ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA
https://doi.org/10.26493/1855-3974.2646.c07
(Also available at http://amc-journal.eu)

Cell reducing and the dimension of the C^1 bivariate spline space

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Received 1 Junij 2021, accepted 8 November 2021

Abstract

In this paper, the problem of determining the dimension of the space $S_n^1(\triangle), n \geq 3$ of bivariate C^1 splines of degree $\leq n$ over a triangulation $\triangle$ is considered. The piecewise polynomials are represented as blossoms, and the smoothness conditions are written as a system of linear equations. The rank of the system matrix is analysed by repeatedly reducing small subtriangulations (cells) at the boundary of a triangulation. It is shown that the dimension of the bivariate spline space $S_n^1(\triangle), n \geq 3$ is equal to Schumaker's lower bound for a large class of triangulations.

Keywords: Dimension, spline space, triangulation, cell.

Math. Subj. Class. (2020): 65D05, 65D07, 65D17, 15A03

1 Introduction

In the last 40 years the problem of determining the dimension of the bivariate spline space has received a considerable attention. For a given triangulation $\triangle$ of a polygonal region $\Omega \subset \mathbb{R}^2$ with N triangles Ω_i, *the bivariate spline space* of degree n and smoothness r is defined as

$$S_n^r(\triangle) := \{f \in C^r(\Omega); \quad f|_{\Omega_i} \in \Pi_n(\mathbb{R}^2), \quad i = 1, 2, \ldots, N\},$$

where $\Pi_n(\mathbb{R}^2)$ denotes the space of bivariate polynomials of total degree $\leq n$. In contrast to the univariate case, the bivariate spline space has a much more complex structure and even such basic problems as determining its dimension or construction of its basis are surprisingly hard to tackle. Even more surprising is the fact that the "simplest" spaces of splines of the lowest degrees are the most complex. For example, for the most interesting case - the space of cubic C^1 splines $S_3^1(\triangle)$, quite frequently used in practical applications,

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the dimension is still unknown in general, even though a great deal of research has been done on the topic.

But it is essential that the dimension is known in advance in some important applications, in particular for Lagrange interpolation by bivariate splines.

In general, the problem has been solved for a spline space of degree n and smoothness r over a regular triangulation $\triangle$, $S_n^r(\triangle)$, where the degree n is large in comparison to the smoothness r ($n \geq 3r+2$ ([6]), $n=4, r=1$ ([1])). Recall that a triangulation is *regular*, if two adjacent triangles Ω_i, Ω_j can have only one vertex or the whole edge in common.

The dimension of the spline space $S_3^1(\triangle)$ is known for particular classes of triangulations only [4, 7, 5], etc. It has been conjectured that the dimension is equal to Schumaker's lower bound ([11, 12])

$$\dim S_3^1(\triangle) \geq 3V_B(\triangle) + 2V_I(\triangle) + \sigma(\triangle) + 1, \tag{1.1}$$

where $V_B(\triangle)$ denotes the number of boundary vertices, $V_I(\triangle)$ the number of internal vertices, and $\sigma(\triangle) = \sum_{i=1}^{V_I(\triangle)} \sigma_i$,

$$\sigma_i = \begin{cases} 1, & \text{if vertex is singular,} \\ 0, & \text{otherwise.} \end{cases}$$

A vertex is *singular* if it is obtained as an intersection of exactly two lines.

Suppose that a triangulation $\triangle$ consists of a set of triangles that all have one common vertex v. Suppose every triangle in $\triangle$ has at least one neighbour with which it shares a common edge. Then we call $\triangle$ *a cell*. If v is an interior vertex, then $\triangle$ is *an interior cell*, otherwise it is *a boundary cell* (see [13]). *Cell degree* is the degree (valency) of the vertex v.

The main obstacle in the study of the dimension problem is the fact that the dimension depends not only on the topology of the triangulation $\triangle$ but also on its geometry. It has been conjectured (see [14]) that the dimension is equal to Schumaker's lower bound for $n \geq 2r+1$ and that the dimension jump occurs only for singular vertices.

Various approaches have been applied to tackle the dimension problem (tools from linear algebra, algebraic topology, graph theory, symbolic computation and computer aided design), but the problem is very hard. It has been compared even with the well-known Four Color Map Problem.

For the space of cubic C^1 splines, the dimension has been determined for special triangulations only (triangulations of type 1 and 2 ([11, 12]), nested polygon triangulations ([4]), reducible triangulations ([7]), etc.). Dimension is known also for generic triangulations, i.e., such triangulations, that if the dimension of the spline space exceeds the lower bound, a small perturbation of vertices of the triangulation causes the dimension to match the lower bound (see [2, 15], and more elementary proof in [13]). In all the cases the dimension equals the lower bound (1.1).

In this paper, the blossoming approach is used (see [3, 7]). The idea is to study the smoothness conditions between polynomial patches, written as their blossoms ([10]). This is a dual approach to the well known classical approach (see [13], e.g.) and brings a new insight to the dimension problem. An overview of cell reduction at the boundary of the triangulation is given. Thus sufficient conditions for an inductive approach for determining whether the dimension of $S_n^1(\triangle), n \geq 3$, is equal to Schumaker's lower bound for a large class of triangulations $\triangle$ are obtained. It is shown that interior cells of degrees

$k = 4, 5, \ldots, 8$ can be tackled, but the reduction can be applied only in the case $k = 4$ and for special cases for $k = 5$. For $k = 6, 7, 8$, a negative result is proven. Furthermore, interior cells with more than 2 free boundary edges are studied. Since it is possible to reduce most of the cases by methods for boundary cells, we focus the study to the cases with collinearities. It is proven that a cell of degree 4 with 1 common edge and a cell of degree 5 with 2 common edges with the rest of the triangulation can be reduced.

