\ell)\delta+\alpha_0},$$

$$E_{j\delta+\alpha_1}E_{i\delta+\alpha_1} = q^{-2}E_{i\delta+\alpha_1}E_{j\delta+\alpha_1} - (q^2 - q^{-2})\sum_{\ell=1}^{r} q^{-2\ell}E_{(i-\ell)\delta+\alpha_1}E_{(j+\ell)\delta+\alpha_1}.$$

(ii) *Assume that $i - j = 2r$ is even. Then*

$$E_{i\delta+\alpha_0}E_{j\delta+\alpha_0} = q^{-2}E_{j\delta+\alpha_0}E_{i\delta+\alpha_0} - q^{j-i+1}(q - q^{-1})E^2_{(r+j)\delta+\alpha_0} - (q^2 - q^{-2})\sum_{\ell=1}^{r-1} q^{-2\ell}E_{(j+\ell)\delta+\alpha_0}E_{(i-\ell)\delta+\alpha_0},$$

$$E_{j\delta+\alpha_1}E_{i\delta+\alpha_1} = q^{-2}E_{i\delta+\alpha_1}E_{j\delta+\alpha_1} - q^{j-i+1}(q - q^{-1})E^2_{(r+j)\delta+\alpha_1} - (q^2 - q^{-2})\sum_{\ell=1}^{r-1} q^{-2\ell}E_{(i-\ell)\delta+\alpha_1}E_{(j+\ell)\delta+\alpha_1}.$$

Lemma 2.5. *The following (i) – (iii) hold in U_q^+.*

(i) (See [16, p. 307].) *For positive $i, j \in \mathbb{N}$,*

$$E_{i\delta}E_{j\delta} = E_{j\delta}E_{i\delta}. \tag{2.6}$$

(ii) (See [16, p. 307].) *For $i, j \in \mathbb{N}$,*

$$[E_{i\delta+\alpha_0}, E_{j\delta+\alpha_1}]_q = -qE_{(i+j+1)\delta}. \tag{2.7}$$

(iii) *For* $i \in \mathbb{N}$,

$$\frac{[W_0, E_{i\delta+\alpha_0}]_q}{q-q^{-1}} = \sum_{\ell=0}^{i} E_{\ell\delta+\alpha_0} E_{(i-\ell)\delta+\alpha_0}, \tag{2.8}$$

$$\frac{[E_{i\delta+\alpha_1}, W_1]_q}{q-q^{-1}} = \sum_{\ell=0}^{i} E_{(i-\ell)\delta+\alpha_1} E_{\ell\delta+\alpha_1}. \tag{2.9}$$

Proof. (iii) To verify (2.8) and (2.9), use Lemma 2.4 to write each term in the PBW basis for U_q^+ from Proposition 2.3. We give the details for (2.8). Referring to (2.8), let Δ denote the right-hand side minus the left-hand side. We show that $\Delta = 0$. This is quickly verified for $i = 0$, so assume that $i \geq 1$. For i even (resp. i odd) write $i = 2r$ (resp. $i = 2r + 1$). Using Lemma 2.4 we obtain $\Delta = \sum_{\ell=0}^{r} \alpha_\ell E_{\ell\delta+\alpha_0} E_{(i-\ell)\delta+\alpha_0}$, where for i even,

$$\alpha_0 = 1 + q^{-2} - \frac{q}{q-q^{-1}} + \frac{q^{-3}}{q-q^{-1}},$$

$$\alpha_\ell = 1 + q^{-2} - (q^2 - q^{-2}) \sum_{k=1}^{\ell} q^{-2k} - (q + q^{-1}) q^{-2\ell-1} \qquad (1 \leq \ell \leq r-1),$$

$$\alpha_r = 1 - (q - q^{-1}) \sum_{k=1}^{r} q^{1-2k} - q^{-i}$$

and for i odd,

$$\alpha_0 = 1 + q^{-2} - \frac{q}{q-q^{-1}} + \frac{q^{-3}}{q-q^{-1}},$$

$$\alpha_\ell = 1 + q^{-2} - (q^2 - q^{-2}) \sum_{k=1}^{\ell} q^{-2k} - (q + q^{-1}) q^{-2\ell-1} \qquad (1 \leq \ell \leq r).$$

For either case $\alpha_\ell = 0$ for $0 \leq \ell \leq r$, so $\Delta = 0$. We have verified (2.8). For (2.9) the details are similar, and omitted. □

3 An extension of U_q^+

In this section we introduce the algebra $\mathcal{U}$. In Section 6 we will show that $\mathcal{U}$ coincides with the alternating central extension $\mathcal{U}_q^+$ of U_q^+.

Definition 3.1. Define the algebra $\mathcal{U}$ by generators W_0, W_1, $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$ and relations

(i) $[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = 0$,

(ii) $[W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = 0$,

(iii) $[\tilde{G}_1, W_1] = q\frac{[[W_0,W_1]_q,W_1]}{q^2-q^{-2}}$,

(iv) $[W_0, \tilde{G}_1] = q\frac{[W_0,[W_0,W_1]_q]}{q^2-q^{-2}}$,

(v) for $k \geq 1$,

$$[\tilde{G}_{k+1}, W_1] = \frac{[[[\tilde{G}_k, W_0]_q, W_1]_q, W_1]}{(1-q^{-2})(q^2-q^{-2})},$$

(vi) for $k \geq 1$,

$$[W_0, \tilde{G}_{k+1}] = \frac{[W_0, [W_0, [W_1, \tilde{G}_k]_q]_q]}{(1-q^{-2})(q^2-q^{-2})},$$

(vii) for $k, \ell \in \mathbb{N}$,

$$[\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0.$$

For notational convenience define $\tilde{G}_0 = 1$.

Note 3.2. Referring to Definition 3.1, the relation (iii) (resp. (iv)) is obtained from (v) (resp. (vi)) by setting $k = 0$.

Lemma 3.3. *There exists a unique algebra homomorphism* $\flat : U_q^+ \to \mathcal{U}$ *that sends* $W_0 \mapsto W_0$ *and* $W_1 \mapsto W_1$.

Proof. Compare Definitions 2.1, 3.1. □

In Corollary 6.7 we will show that $\flat$ is injective. Let $\langle W_0, W_1 \rangle$ denote the subalgebra of $\mathcal{U}$ generated by W_0, W_1. Of course $\langle W_0, W_1 \rangle$ is the $\flat$-image of U_q^+. For the elements (2.2) of U_q^+, the same notation will be used for their $\flat$-images in $\langle W_0, W_1 \rangle$.

4 Augmenting the generating set for $\mathcal{U}$

Some of the relations in Definition 3.1 are nonlinear. Our next goal is to linearize the relations by adding more generators.

Definition 4.1. We define some elements in $\mathcal{U}$ as follows. For $k \in \mathbb{N}$,

$$W_{-k} = \frac{[\tilde{G}_k, W_0]_q}{q - q^{-1}}, \tag{4.1}$$

$$W_{k+1} = \frac{[W_1, \tilde{G}_k]_q}{q - q^{-1}}, \tag{4.2}$$

$$G_{k+1} = \tilde{G}_{k+1} + \frac{[W_1, W_{-k}]}{1 - q^{-2}}. \tag{4.3}$$

For notational convenience define $G_0 = 1$.