An algorithm that extends the results of [7] is presented, and it is proven that the results can be generalized to $S_n^1(\triangle), n > 3$.

The structure of the paper is as follows. In Section 2, an overview of the blossoming approach to the dimension problem is given. In Section 3, the cell reduction is studied and the main results of the paper are presented. In Section 4, an efficient algorithm for determining whether a given triangulation belongs to the observed class of triangulations is presented. Section 5 extends the results from the cubic ($S_3^1(\triangle)$) to the general case $n \geq 3$ ($S_n^1(\triangle)$). The paper is concluded by a proof of one of the main results.

2 Blossoming approach to the dimension problem

First, let us recall the blossoming approach. It is well known that there exists a bijective correspondence between a bivariate polynomial and its blossom: for every polynomial $p \in \Pi_n(\mathbb{R}^m)$ there exists a unique symmetric n-affine polynomial

$$B_n(p)\Big(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\Big), \quad x^{(i)} \in \mathbb{R}^m,$$

with a diagonal property

$$B_n(p)\Big(\underbrace{x, x, \ldots, x}_{n}\Big) = p(x), \quad x \in \mathbb{R}^m.$$

The polynomial $B_n(p)$ is called the *blossom* of the polynomial p.

For example, the blossom of the polynomial

$$p_2(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$$

is

$$B_2(p_2)\Big((u_1, v_1), (u_2, v_2)\Big) = \\ a_0 + a_1 \frac{u_1 + u_2}{2} + a_2 \frac{v_1 + v_2}{2} + a_3 u_1 u_2 + a_4 \frac{u_1 v_2 + u_2 v_1}{2} + a_5 v_1 v_2.$$

The blossom generalizes a given polynomial. One can study the smoothness conditions between polynomial patches over adjacent triangles, expressed in the blossoming form. In order to determine the dimension of the spline space, smoothness conditions over all inner edges of the triangulation need to be studied.

By studying the dual representation of a triangulated graph [3], it can be seen that some smoothness conditions are independent and can be omitted. The study of the rest results into the dimension equation

$$\dim S_n^r(\triangle) = N\binom{n+2}{2} - \binom{r+2}{2}(E_I - V_I) - \operatorname{rank} M_n. \tag{2.1}$$

Here N denotes the number of triangles, E_I the number of inner edges, and V_I the number of inner vertices of the triangulation. The matrix M_n describes the smoothness relations that need to be studied. It has a particular structure:

$$M_n := (M_{km})_{k=1;m=1}^{n-r;n-r} := (M_{r,km})_{k=1;m=1}^{n-r;n-r} \tag{2.2}$$

is a block upper triangular matrix with blocks M_{km} of size $(r+1)E_I \times (m+r+1)(N-1)$. The matrix M_{km} is also a block matrix with E_I block rows and $N-1$ block columns: ℓ-th block row corresponds to smoothness conditions across the edge $e_\ell = (i,j)$ between triangles Ω_i and Ω_j, and it has at most two nonzero blocks $Q_{\ell i} := Q_{km,\ell i}$, $\quad Q_{\ell j} := Q_{km,\ell j}$, with $Q_{km,\ell i} + Q_{km,\ell j} = 0$. Blocks $Q_{km,\ell i}$ are circulant matrices of size $(r+1) \times (m+r+1)$. Theirs first row is defined by

$$\binom{n-r}{k} \sum_{|\gamma|=k} \binom{k}{\gamma} \binom{n-r-k}{\beta-\gamma} v_\ell^\gamma u_\ell^{\beta-\gamma} = f \cdot \sum_{|\gamma|=k} \binom{k}{\gamma} \binom{m-k}{\beta-\gamma} v_\ell^\gamma u_\ell^{\beta-\gamma},$$

$$\beta = (b_1, m-b_1),\ b_1 = m, m-1, \ldots, 0,$$

with

$$f := f_{n,r,k,m} = \binom{n-r-k}{m-k}.$$

The rest r elements of the row are zeros. Here u_ℓ denotes an arbitrary point on the edge e_ℓ with the normalized directional vector v_ℓ. The standard multiindex notation is used. For more details, see [3]. In the following section, the matrix M_n for the case $n=3$ will be described in detail.

Thus the main problem is how to determine the rank of a large symbolic matrix M_n that depends on the geometry and the topology of a triangulation. Such a problem is very hard to tackle in general. A natural idea is to reduce the problem to a smaller one, if particular assumptions are satisfied.

3 Cell reduction

The idea of the blossoming approach to the dimension problem is to inductively reduce the problem from the given triangulation $\triangle$ to its subtriangulation $\triangle \backslash \triangle_1$ for a proper choice of the subtriangulation $\triangle_1$ (see Fig. 1).

Let $\mathcal{B}$ denote the intersection of $\triangle_1$ and $\triangle \backslash \triangle_1$. We call the subtriangulation $\triangle_1$ *proper*, if it is simply connected, has a vertex T_0 on the outer face of $\triangle$ and contains all the triangles in $\triangle$ with the vertex T_0, no triangle in $\triangle_1$ has two edges on $\mathcal{B}$, there are no consecutive pairs of collinear edges at the vertices of degree 3 in $\triangle_1$ on $\mathcal{B}$, and every singular vertex in $\triangle_1$ lies in the interior of $\triangle_1$.

Let $v_\ell = (\alpha_\ell, \beta_\ell)$ denote a normalized directional vector of the edge e_ℓ between triangles Ω_i and Ω_j. Further, let

$$v_i \times v_j := \alpha_i \beta_j - \alpha_j \beta_i$$

be the planar vector product. The matrix $M := M_n$ could be written as

$$M = M(\triangle) = \begin{bmatrix} M(\triangle_1) & 0 \\ M(\triangle_1, \triangle) & M(\triangle, \triangle_1) \\ 0 & M(\triangle \backslash \triangle_1) \end{bmatrix},$$

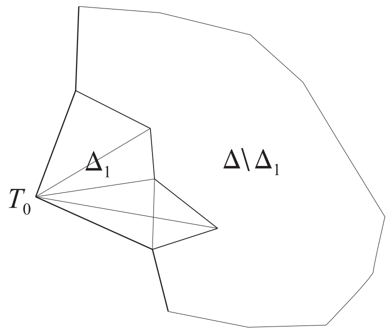

Figure 1: Reduction of a triangulation.

where the matrices $M(\triangle_1,\triangle), M(\triangle,\triangle_1)$ represent the common part of the smoothness conditions between $\triangle_1$ and $\triangle\backslash\triangle_1$, $M(\triangle_1)$ represents the conditions inside $\triangle_1$, and $M(\triangle\backslash\triangle_1)$ the conditions inside $\triangle\backslash\triangle_1$. Let

$$\widetilde{M}(\triangle_1,\triangle) := \begin{bmatrix} M(\triangle_1) \\ M(\triangle_1,\triangle) \end{bmatrix} \in \mathbb{R}^{r\times c}.$$

Theorem 3.1. [7, Thm. 1] *Suppose that $\triangle_1$ is a proper subtriangulation of $\triangle$ that satisfies $V_B(\triangle_1) \le 2V_I(\triangle_1)+6$. If*

$$\operatorname{rank} \widetilde{M}(\triangle_1,\triangle) = r - \sigma(\triangle_1) \tag{3.1}$$

and $\dim S_3^1(\triangle\backslash\triangle_1)$ is equal to the lower bound, then the dimension $\dim S_3^1(\triangle)$ is equal to the lower bound (1.1) too.

The matrices $\widetilde{M}(\triangle_1,\triangle)$ that have to be studied, have a block structure, based on the incidence matrix of the underlying graph of the triangulation. They consist of blocks

$$Q_{11,\ell i} = -Q_{11,\ell j} = \begin{bmatrix} \alpha_\ell & \beta_\ell & 0 \\ 0 & \alpha_\ell & \beta_\ell \end{bmatrix},$$
$$Q_{22,\ell i} = -Q_{22,\ell j} = \begin{bmatrix} \alpha_\ell^2 & 2\alpha_\ell\beta_\ell & \beta_\ell^2 & 0 \\ 0 & \alpha_\ell^2 & 2\alpha_\ell\beta_\ell & \beta_\ell^2 \end{bmatrix},$$

and blocks $Q_{12,\ell j}$ that depend not only on the directions but also on the vertices of the triangulation and some arbitrary additional points. More precisely, by using [3, Lemma 3.1] it is possible to simplify some of the blocks in M_{12} without changing the rank of M by choosing the points $t_\ell := (c_\ell, d_\ell)$ as the inner vertices of the triangulation, and by choosing some arbitrary additional points $z_k := (x_k, y_k) \in \mathbb{R}^2$, $k = 1, 2, \ldots, N$ for the faces Ω_k. The block $Q_{12,\ell i}$ is of the form

$$\begin{bmatrix} \alpha_\ell(c_\ell - x_i) & \alpha_\ell(d_\ell - y_i) + \beta_\ell(c_\ell - x_i) & \beta_\ell(d_\ell - y_i) & 0 \\ 0 & \alpha_\ell(c_\ell - x_i) & \alpha_\ell(d_\ell - y_i) + \beta_\ell(c_\ell - x_i) & \beta_\ell(d_\ell - y_i) \end{bmatrix}, \tag{3.2}$$

and the block $Q_{12,\ell j}$ reads

$$-\begin{bmatrix} \alpha_\ell(c_\ell - x_j) & \alpha_\ell(d_\ell - y_j) + \beta_\ell(c_\ell - x_j) & \beta_\ell(d_\ell - y_j) & 0 \\ 0 & \alpha_\ell(c_\ell - x_j) & \alpha_\ell(d_\ell - y_j) + \beta_\ell(c_\ell - x_j) & \beta_\ell(d_\ell - y_j) \end{bmatrix}, \tag{3.3}$$

so the choice $z_k = t_\ell$, $k \in \{i, j\}$ reduces (3.2) or (3.3) to zero block. Of course, not all blocks can be simplified in this way.

The following theorem gives conditions on boundary cells $\triangle_1$ where the reduction can be applied.

Theorem 3.2. [7, Thm. 2] *Let $\triangle_1$ be a proper subtriangulation of $\triangle$ with a vertex T_0 of degree $s+1$ on the outer face of $\triangle$. If*
1) $s \leq 4$, $|\triangle_1| = s$, or
2) $s = 2, 3$, $|\triangle_1| = s+2$, and $\triangle_1$ includes an interior cell,
and $\dim S_3^1(\triangle \backslash \triangle_1)$ is equal to the lower bound, then the dimension $\dim S_3^1(\triangle)$ is equal to the lower bound (1.1) too.