Lemma 4.2. *For* $k \in \mathbb{N}$ *the following hold in* $\mathcal{U}$*:*

$$\tilde{G}_k W_0 = q^{-2} W_0 \tilde{G}_k + (1 - q^{-2}) W_{-k},$$
$$\tilde{G}_k W_1 = q^2 W_1 \tilde{G}_k + (1 - q^2) W_{k+1}.$$

Proof. These are reformulations of (4.1) and (4.2). □

The following is a generating set for $\mathcal{U}$:

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}. \tag{4.4}$$

The elements of this set will be called *alternating*. We seek a presentation of $\mathcal{U}$, that has the above generating set and all relations linear. We will obtain this presentation in Theorem 6.2.

Next we obtain some formulas that will help us prove Theorem 6.2. We will show that for $n \in \mathbb{N}$,

$$W_{n+1} = \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}}, \tag{4.5}$$

$$W_{-n} = \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}}. \tag{4.6}$$

We will prove (4.5), (4.6) by induction on n. Note that (4.5), (4.6) hold for $n = 0$, since $W_1 = E_{\alpha_1}$ and $W_0 = E_{\alpha_0}$. We will give the main induction argument after a few lemmas. For the rest of this section k and ℓ are understood to be in $\mathbb{N}$.

Lemma 4.3. *Pick $n \in \mathbb{N}$, and assume that (4.5), (4.6) hold for $n, n-1, \dots, 1, 0$. Then*

$$[W_0, W_{n+1}] = [W_{-n}, W_1]. \tag{4.7}$$

Proof. The commutator $[W_0, W_{n+1}]$ is equal to

$$\begin{aligned}
&W_0W_{n+1} - W_{n+1}W_0 \\
&= \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n-k}W_0(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_1}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}\big(q^{-2}W_0\tilde{G}_{n-k} + (1-q^{-2})W_{k-n}\big)(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{\big(W_0E_{k\delta+\alpha_1} - q^{-2}E_{k\delta+\alpha_1}W_0\big)\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}W_{k-n}(-1)^k q^{k-1}}{(q-q^{-1})^{2k-1}} \\
&= -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}W_{k-n}(-1)^k q^{k-1}}{(q-q^{-1})^{2k-1}} \\
&= -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1}(-1)^k q^{k-1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_0}\tilde{G}_{n-k-\ell}(-1)^\ell q^{3\ell}}{(q-q^{-1})^{2\ell}} \\
&= -\sum_{p=0}^{n} \frac{E_{(p+1)\delta}\tilde{G}_{n-p}(-1)^p q^p}{(q-q^{-1})^{2p}} - \sum_{p=0}^{n} \Bigg(\sum_{k+\ell=p} q^{2\ell} E_{k\delta+\alpha_1}E_{\ell\delta+\alpha_0} \Bigg) \frac{\tilde{G}_{n-p}(-1)^p q^{p-1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}$$

The commutator $[W_{-n}, W_1]$ is equal to

$$
\begin{aligned}
& W_{-n}W_1 - W_1W_{-n} \\
& = \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}W_1(-1)^k q^{3k}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{W_1E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
& = \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\big(q^2W_1\tilde{G}_{n-k} + (1-q^2)W_{n-k+1}\big)(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
& \qquad\qquad - \sum_{k=0}^{n} \frac{W_1E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
& = \sum_{k=0}^{n} \frac{\big(q^2E_{k\delta+\alpha_0}W_1 - W_1E_{k\delta+\alpha_0}\big)\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
& \qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}W_{n-k+1}(-1)^k q^{3k+1}}{(q-q^{-1})^{2k-1}} \\
& = -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^{3k+2}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}W_{n-k+1}(-1)^k q^{3k+1}}{(q-q^{-1})^{2k-1}} \\
& = -\sum_{k=0}^{n} \frac{E_{(k+1)\delta}\tilde{G}_{n-k}(-1)^k q^{3k+2}}{(q-q^{-1})^{2k}} \\
& \qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}(-1)^k q^{3k+1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_1}\tilde{G}_{n-k-\ell}(-1)^\ell q^\ell}{(q-q^{-1})^{2\ell}} \\
& = -\sum_{p=0}^{n} \frac{E_{(p+1)\delta}\tilde{G}_{n-p}(-1)^p q^{3p+2}}{(q-q^{-1})^{2p}} \\
& \qquad\qquad - \sum_{p=0}^{n} \left(\sum_{k+\ell=p} q^{2k} E_{k\delta+\alpha_0}E_{\ell\delta+\alpha_1} \right) \frac{\tilde{G}_{n-p}(-1)^p q^{p+1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}
$$

By these comments

$$
[W_{-n}, W_1] - [W_0, W_{n+1}] = \sum_{p=0}^{n} \frac{C_p\tilde{G}_{n-p}(-1)^p q^p}{(q-q^{-1})^{2p}},
$$

where for $0 \le p \le n$,

$$
\begin{aligned}
C_p &= E_{(p+1)\delta} + q^{-1}(q-q^{-1}) \sum_{k+\ell=p} q^{2\ell} E_{k\delta+\alpha_1}E_{\ell\delta+\alpha_0} \\
& \qquad - q^{2p+2}E_{(p+1)\delta} - q(q-q^{-1}) \sum_{k+\ell=p} q^{2k} E_{k\delta+\alpha_0}E_{\ell\delta+\alpha_1} \\
& = (1-q^{2p+2})E_{(p+1)\delta} - (1-q^2) \sum_{k+\ell=p} q^{2\ell}\big(q^{-2}E_{k\delta+\alpha_1}E_{\ell\delta+\alpha_0} - E_{\ell\delta+\alpha_0}E_{k\delta+\alpha_1}\big) \\
& = (1-q^{2p+2})E_{(p+1)\delta} - (1-q^2) \sum_{k+\ell=p} q^{2\ell} E_{(p+1)\delta}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - q^{2p+2} - (1-q^2)\sum_{\ell=0}^{p} q^{2\ell}\right) E_{(p+1)\delta} \\
&= 0.
\end{aligned}
$$

The result follows. □

Lemma 4.4. *Pick $n \in \mathbb{N}$, and assume that (4.5), (4.6) hold for $n, n-1, \ldots, 1, 0$. Then*

$$
[\tilde{G}_n, E_\delta] = 0. \tag{4.8}
$$

Proof. Using Lemma 4.3,

$$
\begin{aligned}
0 &= (q - q^{-1})\big([W_{-n}, W_1] - [W_0, W_{n+1}]\big) \\
&= [[\tilde{G}_n, W_0]_q, W_1] - [W_0, [W_1, \tilde{G}_n]_q] \\
&= [\tilde{G}_n, [W_0, W_1]_q] \\
&= -q[\tilde{G}_n, E_\delta].
\end{aligned}
$$