Therefore, as a natural extension, it is interesting to study interior cell reduction at the boundary of the triangulation $\triangle$. Usually, the main obstacle are collinear edges. Those are the only special cases for $n \geq 3r+2$ and for known results for $n = 2r+1$. For $n = 2r$, there exist other geometric configurations, that result in a change of the dimension (Morgan-Scott triangulation [3], e.g.).

By studying interior cells, the collinear edges can be included in the subtriangulation $\triangle_1$. This will enable us to study the dimension problem on a wider class of triangulations.

The conditions of Theorem 3.1 allow us to study cells of degree k, $k = 4, 5, \ldots, 8$. In light of the previous discussion, the study will be limited to particularly interesting cells where there is a collinearity of edges between the inner vertex and the vertices, adjacent to the boundary vertex.

Theorem 3.3. *Let $\triangle_1$ be an interior cell of degree k, $k = 5, 6, 7, 8$, and let $\triangle_1$ be a proper subtriangulation of the triangulation $\triangle$ with a vertex T_0 of degree 3 at the outer face of the triangulation $\triangle$ and an inner vertex T_1. Let the edges adjacent to T_1 be denoted in clockwise direction as $e_1 = T_0T_1, e_2, \ldots, e_k$ (see Fig. 3, 5, 6, 7). Let there be the collinearity $e_2 || e_k$.*

1. *If $k = 5$ and there is a collinearity of edges $e_1 || e_3$ or $e_1 || e_4$ (Fig. 3), and $\dim S_3^1(\triangle \backslash \triangle_1)$ is equal to Schumaker's lower bound, then also the dimension $\dim S_3^1(\triangle)$ is equal to the lower bound (1.1).*
2. *If $k = 6, 7, 8$, then Theorem 3.1 can not be applied (Fig. 5, 6, 7).*

Proof. Proof of the theorem is given as the last section of the paper. □

By Theorem 3.1 we have to study ranks of certain matrices $\widetilde{M}(\triangle_1, \triangle)$ that belong to cells considered. Ranks will be obtained by studying appropriate minors. Since the matrices are large and symbolic, symbolic computer algebra tools and [9] will be used for the computation of determinants and the simplification of huge symbolic expressions. Here, properties of triangulation's topology and geometry have to be used very carefully, since otherwise computations are futile because of the huge time and memory requirements and enormous symbolic expressions. Note that the determinants obtained can be easily verified by evaluating the polynomials in enough number of points, exact numerical determinant calculation, and the well-known results on multivariate Lagrange polynomial interpolation (see [8, Lemma 1]).

Remark 3.4. Interior cells of degree 4 and boundary cells of degree ≤ 5 are covered by Theorem 3.2.

Remark 3.5. The study of interior cells in the general position is much more difficult.

1. For $k = 5, 6$, the appropriate minors can be computed, but the obtained polynomial expressions are difficult to simplify enough to be able to prove that the polynomial is nonzero for all geometrically admissible edge directions and vertex positions.
2. For $k = 7$ and $k = 8$ also in the general case the appropriate matrices are not of the full rank and the considered cell reducing approach can not be applied. The proof of Theorem 3.3 holds in general, since the collinearity of the edges is not used.

Remark 3.6. In general, it is interesting to consider interior cells of degree k at the boundary of triangulation, where there are $\ell = 1, 2, \ldots, k-2$ edges in the common border $\mathcal{B}$. It can be easily seen from (3.1), that the matrix dimensions limit the consideration to $\ell \leq \min\{k-2, \lfloor 3/4k \rfloor\}$. It is important to notice that most of the cases can be reduced by methods for boundary cells (Theorem 3.3). As expected, the most interesting cases with collinearities are left for the study.

Theorem 3.7. *For interior cells of small degrees $k = 4, 5, 6$ there is only one problematic configuration, namely, an interior cell with $\ell = k-3$ edges in $\mathcal{B}$ and two collinearities in the interior. For $k = 4, 5$ such configurations can be reduced.*

Proof. For $k = 4$, by [7] the matrix obtained is not of full rank because of the interior singular vertex. Consider Fig. 2. Let $T_1 = (c_1, d_1)$ be the interior point of the cell, and $T_2 = (c_2, d_2)$ be an additional point on $\mathcal{B}$, the join of e_5 and e_6. Further, let $w_6 := (\beta_6, -\alpha_6)$. The minor, obtained by omitting rows, corresponding to the boundary edge e_5 and additionally row 1 and columns $1, 2, 3, 4, 13, 14, 15, 16, 26$, is

$$\alpha_3\alpha_6\beta_4^2\beta_6^2(v_3 \times v_4)^9(v_6 \times v_3)^2(\beta_6(-c_1 + c_2) + \alpha_6(d_1 - d_2)),$$

which is nonzero, since $(\beta_6(-c_1 + c_2) + \alpha_6(d_1 - d_2)) = \langle w_6, T_1 - T_2 \rangle$ and w_6 is perpendicular to $v_6 = (\alpha_6, \beta_6)$.

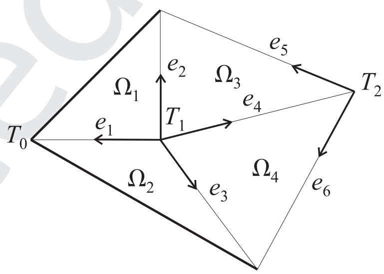

Figure 2: An interior cell of degree 4 at the boundary of the triangulation.

For $k = 5$ we similarly first omit the block rows and columns belonging to e_8 (see Fig. 3), apply the same simplification as in the proof of [7, Thm. 2], and then omit the columns $15, 16, 19, 20$ and further the columns $3, 14, 15, 16, 18, 19, 24$ in the obtained matrix. The resulting minor is

$$6g^2\alpha_1\alpha_2^3\alpha_7^2\beta_1^2\beta_2^2(v_2 \times v_1)^2(v_6 \times v_3)^5(v_7 \times v_3)^3(v_7 \times v_6)(v_1 \times v_8)(v_2 \times v_8). \quad (3.4)$$

Here g denotes the length of $T_2 - T_1 = gv_3$. Since by assumption the cell is proper, $v_7 \times v_6 \neq 0$, thus the expression (3.4) is nonzero. $\square$

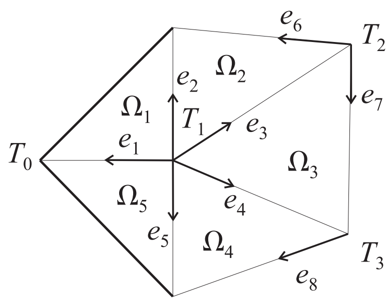

Figure 3: A cell of degree 5 at the boundary of the triangulation.

Remark 3.8. For interior cells of higher degrees $k \geq 6$ the matrices obtained are unfortunately too large to study with current computational facilities.

Numerical computations suggest that in the case $k = 5$ the interior cell reduction can be applied for all but one configuration. Unfortunately this seems hard to prove, and we will state it as the following conjecture with a discussion on its possible proof.

Conjecture. Let $k = 5$ and let the assumptions of Theorem 3.3 be fulfilled. Then the cell reduction can be applied for almost every position of edges of the interior cell. There exists a unique configuration of edges where the reduction can not be used.

A computation of all possible $\binom{35}{32} = 6545$ minors of size 32 and a careful simplification of nonzero expressions reveals that all contain the same linear expression in α_1 and β_1. This results in a unique condition on the positions of the edges of the cell, where the matrix considered, $\widetilde{M}(\triangle_1, \triangle)$, is not of full rank. This configuration is geometrically admissible. Thus the reduction of the interior cell can not be applied in this special case. Unfortunately, the symbolic polynomial expressions are huge and it is not feasible to write them down. Thus it is difficult to find a proper minor and prove that it is nonzero for all possible geometrically admissible edge directions, with the exception of the considered one.

4 An algorithm for the reduction of the triangulation

By using Theorem 3.2, Theorem 3.3 and Theorem 3.7, we can construct an algorithm that determines if the triangulation $\triangle$ belongs to the class of triangulations where the dimension $\dim S_3^1(\triangle)$ can be obtained by sequential reductions of the triangulation $\triangle$. This algorithm improves the algorithm in [7].

We are given a triangulation $\triangle$. First, it is rotated to a general position, such that no edge lies on the coordinate axes. Then the reduction step can be used on every boundary vertex that satisfies one of the conditions of Theorem 3.2, Theorem 3.3, item 1 or Theorem 3.7. In the algorithm, let B denote the intersection of a current boundary or interior cell $\triangle_1$ and $\triangle \backslash \triangle_1$. Let $|B|$ denote the number of edges in B.

An algorithm for the reduction of the triangulation

```
// T - a given triangulation
```

```
// deg(v) - degree of the vertex v
list<vertex> L = {list of all vertices of degree <=5 on the
outer face of the triangulation T};
while (!L.empty()) {
    if (T= interior cell or T=triangle) break(success);
    //check for possible reductions using L
    v=L.pop();
    check for collinearities between neighbours of v in B;
    //boundary cells
    if (no collinearities && all neighbours of v are in B){
        T.delete_vertex(v);
        if any neighbour(v) has new degree <=5, add it to L;
        }
    //interior cells
    else if ((collinearity at the inner vertex z, deg(z)=4,
    deg(v)<5), |B|=1 or 2 ||
    (collinearity at the inner vertex z, deg(v)=3, deg(z)=5,
    edge vz is collinear with another edge from z,
    |B|=2 or 3)){
        T.delete_vertex(v);
        T.delete_vertex(z);
        add vertices in B with new degree <=5 to L;
        }
    else {
        L.append(v);
        }
    }
if (success) {
    print("Dimension equals Schumaker's lower bound");
    }
else {
    print("Algorithm can not be used");
    }
}
```

If all the vertices in the list `L` were checked, and no reduction could be applied, the list `L` remains the same, and the algorithm stops. In such a case (because of too large degrees of the boundary vertices or because of the collinearities) this method can not be applied for the study of the dimension. If the while loop successfully terminates, the dimension $\dim S_3^1(\triangle)$ is equal to Schumaker's lower bound (1.1). The answer of the algorithm is quite clearly independent of the enumeration of the vertices, i.e., on a particular sequence of reductions.

As an example, consider the triangulation in Fig. 4. It is derived from a nested polygonal configuration (see [4] and an example in [7]). Because of a slight modification (deletion of some vertices and edges, and perturbation of vertices), the tools in [4] can not be applied. Similarly, the algorithm from [7] can not tackle it because of collinearities along reduction boundaries at all boundary vertices. But the algorithm, presented in this paper, enables the reduction of the triangulation. Thus the dimension of the C^1 cubic spline space on the triangulation in Fig. 4 is equal to Schumaker's lower bound.