□

Lemma 4.5. *Pick $n \in \mathbb{N}$, and assume that (4.5), (4.6) hold for $n, n-1, \ldots, 1, 0$. Then*

$$
[W_{-n}, W_0] = 0. \tag{4.9}
$$

Proof. The commutator $[W_{-n}, W_0]$ is equal to

$$
\begin{aligned}
&W_{-n}W_0 - W_0W_{-n} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\tilde{G}_{n-k}W_0(-1)^k q^{3k}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}\big(q^{-2}W_0\tilde{G}_{n-k} + (1-q^{-2})W_{k-n}\big)(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{W_0E_{k\delta+\alpha_0}\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{[W_0, E_{k\delta+\alpha_0}]_q\tilde{G}_{n-k}(-1)^{k-1} q^{3k-1}}{(q-q^{-1})^{2k}} + \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}W_{k-n}(-1)^k q^{3k-1}}{(q-q^{-1})^{2k-1}} \\
&= \sum_{k=0}^{n} \frac{[W_0, E_{k\delta+\alpha_0}]_q\tilde{G}_{n-k}(-1)^{k-1} q^{3k-1}}{(q-q^{-1})^{2k}} \\
&\qquad\qquad + \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_0}(-1)^k q^{3k-1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_0}\tilde{G}_{n-k-\ell}(-1)^\ell q^{3\ell}}{(q-q^{-1})^{2\ell}} \\
&= \sum_{p=0}^{n} \frac{[W_0, E_{p\delta+\alpha_0}]_q\tilde{G}_{n-p}(-1)^{p-1} q^{3p-1}}{(q-q^{-1})^{2p}} \\
&\qquad\qquad + \sum_{p=0}^{n} \left(\sum_{k+\ell=p} E_{k\delta+\alpha_0}E_{\ell\delta+\alpha_0}\right) \frac{\tilde{G}_{n-p}(-1)^p q^{3p-1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}
$$

By these comments

$$[W_{-n}, W_0] = \sum_{p=0}^{n} \frac{S_p \tilde{G}_{n-p}(-1)^{p-1} q^{3p-1}}{(q-q^{-1})^{2p-1}}$$

where

$$S_p = \frac{[W_0, E_{p\delta+\alpha_0}]_q}{q-q^{-1}} - \sum_{k+\ell=p} E_{k\delta+\alpha_0} E_{\ell\delta+\alpha_0} \qquad (0 \le p \le n).$$

By (2.8) we have $S_p = 0$ for $0 \le p \le n$. The result follows. □

Lemma 4.6. *Pick $n \in \mathbb{N}$, and assume that (4.5), (4.6) hold for $n, n-1, \ldots, 1, 0$. Then*

$$[W_{n+1}, W_1] = 0. \tag{4.10}$$

Proof. The commutator $[W_{n+1}, W_1]$ is equal to

$$\begin{aligned}
&W_{n+1}W_1 - W_1 W_{n+1} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} \tilde{G}_{n-k} W_1 (-1)^k q^k}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{W_1 E_{k\delta+\alpha_1} \tilde{G}_{n-k} (-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} \big(q^2 W_1 \tilde{G}_{n-k} + (1-q^2) W_{n-k+1}\big)(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{W_1 E_{k\delta+\alpha_1} \tilde{G}_{n-k} (-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n} \frac{[E_{k\delta+\alpha_1}, W_1]_q \tilde{G}_{n-k} (-1)^k q^{k+1}}{(q-q^{-1})^{2k}} - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} W_{n-k+1} (-1)^k q^{k+1}}{(q-q^{-1})^{2k-1}} \\
&= \sum_{k=0}^{n} \frac{[E_{k\delta+\alpha_1}, W_1]_q \tilde{G}_{n-k} (-1)^k q^{k+1}}{(q-q^{-1})^{2k}} \\
&\qquad\qquad - \sum_{k=0}^{n} \frac{E_{k\delta+\alpha_1} (-1)^k q^{k+1}}{(q-q^{-1})^{2k-1}} \sum_{\ell=0}^{n-k} \frac{E_{\ell\delta+\alpha_1} \tilde{G}_{n-k-\ell} (-1)^\ell q^\ell}{(q-q^{-1})^{2\ell}} \\
&= \sum_{p=0}^{n} \frac{[E_{p\delta+\alpha_1}, W_1]_q \tilde{G}_{n-p} (-1)^p q^{p+1}}{(q-q^{-1})^{2p}} \\
&\qquad\qquad - \sum_{p=0}^{n} \left(\sum_{k+\ell=p} E_{k\delta+\alpha_1} E_{\ell\delta+\alpha_1} \right) \frac{\tilde{G}_{n-p} (-1)^p q^{p+1}}{(q-q^{-1})^{2p-1}}.
\end{aligned}$$

By these comments

$$[W_{n+1}, W_1] = \sum_{p=0}^{n} \frac{T_p \tilde{G}_{n-p} (-1)^p q^{p+1}}{(q-q^{-1})^{2p-1}}$$

where

$$T_p = \frac{[E_{p\delta+\alpha_1}, W_1]_q}{q-q^{-1}} - \sum_{k+\ell=p} E_{k\delta+\alpha_1} E_{\ell\delta+\alpha_1} \qquad (0 \leq p \leq n).$$

By (2.9) we have $T_p = 0$ for $0 \leq p \leq n$. The result follows. □

Proposition 4.7. *The equations (4.5), (4.6) hold in $\mathcal{U}$ for $n \in \mathbb{N}$.*

Proof. The proof is by induction on n. We assume that (4.5), (4.6) hold for $n, n-1, \ldots, 1, 0$, and show that (4.5), (4.6) hold for $n+1$. Concerning (4.5),

$$\begin{aligned}
W_{n+2} &= \frac{qW_1\tilde{G}_{n+1} - q^{-1}\tilde{G}_{n+1}W_1}{q-q^{-1}} && \text{by (4.2)} \\
&= W_1\tilde{G}_{n+1} - q^{-1}\frac{[\tilde{G}_{n+1}, W_1]}{q-q^{-1}} \\
&= W_1\tilde{G}_{n+1} - \frac{[[[\tilde{G}_n, W_0]_q, W_1]_q, W_1]}{(q-q^{-1})^2(q^2-q^{-2})} && \text{by Definition 3.1(v)} \\
&= W_1\tilde{G}_{n+1} - \frac{[[W_{-n}, W_1]_q, W_1]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (4.1)} \\
&= W_1\tilde{G}_{n+1} - \frac{[[W_{-n}, W_1], W_1]_q}{(q-q^{-1})(q^2-q^{-2})} \\
&= W_1\tilde{G}_{n+1} - \frac{[[W_0, W_{n+1}], W_1]_q}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.3} \\
&= W_1\tilde{G}_{n+1} - \frac{[[W_0, W_1]_q, W_{n+1}]}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.6} \\
&= W_1\tilde{G}_{n+1} + \frac{q[E_\delta, W_{n+1}]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (2.3)} \\
&= W_1\tilde{G}_{n+1} + q\sum_{k=0}^{n} \frac{[E_\delta, E_{k\delta+\alpha_1}\tilde{G}_{n-k}](-1)^k q^k}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by (4.5) and induction} \\
&= W_1\tilde{G}_{n+1} + q\sum_{k=0}^{n} \frac{[E_\delta, E_{k\delta+\alpha_1}]\tilde{G}_{n-k}(-1)^k q^k}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by Lemma 4.4} \\
&= W_1\tilde{G}_{n+1} + \sum_{k=0}^{n} \frac{E_{(k+1)\delta+\alpha_1}\tilde{G}_{n-k}(-1)^{k+1}q^{k+1}}{(q-q^{-1})^{2k+2}} && \text{by (2.4)} \\
&= E_{\alpha_1}\tilde{G}_{n+1} + \sum_{k=1}^{n+1} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n+1-k}(-1)^k q^k}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n+1} \frac{E_{k\delta+\alpha_1}\tilde{G}_{n+1-k}(-1)^k q^k}{(q-q^{-1})^{2k}}.
\end{aligned}$$