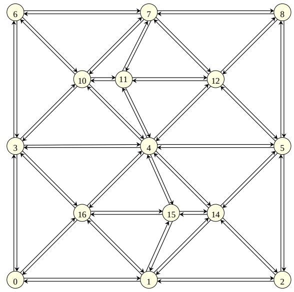

Figure 4: A triangulation $\triangle$, where the algorithm determines $\dim S_3^1(\triangle)$.

5 Generalization to $n \geq 3$

In [3], it has been shown how particular triangulations can be tackled under the assumption, that no inner edges that share a common vertex have the same slope. We will show that this assumption can be omitted, and apply this to generalize our results to $n \geq 3$. For the sake of completeness, we will include the relevant results from [3].

First, let us recall Schumaker's lower bound in general,

$$\dim S_n^r(\triangle) \geq LB_n^r(\triangle) := \binom{n+2}{2} + \binom{n-r+1}{2} E_I - \left(\binom{n+2}{2} - \binom{r+2}{2} \right) V_I + \sum_{i=1}^{V_I} \sigma_i, \quad (5.1)$$

where

$$\sigma_i = \sum_{j=1}^{n-r} (r+j+1-j\, e_i)_+, \quad i = 1, 2, \ldots, V_I, \quad (5.2)$$

and E_I denotes the number of interior edges, V_I the number of inner vertices, and e_i the number of edges with distinct slopes in an inner vertex v_i.

Let $\sigma_i(n)$ denote the number σ_i, defined in (5.2), that belong to the vertex i and the considered spline space $S_n^r(\triangle)$.

Lemma 5.1. *Let $n > 2r$. Then $\sigma_i(n) = \sigma_i(2r)$.*

Proof. The number $\sigma_i(n)$ can be written as

$$\sigma_i(n) = \sum_{j=1}^{n-r} (r+j+1-j\, e_i)_+ = \sigma_i(2r) + \sum_{j=r+1}^{n-r} (r+j+1-j\, e_i)_+,$$

where e_i is the number of edges with different edge slopes at the inner vertex i. Note that in the second term $j \geq r+1$. Now let us consider the expression

$$r + j + 1 - j\, e_i = r + 1 - j(e_i - 1) \leq r + 1 - (r+1)(e_i - 1) = (r+1)(2 - e_i).$$

Since for the inner vertices of the triangulation in general position $e_i \geq 3$ and for the singular vertices $e_i = 2$,

$$(r+1)(2-e_i) \leq 0.$$

Thus σ_i does not change if the polynomial degree increases. □

Now we can show that [3, Thm. 3.2] and [3, Thm. 3.3] hold also for triangulations that contain collinearities.

Recall the definition of the matrix M_n in (2.2).

Theorem 5.2. [3, Thm. 3.2] *Let $\triangle$ be a regular triangulation and $n \geq n_0 \geq 2r$. Then*

$$\operatorname{rank} M_n \geq \operatorname{rank} M_{n_0} + (r+1)E_I(n-n_0). \tag{5.3}$$

Proof. The matrix M_n can be written as

$$M_n = \begin{bmatrix} M_{n-1} & X \\ 0 & M_{n-k,n-k} \end{bmatrix}.$$

Clearly, $\operatorname{rank} M_n \geq \operatorname{rank} M_{n-1} + \operatorname{rank} M_{n-k,n-k}$. We need to prove that

$$\operatorname{rank} M_{n-k,n-k} = (r+1)E_I,$$

i.e., the rank of the matrix $M_{n-k,n-k}$ is equal to the number of rows. The number of columns of $M_{n-k,n-k}$ is equal to

$$(n+1)(N-1) \geq 2(r+1)(N-1) \geq \frac{4}{3}(r+1)E_I > (r+1)E_I,$$

since $3\,N \geq 2\,E_I + 3$, thus it suffices to prove, that the rows are linearly independent.

Suppose that the rows are not independent. Then there exists a vector $x \in \mathbb{R}^{(r+1)E_I}$, $x \neq 0$, such that $x^T M_{n-k,n-k} = 0$. Since the rows that correspond to boundary triangles with only one inner edge are clearly independent, the corresponding components of the vector x are zero. Thus we can assume, that all boundary triangles have only one outer edge. Let Ω_i be such a boundary triangle in $\triangle$. Then $e_{\ell_1} = (j_1, i)$, $e_{\ell_2} = (i, j_2)$ for some j_1, j_2, and the block matrices $Q_{\ell_1 i}, Q_{\ell_2 i}$ are the only nonzero blocks in the block column i. Let $x|_{\ell_j}$ denote the ℓ_j-th block row in x. Then

$$[x|_{\ell_1}, x|_{\ell_2}]^T \begin{bmatrix} Q_{\ell_1 i} \\ Q_{\ell_2 i} \end{bmatrix} = 0.$$