We have shown that (4.5) holds for $n+1$. Concerning (4.6),

$$W_{-n-1} = \frac{q\tilde{G}_{n+1}W_0 - q^{-1}W_0\tilde{G}_{n+1}}{q-q^{-1}} \qquad \text{by (4.1)}$$

$$
\begin{aligned}
&= W_0\tilde{G}_{n+1} - q\frac{[W_0, \tilde{G}_{n+1}]}{q-q^{-1}} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_0, [W_1, \tilde{G}_n]_q]_q]}{(q-q^{-1})^2(q^2-q^{-2})} && \text{by Definition 3.1(vi)} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_0, W_{n+1}]_q]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (4.2)} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_0, W_{n+1}]]_q}{(q-q^{-1})(q^2-q^{-2})} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_0, [W_{-n}, W_1]]_q}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.3} \\
&= W_0\tilde{G}_{n+1} - q^2\frac{[W_{-n}, [W_0, W_1]_q]}{(q-q^{-1})(q^2-q^{-2})} && \text{by Lemma 4.5} \\
&= W_0\tilde{G}_{n+1} + q^3\frac{[W_{-n}, E_\delta]}{(q-q^{-1})(q^2-q^{-2})} && \text{by (2.3)} \\
&= W_0\tilde{G}_{n+1} + q^3\sum_{k=0}^{n}\frac{[E_{k\delta+\alpha_0}\tilde{G}_{n-k}, E_\delta](-1)^k q^{3k}}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by (4.6) and induction} \\
&= W_0\tilde{G}_{n+1} + q^3\sum_{k=0}^{n}\frac{[E_{k\delta+\alpha_0}, E_\delta]\tilde{G}_{n-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k+1}(q^2-q^{-2})} && \text{by Lemma 4.4} \\
&= W_0\tilde{G}_{n+1} + \sum_{k=0}^{n}\frac{E_{(k+1)\delta+\alpha_0}\tilde{G}_{n-k}(-1)^{k+1} q^{3k+3}}{(q-q^{-1})^{2k+2}} && \text{by (2.4)} \\
&= E_{\alpha_0}\tilde{G}_{n+1} + \sum_{k=1}^{n+1}\frac{E_{k\delta+\alpha_0}\tilde{G}_{n+1-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}} \\
&= \sum_{k=0}^{n+1}\frac{E_{k\delta+\alpha_0}\tilde{G}_{n+1-k}(-1)^k q^{3k}}{(q-q^{-1})^{2k}}.
\end{aligned}
$$

We have shown that (4.6) holds for $n+1$. □

Lemma 4.8. *The following relations hold in $\mathcal{U}$. For $n \in \mathbb{N}$,*

$$
\begin{aligned}
[W_0, W_{n+1}] &= [W_{-n}, W_1], & [\tilde{G}_n, E_\delta] &= 0, \\
[W_{-n}, W_0] &= 0, & [W_{n+1}, W_1] &= 0.
\end{aligned}
$$

Proof. By Lemmas 4.3 – 4.6 and Proposition 4.7. □

Lemma 4.9. *The following relations hold in $\mathcal{U}$. For $k \in \mathbb{N}$,*

(i) $[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q$;

(ii) $[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q$.

Proof. (i) We have

$$
[G_{k+1}, W_1]_q - [W_1, \tilde{G}_{k+1}]_q = \left[\tilde{G}_{k+1} + \frac{[W_1, W_{-k}]}{1-q^{-2}}, W_1\right]_q - [W_1, \tilde{G}_{k+1}]_q
$$

$$\begin{aligned}
&= (q+q^{-1})[\tilde{G}_{k+1}, W_1] - \frac{[[W_{-k}, W_1], W_1]_q}{1-q^{-2}} \\
&= (q+q^{-1})[\tilde{G}_{k+1}, W_1] - \frac{[[W_{-k}, W_1]_q, W_1]}{1-q^{-2}} \\
&= (q+q^{-1})[\tilde{G}_{k+1}, W_1] - \frac{[[[\tilde{G}_k, W_0]_q, W_1]_q, W_1]}{(1-q^{-2})(q-q^{-1})} \\
&= 0.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
[W_0, G_{k+1}]_q - [\tilde{G}_{k+1}, W_0]_q &= \left[W_0, \tilde{G}_{k+1} + \frac{[W_{k+1}, W_0]}{1-q^{-2}}\right]_q - [\tilde{G}_{k+1}, W_0]_q \\
&= (q+q^{-1})[W_0, \tilde{G}_{k+1}] - \frac{[W_0, [W_0, W_{k+1}]]_q}{1-q^{-2}} \\
&= (q+q^{-1})[W_0, \tilde{G}_{k+1}] - \frac{[W_0, [W_0, W_{k+1}]_q]}{1-q^{-2}} \\
&= (q+q^{-1})[W_0, \tilde{G}_{k+1}] - \frac{[W_0, [W_0, [W_1, \tilde{G}_k]_q]_q]}{(1-q^{-2})(q-q^{-1})} \\
&= 0.
\end{aligned}$$

□

5 Generating functions

The alternating generators of $\mathcal{U}$ are displayed in (4.4). In the previous section we described how these generators are related to W_0 and W_1. Our next goal is to describe how the alternating generators are related to each other. It is convenient to use generating functions.

Definition 5.1. We define some generating functions in an indeterminate t. Referring to (4.4),

$$G(t) = \sum_{n\in\mathbb{N}} G_n t^n, \qquad \tilde{G}(t) = \sum_{n\in\mathbb{N}} \tilde{G}_n t^n,$$

$$W^-(t) = \sum_{n\in\mathbb{N}} W_{-n} t^n, \qquad W^+(t) = \sum_{n\in\mathbb{N}} W_{n+1} t^n.$$

Lemma 5.2. *For the algebra* $\mathcal{U}$*,*

$$\frac{[W_0, G(t)]_q}{q-q^{-1}} = W^-(t), \qquad \frac{[\tilde{G}(t), W_0]_q}{q-q^{-1}} = W^-(t),$$

$$[W_0, W^-(t)] = 0, \qquad \frac{[W_0, W^+(t)]}{1-q^{-2}} = t^{-1}(\tilde{G}(t) - G(t))$$

and

$$\frac{[G(t), W_1]_q}{q-q^{-1}} = W^+(t), \qquad \frac{[W_1, \tilde{G}(t)]_q}{q-q^{-1}} = W^+(t),$$

$$[W_1, W^+(t)] = 0, \qquad \frac{[W_1, W^-(t)]}{1-q^{-2}} = t^{-1}(G(t) - \tilde{G}(t)).$$

Proof. Use Definition 4.1 and Lemmas 4.8, 4.9. □

For the rest of this section, let s denote an indeterminate that commutes with t.