It can easily be seen that the matrix

$$\begin{bmatrix} Q_{\ell_1 i} \\ Q_{\ell_2 i} \end{bmatrix} \in \mathbb{R}^{2(r+1)\times(n+1)} \tag{5.4}$$

where $n+1 \geq 2r+1+1 = 2(r+1)$, is of full rank. Indeed, the rank of the matrix (5.4) stays unchanged, if $n-2r-1$ zero columns are added. If we append another $n-2r-1$ rows, such that they cyclically continue $Q_{\ell_1 i}$, and $n-2r-1$ rows by cyclical continuation of $Q_{\ell_2 i}$, the rank of the matrix (5.4) increases at most by $2(n-2r-1)$. The resulting matrix is of dimension $2(n-r) \times 2(n-r)$. Its determinant is equal to the resultant of polynomials

$$p_{\ell_1}(x) := (v_{1,\ell_1}x + v_{2,\ell_1})^{n-r}, \quad p_{\ell_2}(x) := (v_{1,\ell_2}x + v_{2,\ell_2})^{n-r},$$

where $v_{\ell_i} := (v_{1,\ell_i}, v_{2,\ell_i})$ is the direction of e_{ℓ_i}. Since the directions v_{ℓ_1} and v_{ℓ_2} are different, the polynomials p_{ℓ_1} and p_{ℓ_2} can not have common zeros. Thus theirs resultant $(v_{1,\ell_1}v_{2,\ell_2} - v_{2,\ell_1}v_{1,\ell_2})^{(n-r)^2}$ is nonzero, hence the matrix (5.4) is of full rank. Thus $x|_{\ell_1} = x|_{\ell_2} = 0$. So one can study $\triangle\backslash\{\Omega_i\}$ only, and use the previous result on the reduced triangulation. The procedure can be repeated until only one triangle is left. This concludes the proof. $\square$

Theorem 5.3. [3, Thm. 3.3] *Let $\triangle$ be a triangulation, and $n \geq n_0 \geq 2r$. The function*

$$\theta(n, n_0, r) := \dim S_n^r(\triangle) - N\left(\binom{n+2}{2} - \binom{n_0+2}{2}\right) + (r+1)E_I(n-n_0)$$

is nonincreasing function of n, and

$$LB_{n_0}^r(\triangle) \leq \theta(n, n_0, r) \leq \dim S_{n_0}^r(\triangle). \tag{5.5}$$

Proof. It is enough to consider $n > n_0$. The first claim follows from (2.1) and Theorem 5.2. Recall Schumaker's lower bound (5.1). By Lemma 5.1, σ_i stays unchanged for any $n > n_0$. Now it is straightforward to apply (2.1) and (5.3) to obtain (5.5). $\square$

This approach has been used in [3] in order to determine the dimension of $S_n^2(\triangle_{MS})$, where $\triangle_{MS}$ denotes the Morgan-Scott triangulation. The key observation is the fact, that if in (5.5) the right inequality reduces to equality, so does the left. Of course, the main problem is how to determine rank M_n.

From Theorem 5.3 it follows that the results of Theorem 3.2 and Theorem 3.3 hold not only for the cubic case, but also for spline spaces of higher degrees.

Remark 5.4. The algorithm, given in Section 4, determines whether the dimension $\dim S_n^1(\triangle)$, $n \geq 3$, equals Schumaker's lower bound for a large class of triangulations $\triangle$. If the answer is in the affirmative, $\dim S_3^1(\triangle)$ agrees with Schumaker's lower bound, and we can apply Theorem 5.3. Of course, for $n \geq 4$, the dimension $S_n^1(\triangle)$ is known for any triangulation $\triangle$ (see [1] and [6]). In this case, the blossoming approach does not yield anything new. It is promising for use on the spline spaces with higher degrees of smoothness $r \geq 1$, as observed for the special case $S_{2r}^r(\triangle)$ in [3].

6 Proof of Theorem 3.3

Proof. Let us shorten the notation by $Q_{ij}^{k\ell} := Q_{k\ell,ij}$. It turns out that instead of considering all the cells at once, it is more reasonable to choose different points for each case $k = 5, 6, \ldots, 8$ in order to simplify non-diagonal blocks as much as possible. Thus each case will be studied separately.

First, let us consider the case $k = 5$. The points on the edges are chosen as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_2,\ e_7 : T_2,\ e_8 : T_3$, and the points for the faces as $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1$ (Fig. 3). This implies that in the matrix M_{12} only three nonzero blocks remain: $Q_{62}^{12}, Q_{73}^{12}, Q_{84}^{12}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$\begin{bmatrix}
Q_{11}^{11} & 0 & 0 & 0 & Q_{15}^{11} & 0 & 0 & 0 & 0 & 0 \\
Q_{21}^{11} & Q_{22}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{32}^{11} & Q_{33}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_{43}^{11} & Q_{44}^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{54}^{11} & Q_{55}^{11} & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{62}^{11} & 0 & 0 & 0 & 0 & Q_{62}^{12} & 0 & 0 & 0 \\
0 & 0 & Q_{73}^{11} & 0 & 0 & 0 & 0 & Q_{73}^{12} & 0 & 0 \\
0 & 0 & 0 & Q_{84}^{11} & 0 & 0 & 0 & 0 & Q_{84}^{12} & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{11}^{22} & 0 & 0 & 0 & Q_{15}^{22} \\
0 & 0 & 0 & 0 & 0 & Q_{21}^{22} & Q_{22}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q_{32}^{22} & Q_{33}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{43}^{22} & Q_{44}^{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{54}^{22} & Q_{55}^{22} \\
0 & 0 & 0 & 0 & 0 & 0 & Q_{62}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{73}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{84}^{22} & 0
\end{bmatrix}_{32\times 35}.$$