Lemma 5.3. *For the algebra $\mathcal{U}$,*

$$
\begin{aligned}
&[W^-(s), W^-(t)] = 0, \qquad [W^+(s), W^+(t)] = 0,\\
&[W^-(s), W^+(t)] + [W^+(s), W^-(t)] = 0,\\
&s[W^-(s), G(t)] + t[G(s), W^-(t)] = 0,\\
&s[W^-(s), \tilde{G}(t)] + t[\tilde{G}(s), W^-(t)] = 0,\\
&s[W^+(s), G(t)] + t[G(s), W^+(t)] = 0,\\
&s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)] = 0,\\
&[G(s), G(t)] = 0, \qquad [\tilde{G}(s), \tilde{G}(t)] = 0,\\
&[\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)] = 0
\end{aligned}
$$

and also

$$
\begin{aligned}
&[W^-(s), G(t)]_q = [W^-(t), G(s)]_q, \qquad [G(s), W^+(t)]_q = [G(t), W^+(s)]_q,\\
&[\tilde{G}(s), W^-(t)]_q = [\tilde{G}(t), W^-(s)]_q, \qquad [W^+(s), \tilde{G}(t)]_q = [W^+(t), \tilde{G}(s)]_q,\\
&t^{-1}[G(s), \tilde{G}(t)] - s^{-1}[G(t), \tilde{G}(s)] = q[W^-(t), W^+(s)]_q - q[W^-(s), W^+(t)]_q,\\
&t^{-1}[\tilde{G}(s), G(t)] - s^{-1}[\tilde{G}(t), G(s)] = q[W^+(t), W^-(s)]_q - q[W^+(s), W^-(t)]_q,\\
&[G(s), \tilde{G}(t)]_q - [G(t), \tilde{G}(s)]_q = qt[W^-(t), W^+(s)] - qs[W^-(s), W^+(t)],\\
&[\tilde{G}(s), G(t)]_q - [\tilde{G}(t), G(s)]_q = qt[W^+(t), W^-(s)] - qs[W^+(s), W^-(t)].
\end{aligned}
$$

Proof. We refer to the generating functions $A(s,t), B(s,t), \ldots, S(s,t)$ from Appendix A. The present lemma asserts that for the algebra $\mathcal{U}$ these generating functions are all zero. To verify this assertion, we refer to the canonical relations in Appendix A. We will use induction with respect to the linear order

$$
\begin{aligned}
&I(s,t), M(s,t), N(s,t), A(s,t), B(s,t), Q(s,t), D(s,t), E(s,t), F(s,t),\\
&G(s,t), R(s,t), S(s,t), H(s,t), K(s,t), L(s,t), P(s,t), C(s,t), J(s,t).
\end{aligned}
$$

For each element in this linear order besides $I(s,t)$, there exists a canonical relation that expresses the given element in terms of the previous elements in the linear order. So in $\mathcal{U}$ the given element is zero, provided that in $\mathcal{U}$ every previous element is zero. Note that in $\mathcal{U}$ we have $I(s,t) = 0$ by Definition 3.1(vii). By these comments and induction, in $\mathcal{U}$ every element in the linear order is zero. We have shown that in $\mathcal{U}$ each of $A(s,t)$, $B(s,t), \ldots, S(s,t)$ is zero. □

6 The main results

In this section we present our main results, which are Theorem 6.2 and Corollary 6.5. Recall the alternating generators (4.4) for $\mathcal{U}$.

Lemma 6.1. *The following relations hold in $\mathcal{U}$. For $k, \ell \in \mathbb{N}$ we have*

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \tag{6.1}$$
$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1}, \tag{6.2}$$
$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}, \tag{6.3}$$
$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \tag{6.4}$$
$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \tag{6.5}$$
$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \tag{6.6}$$
$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \tag{6.7}$$
$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \tag{6.8}$$
$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \tag{6.9}$$
$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \tag{6.10}$$
$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0 \tag{6.11}$$

and also

$$[W_{-k}, G_\ell]_q = [W_{-\ell}, G_k]_q, \qquad [G_k, W_{\ell+1}]_q = [G_\ell, W_{k+1}]_q, \tag{6.12}$$
$$[\tilde{G}_k, W_{-\ell}]_q = [\tilde{G}_\ell, W_{-k}]_q, \qquad [W_{\ell+1}, \tilde{G}_k]_q = [W_{k+1}, \tilde{G}_\ell]_q, \tag{6.13}$$
$$[G_k, \tilde{G}_{\ell+1}] - [G_\ell, \tilde{G}_{k+1}] = q[W_{-\ell}, W_{k+1}]_q - q[W_{-k}, W_{\ell+1}]_q, \tag{6.14}$$
$$[\tilde{G}_k, G_{\ell+1}] - [\tilde{G}_\ell, G_{k+1}] = q[W_{\ell+1}, W_{-k}]_q - q[W_{k+1}, W_{-\ell}]_q, \tag{6.15}$$
$$[G_{k+1}, \tilde{G}_{\ell+1}]_q - [G_{\ell+1}, \tilde{G}_{k+1}]_q = q[W_{-\ell}, W_{k+2}] - q[W_{-k}, W_{\ell+2}], \tag{6.16}$$
$$[\tilde{G}_{k+1}, G_{\ell+1}]_q - [\tilde{G}_{\ell+1}, G_{k+1}]_q = q[W_{\ell+1}, W_{-k-1}] - q[W_{k+1}, W_{-\ell-1}]. \tag{6.17}$$

Proof. The relations (6.1) – (6.3) are from Definition 4.1 and Lemmas 4.8, 4.9. The relations (6.4) – (6.17) follow from Definition 5.1 and Lemma 5.3. □

Theorem 6.2. *The algebra $\mathcal{U}$ has a presentation by generators*

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations in Lemma 6.1.

Proof. It suffices to show that the relations in Definition 3.1 are implied by the relations in Lemma 6.1. The relation (iii) in Definition 3.1 is obtained from the equation on the left in (6.3) at $k = 0$, by eliminating G_1 using $[W_0, W_1] = (1 - q^{-2})(\tilde{G}_1 - G_1)$. The relation (iv) in Definition 3.1 is obtained from the equation on the left in (6.2) at $k = 0$, by eliminating G_1 using $[W_0, W_1] = (1 - q^{-2})(\tilde{G}_1 - G_1)$. For $k \geq 1$ the relation (v) in Definition 3.1 is obtained from the equation on the left in (6.3), by eliminating G_{k+1} using $[W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1})$ and evaluating the result using $[\tilde{G}_k, W_0]_q = (q - q^{-1})W_{-k}$. For $k \geq 1$ the relation (vi) in Definition 3.1 is obtained from the equation on the left in (6.2), by eliminating G_{k+1} using $[W_0, W_{k+1}] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1})$ and evaluating the result using $[W_1, \tilde{G}_k]_q = (q - q^{-1})W_{k+1}$. The relation (vii) in Definition 3.1 is from (6.10). The relation (i) in Definition 3.1 is obtained from $[W_0, W_{-1}] = 0$, by eliminating W_{-1} using $[\tilde{G}_1, W_0]_q = (q - q^{-1})W_{-1}$ and evaluating the result using Definition 3.1(iv). The relation (ii) in Definition 3.1 is obtained from $[W_1, W_2] = 0$, by eliminating W_2 using $[W_1, \tilde{G}_1]_q = (q - q^{-1})W_2$ and evaluating the result using Definition 3.1(iii). □

It is apparent from the proof of Theorem 6.2 that the relations in Lemma 6.1 are redundant in the following sense.

Corollary 6.3. The relations in Lemma 6.1 are implied by the relations listed in (i) – (iii) below:

(i) (6.1) – (6.3);

(ii) (6.4) with $k = 0$ and $\ell = 1$;

(iii) the relations on the right in (6.10).

Proof. By Lemma 6.1 the relations (6.1) – (6.17) are implied by the relations in Definitions 3.1, 4.1. The relations listed in (i) – (iii) are used in the proof of Theorem 6.2 to obtain the relations in Definition 3.1. The relations listed in (i) imply the relations in Definition 4.1. The result follows. □

The relations in Lemma 6.1 first appeared in [30, Propositions 5.7, 5.10, 5.11]. It was observed in [2, Propositions 3.1, 3.2] and [5, Remark 2.5] that the relations (6.1) – (6.11) imply the relations (6.12) – (6.17). This observation motivated the following definition.

Definition 6.4 ([29, Definition 3.1])**.** Define the algebra $\mathcal{U}_q^+$ by generators

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the relations (6.1) – (6.11). The algebra $\mathcal{U}_q^+$ is called the *alternating central extension of* U_q^+.

Corollary 6.5. *We have* $\mathcal{U} = \mathcal{U}_q^+$.

Proof. By Theorem 6.2, Corollary 6.3, and Definition 6.4. □

Definition 6.6. By the *compact* presentation of $\mathcal{U}_q^+$ we mean the presentation given in Definition 3.1. By the *expanded* presentation of $\mathcal{U}_q^+$ we mean the presentation given in Theorem 6.2.

Corollary 6.7. *The map* $\flat$ *from* Lemma 3.3 *is injective.*

Proof. By Corollary 6.5 and [29, Proposition 6.4]. □

7 The subalgebra of $\mathcal{U}_q^+$ generated by $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$

Let $\tilde{G}$ denote the subalgebra of $\mathcal{U}_q^+$ generated by $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$. In this section we describe $\tilde{G}$ and its relationship to $\langle W_0, W_1\rangle$.

The following notation will be useful. Let $z_1, z_2, \dots$ denote mutually commuting indeterminates. Let $\mathbb{F}[z_1, z_2, \dots]$ denote the algebra consisting of the polynomials in $z_1, z_2, \dots$ that have all coefficients in $\mathbb{F}$. For notational convenience define $z_0 = 1$.

Lemma 7.1 ([29, Lemma 3.5])**.** *There exists an algebra homomorphism* $\mathcal{U}_q^+ \to \mathbb{F}[z_1, z_2, \dots]$ *that sends*

$$W_{-n} \mapsto 0, \qquad W_{n+1} \mapsto 0, \qquad G_n \mapsto z_n, \qquad \tilde{G}_n \mapsto z_n$$

for $n \in \mathbb{N}$.

Proof. By Theorem 6.2 and the nature of the relations in Lemma 6.1. □

Corollary 7.2 ([29, Theorem 10.2])**.** *The generators* $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$ *of* $\tilde{G}$ *are algebraically independent.*

Proof. By Lemma 7.1 and since $\{z_{k+1}\}_{k\in\mathbb{N}}$ are algebraically independent. □

The following result will help us describe how $\tilde{G}$ is related to $\langle W_0, W_1\rangle$.

Lemma 7.3. *For* $n \in \mathbb{N}$,

$$\tilde{G}_n W_1 = W_1 \tilde{G}_n + \sum_{k=1}^{n} \frac{E_{k\delta+\alpha_1} \tilde{G}_{n-k} (-1)^{k+1} q^{k+1}}{(q-q^{-1})^{2k-1}}, \tag{7.1}$$

$$\tilde{G}_n W_0 = W_0 \tilde{G}_n + \sum_{k=1}^{n} \frac{E_{k\delta+\alpha_0} \tilde{G}_{n-k} (-1)^{k} q^{3k-1}}{(q-q^{-1})^{2k-1}}. \tag{7.2}$$

Proof. To obtain (7.1), eliminate W_{n+1} from (4.5) using (4.2), and solve the resulting equation for $\tilde{G}_n W_1$. To obtain (7.2), eliminate W_{-n} from (4.6) using (4.1), and solve the resulting equation for $\tilde{G}_n W_0$. □

Shortly we will describe how $\tilde{G}$ is related to $\langle W_0, W_1\rangle$. This description involves the center $\mathcal{Z}$ of $\mathcal{U}_q^+$. To prepare for this description, we have some comments about $\mathcal{Z}$. In [29, Sections 5, 6] we introduced some algebraically independent elements $Z_1, Z_2, \ldots$ that generate the algebra $\mathcal{Z}$. For notational convenience define $Z_0 = 1$. Using $\{Z_n\}_{n\in\mathbb{N}}$ we obtain a basis for $\mathcal{Z}$ that is described as follows. For $n \in \mathbb{N}$, a *partition of* n is a sequence $\lambda = \{\lambda_i\}_{i=1}^{\infty}$ of natural numbers such that $\lambda_i \geq \lambda_{i+1}$ for $i \geq 1$ and $n = \sum_{i=1}^{\infty} \lambda_i$. The set Λ_n consists of the partitions of n. Define $\Lambda = \cup_{n\in\mathbb{N}} \Lambda_n$. For $\lambda \in \Lambda$ define $Z_\lambda = \prod_{i=1}^{\infty} Z_{\lambda_i}$. The elements $\{Z_\lambda\}_{\lambda\in\Lambda}$ form a basis for the vector space $\mathcal{Z}$. Next we describe a grading for $\mathcal{Z}$. For $n \in \mathbb{N}$ let $\mathcal{Z}_n$ denote the subspace of $\mathcal{Z}$ with basis $\{Z_\lambda\}_{\lambda\in\Lambda_n}$. For example $\mathcal{Z}_0 = \mathbb{F}1$. The sum $\mathcal{Z} = \sum_{n\in\mathbb{N}} \mathcal{Z}_n$ is direct. Moreover $\mathcal{Z}_r\mathcal{Z}_s \subseteq \mathcal{Z}_{r+s}$ for $r, s \in \mathbb{N}$. By these comments the subspaces $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ form a grading of $\mathcal{Z}$. Note that $Z_n \in \mathcal{Z}_n$ for $n \in \mathbb{N}$. Next we describe how $\mathcal{Z}$ is related to $\langle W_0, W_1\rangle$.

Lemma 7.4 ([29, Proposition 6.5])**.** *The multiplication map*

$$\begin{aligned} \langle W_0, W_1\rangle \otimes \mathcal{Z} &\to \mathcal{U}_q^+ \\ w \otimes z &\mapsto wz \end{aligned}$$

is an algebra isomorphism.

For $n \in \mathbb{N}$ let $\mathcal{U}_n$ denote the image of $\langle W_0, W_1\rangle \otimes \mathcal{Z}_n$ under the multiplication map. By construction the sum $\mathcal{U}_q^+ = \sum_{n\in\mathbb{N}} \mathcal{U}_n$ is direct.

In the next two lemmas we describe how $\tilde{G}$ is related to $\mathcal{Z}$.

Lemma 7.5 ([29, Lemmas 3.6, 5.9])**.** *For* $n \in \mathbb{N}$,

$$\tilde{G}_n \in \sum_{k=0}^{n} \langle W_0, W_1\rangle Z_k, \qquad \tilde{G}_n - Z_n \in \sum_{k=0}^{n-1} \langle W_0, W_1\rangle Z_k.$$

For $\lambda \in \Lambda$ define $\tilde{G}_\lambda = \prod_{i=1}^{\infty} \tilde{G}_{\lambda_i}$. By Corollary 7.2 the elements $\{\tilde{G}_\lambda\}_{\lambda \in \Lambda}$ form a basis for the vector space $\tilde{G}$.

Lemma 7.6. *For $n \in \mathbb{N}$ and $\lambda \in \Lambda_n$,*

$$\tilde{G}_\lambda \in \sum_{k=0}^{n} \mathcal{U}_k, \qquad\qquad \tilde{G}_\lambda - Z_\lambda \in \sum_{k=0}^{n-1} \mathcal{U}_k.$$

Proof. By Lemma 7.5 and our comments above Lemma 7.4 about the grading of $\mathcal{Z}$. □

Next we describe how $\tilde{G}$ is related to $\langle W_0, W_1 \rangle$.

Proposition 7.7. *The multiplication map*

$$\begin{aligned} \langle W_0, W_1 \rangle \otimes \tilde{G} &\to \mathcal{U}_q^+ \\ w \otimes g &\mapsto wg \end{aligned}$$

is an isomorphism of vector spaces.

Proof. The multiplication map is $\mathbb{F}$-linear. The multiplication map is surjective by Lemma 7.3 and since $\mathcal{U}_q^+$ is generated by W_0, W_1, $\tilde{G}$. We now show that the multiplicaton map is injective. Consider a vector $v \in \langle W_0, W_1 \rangle \otimes \tilde{G}$ that is sent to zero by the multiplication map. We show that $v = 0$. Write $v = \sum_{\lambda \in \Lambda} a_\lambda \otimes \tilde{G}_\lambda$, where $a_\lambda \in \langle W_0, W_1 \rangle$ for $\lambda \in \Lambda$ and $a_\lambda = 0$ for all but finitely many $\lambda \in \Lambda$. To show that $v = 0$, we must show that $a_\lambda = 0$ for all $\lambda \in \Lambda$. Suppose that there exists $\lambda \in \Lambda$ such that $a_\lambda \neq 0$. Let C denote the set of natural numbers m such that Λ_m contains a partition λ with $a_\lambda \neq 0$. The set C is nonempty and finite. Let n denote the maximal element of C. By construction $\sum_{\lambda \in \Lambda_n} a_\lambda \otimes Z_\lambda$ is nonzero. By Lemma 7.4,

$$\sum_{\lambda \in \Lambda_n} a_\lambda Z_\lambda \neq 0. \tag{7.3}$$

By construction

$$0 = \sum_{\lambda \in \Lambda} a_\lambda \tilde{G}_\lambda = \sum_{k=0}^{n} \sum_{\lambda \in \Lambda_k} a_\lambda \tilde{G}_\lambda = \sum_{\lambda \in \Lambda_n} a_\lambda \tilde{G}_\lambda + \sum_{k=0}^{n-1} \sum_{\lambda \in \Lambda_k} a_\lambda \tilde{G}_\lambda. \tag{7.4}$$

Using (7.4),

$$\sum_{\lambda \in \Lambda_n} a_\lambda Z_\lambda = \sum_{\lambda \in \Lambda_n} a_\lambda (Z_\lambda - \tilde{G}_\lambda) - \sum_{k=0}^{n-1} \sum_{\lambda \in \Lambda_k} a_\lambda \tilde{G}_\lambda. \tag{7.5}$$

The left-hand side of (7.5) is contained in $\mathcal{U}_n$. By Lemma 7.6 the right-hand side of (7.5) is contained in $\sum_{k=0}^{n-1} \mathcal{U}_k$. The subspaces $\mathcal{U}_n$ and $\sum_{k=0}^{n-1} \mathcal{U}_k$ have zero intersection because the sum $\sum_{k=0}^{n} \mathcal{U}_k$ is direct. This contradicts (7.3), so $a_\lambda = 0$ for $\lambda \in \Lambda$. Consequently $v = 0$, as desired. We have shown that the multiplication map is injective. By the above comments the multiplication map is an isomorphism of vector spaces. □

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Appendix A

Recall the algebra $\mathcal{U}$ from Definition 3.1. In this appendix we list some relations that hold in $\mathcal{U}$. We will define an algebra $\mathcal{U}^\vee$ that is a homomorphic preimage of $\mathcal{U}$. All the results in this appendix are about $\mathcal{U}^\vee$.

Define the algebra $\mathcal{U}^\vee$ by generators

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$$

and the following relations. For $k \in \mathbb{N}$,

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (1-q^{-2})(\tilde{G}_{k+1} - G_{k+1}), \tag{A.1}$$

$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q-q^{-1})W_{-k-1}, \tag{A.2}$$

$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q-q^{-1})W_{k+2}, \tag{A.3}$$

$$[W_0, W_{-k}] = 0, \qquad [W_1, W_{k+1}] = 0. \tag{A.4}$$

For notational convenience, define $G_0 = 1$ and $\tilde{G}_0 = 1$.

For $\mathcal{U}^\vee$ we define the generating functions $W^-(t)$, $W^+(t)$, $G(t)$, $\tilde{G}(t)$ as in Definition 5.1. In terms of these generating functions, the relations (A.1) – (A.4) become the relations in Lemma 5.2. Let s denote an indeterminate that commutes with t. Define

$$\begin{aligned}
A(s,t) &= [W^-(s), W^-(t)],\\
B(s,t) &= [W^+(s), W^+(t)],\\
C(s,t) &= [W^-(s), W^+(t)] + [W^+(s), W^-(t)],\\
D(s,t) &= s[W^-(s), G(t)] + t[G(s), W^-(t)],\\
E(s,t) &= s[W^-(s), \tilde{G}(t)] + t[\tilde{G}(s), W^-(t)],\\
F(s,t) &= s[W^+(s), G(t)] + t[G(s), W^+(t)],\\
G(s,t) &= s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)],\\
H(s,t) &= [G(s), G(t)],\\
I(s,t) &= [\tilde{G}(s), \tilde{G}(t)],\\
J(s,t) &= [\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)]
\end{aligned}$$

and also

$$\begin{aligned}
K(s,t) &= [W^-(s), G(t)]_q - [W^-(t), G(s)]_q,\\
L(s,t) &= [G(s), W^+(t)]_q - [G(t), W^+(s)]_q,\\
M(s,t) &= [\tilde{G}(s), W^-(t)]_q - [\tilde{G}(t), W^-(s)]_q,\\
N(s,t) &= [W^+(s), \tilde{G}(t)]_q - [W^+(t), \tilde{G}(s)]_q,\\
P(s,t) &= t^{-1}[G(s), \tilde{G}(t)] - s^{-1}[G(t), \tilde{G}(s)] - q[W^-(t), W^+(s)]_q + q[W^-(s), W^+(t)]_q,\\
Q(s,t) &= t^{-1}[\tilde{G}(s), G(t)] - s^{-1}[\tilde{G}(t), G(s)] - q[W^+(t), W^-(s)]_q + q[W^+(s), W^-(t)]_q,\\
R(s,t) &= [G(s), \tilde{G}(t)]_q - [G(t), \tilde{G}(s)]_q - qt[W^-(t), W^+(s)] + qs[W^-(s), W^+(t)],\\
S(s,t) &= [\tilde{G}(s), G(t)]_q - [\tilde{G}(t), G(s)]_q - qt[W^+(t), W^-(s)] + qs[W^+(s), W^-(t)].
\end{aligned}$$

By linear algebra,

$$C(s,t) = \frac{(q+q^{-1})(P(s,t)+Q(s,t)) - (s^{-1}+t^{-1})(R(s,t)+S(s,t))}{(q^2-s^{-1}t)(q^2-st^{-1})q^{-1}}, \tag{A.5}$$

$$J(s,t) = \frac{(q+q^{-1})(R(s,t)+S(s,t)) - (s+t)(P(s,t)+Q(s,t))}{(q^2-s^{-1}t)(q^2-st^{-1})q^{-2}}. \tag{A.6}$$

Using Lemma 5.2 we routinely obtain

$$[W_0, A(s,t)] = 0, \qquad \frac{[W_0, B(s,t)]}{1-q^{-2}} = \frac{G(s,t)-F(s,t)}{st},$$

$$\frac{[W_0, C(s,t)]}{1-q^{-2}} = \frac{E(s,t)-D(s,t)}{st}, \qquad \frac{[W_0, D(s,t)]_q}{q-q^{-1}} = (s+t)A(s,t),$$

$$\frac{[E(s,t), W_0]_q}{q-q^{-1}} = (s+t)A(s,t), \qquad \frac{[W_0, F(s,t)]_q}{1-q^{-2}} = S(s,t) - (q+q^{-1})H(s,t),$$

$$\frac{[G(s,t), W_0]_q}{1-q^{-2}} = S(s,t) - (q+q^{-1})I(s,t), \qquad \frac{[W_0, H(s,t)]_{q^2}}{q-q^{-1}} = K(s,t),$$

$$\frac{[I(s,t), W_0]_{q^2}}{q-q^{-1}} = M(s,t), \qquad \frac{[W_0, J(s,t)]}{q-q^{-1}} = M(s,t) - K(s,t)$$

and

$$\frac{[W_0, K(s,t)]_q}{q^2-q^{-2}} = A(s,t), \qquad \frac{[W_0, L(s,t)]_q}{q-q^{-1}} = P(s,t) - (s^{-1}+t^{-1})H(s,t),$$

$$\frac{[M(s,t), W_0]_q}{q^2-q^{-2}} = A(s,t), \qquad \frac{[N(s,t), W_0]_q}{q-q^{-1}} = Q(s,t) - (s^{-1}+t^{-1})I(s,t),$$

$$\frac{[P(s,t), W_0]}{q-q^{-1}} = (s^{-1}+t^{-1})K(s,t) - (q+q^{-1})s^{-1}t^{-1}E(s,t),$$

$$\frac{[W_0, Q(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})M(s,t) - (q+q^{-1})s^{-1}t^{-1}D(s,t),$$

$$\frac{[W_0, R(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})(E(s,t)-D(s,t)),$$

$$\frac{[W_0, S(s,t)]}{q^2-q^{-2}} = M(s,t) - K(s,t)$$

and

$$\frac{[W_1, A(s,t)]}{1-q^{-2}} = \frac{D(s,t)-E(s,t)}{st}, \qquad [W_1, B(s,t)] = 0,$$

$$\frac{[W_1, C(s,t)]}{1-q^{-2}} = \frac{F(s,t)-G(s,t)}{st}, \qquad \frac{[D(s,t), W_1]_q}{1-q^{-2}} = R(s,t) - (q+q^{-1})H(s,t),$$

$$\frac{[W_1, E(s,t)]_q}{1-q^{-2}} = R(s,t) - (q+q^{-1})I(s,t), \qquad \frac{[F(s,t), W_1]_q}{q-q^{-1}} = (s+t)B(s,t),$$

$$\frac{[W_1, G(s,t)]_q}{q-q^{-1}} = (s+t)B(s,t), \qquad \frac{[H(s,t), W_1]_{q^2}}{q-q^{-1}} = L(s,t),$$

$$\frac{[W_1, I(s,t)]_{q^2}}{q-q^{-1}} = N(s,t), \qquad \frac{[W_1, J(s,t)]}{q-q^{-1}} = L(s,t) - N(s,t)$$

and

$$\frac{[K(s,t),W_1]_q}{q-q^{-1}} = P(s,t) - (s^{-1}+t^{-1})H(s,t), \qquad \frac{[L(s,t),W_1]_q}{q^2-q^{-2}} = B(s,t),$$
$$\frac{[W_1,M(s,t)]_q}{q-q^{-1}} = Q(s,t) - (s^{-1}+t^{-1})I(s,t), \qquad \frac{[W_1,N(s,t)]_q}{q^2-q^{-2}} = B(s,t),$$
$$\frac{[W_1,P(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})L(s,t) - (q+q^{-1})s^{-1}t^{-1}G(s,t),$$
$$\frac{[Q(s,t),W_1]}{q-q^{-1}} = (s^{-1}+t^{-1})N(s,t) - (q+q^{-1})s^{-1}t^{-1}F(s,t),$$
$$\frac{[W_1,R(s,t)]}{q^2-q^{-2}} = L(s,t) - N(s,t), \qquad \frac{[W_1,S(s,t)]}{q-q^{-1}} = (s^{-1}+t^{-1})(F(s,t)-G(s,t)).$$

We just listed 38 relations, including (A.5), (A.6). These 38 relations are called *canonical*.