Since

$$\det \begin{bmatrix} Q_{11}^{22} \\ Q_{21}^{22} \end{bmatrix}_{4\times 4} = (v_2 \times v_1)^4 \neq 0,$$

the rows 17, 18, 19, 20, and columns 16, 17, 18, 19, can be omitted without changing the rank of the matrix $\widetilde{M}(\triangle_1, \triangle)$. If the columns 3, 30 and 31 in the new matrix are omitted, in the case of the collinearity $e_1||e_3$ we obtain the minor

$$\|e_4\|^2 \alpha_2^6 (v_2 \times v_3)(v_2 \times v_4)(v_4 \times v_3)^2 (v_3 \times v_6)^5 \cdot \\ (v_7 \times v_4)^2 (v_6 \times v_7)(v_8 \times v_4)^3 (v_8 \times v_7)^3 \neq 0,$$

and in the case of the collinearity $e_1||e_4$ the minor

$$\|e_4\|^2 \alpha_2^6 (v_2 \times v_3)(v_2 \times v_4)(v_4 \times v_3)^2 (v_3 \times v_6)^3 \cdot \\ (v_7 \times v_4)^2 (v_6 \times v_7)^3 (v_4 \times v_8)^5 (v_7 \times v_8) \neq 0.$$

Therefore the matrix $\widetilde{M}(\triangle_1, \triangle)$ is of full rank in the considered special cases. The conditions of Theorem 3.1 are fulfilled.

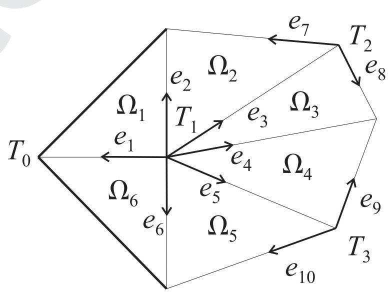

Figure 5: A cell of degree 6 at the boundary of the triangulation.

In the case $k = 6$ we pick the points on the edges as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_1,\ e_7 : T_2,\ e_8 : T_2,\ e_9 : T_3,\ e_{10} : T_3$, and the points for the faces $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1,\ \Omega_6 : T_1$ (Fig. 5). This choice implies that in the matrix M_{12} only 4 nonzero blocks remain: $Q^{12}_{72}, Q^{12}_{83}, Q^{12}_{94}, Q^{12}_{10,5}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$\begin{bmatrix}
Q^{11}_{11} & 0 & 0 & 0 & 0 & Q^{11}_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
Q^{11}_{21} & Q^{11}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{32} & Q^{11}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{43} & Q^{11}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{54} & Q^{11}_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{65} & Q^{11}_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{72} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{72} & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{83} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{83} & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{94} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{94} & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{10,5} & 0 & 0 & 0 & 0 & 0 & Q^{12}_{10,5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11} & 0 & 0 & 0 & 0 & Q^{22}_{16} \\
0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{21} & Q^{22}_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{32} & Q^{22}_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{43} & Q^{22}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{54} & Q^{22}_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{65} & Q^{22}_{66} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{72} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{83} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{94} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{10,5} & 0
\end{bmatrix}_{40 \times 42}$$

Since the submatrix

$$\begin{bmatrix} Q^{22}_{11} \\ Q^{22}_{21} \end{bmatrix},$$

that belong to the block column for Ω_1 in M_{22}, is nonsingular, the block rows for e_1, e_2 and the block column for Ω_1 in M_{12} and M_{22} can be omitted. A matrix of dimension 36×38 remains. It is enough to compute 6 minors, where 2 of the columns that belong to the last block column of the matrix are omitted (the rest of the minors are 0 because of the linearly dependent columns in the last block column). A quick computation shows that all minors are 0. Therefore the matrix $\widetilde{M}(\triangle_1, \triangle)$ is not of full rank.

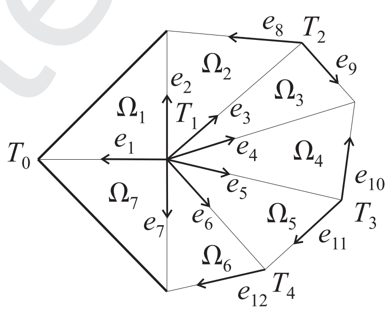

Figure 6: A cell of degree 7 at the boundary of the triangulation.

In the case $k = 7$ we choose the points on the edges as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_1,\ e_7 : T_1,\ e_8 : T_2,\ e_9 : T_2,\ e_{10} : T_3,\ e_{11} : T_3,\ e_{12} : T_4,$ and the points for the faces $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1,\ \Omega_6 : T_1,\ \Omega_7 : T_1$ (Fig. 6). Therefore, in the matrix M_{12} only 5 nonzero blocks remain:

$Q^{12}_{82}, Q^{12}_{93}, Q^{12}_{10,4}, Q^{12}_{11,5}, Q^{12}_{12,6}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$
\begin{bmatrix}
Q^{11}_{11} & 0 & 0 & 0 & 0 & 0 & Q^{11}_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q^{11}_{21} & Q^{11}_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{32} & Q^{11}_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{43} & Q^{11}_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{54} & Q^{11}_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{65} & Q^{11}_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q^{11}_{76} & Q^{11}_{77} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q^{11}_{82} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{82} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q^{11}_{93} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{93} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q^{11}_{10,4} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{10,4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q^{11}_{11,5} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{11,5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q^{11}_{12,6} & 0 & 0 & 0 & 0 & 0 & 0 & Q^{12}_{12,6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11} & 0 & 0 & 0 & 0 & 0 & Q^{22}_{17} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{21} & Q^{22}_{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{32} & Q^{22}_{33} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{43} & Q^{22}_{44} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{54} & Q^{22}_{55} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{65} & Q^{22}_{66} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{76} & Q^{22}_{77} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{82} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{93} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{10,4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{11,5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q^{22}_{12,6} & 0
\end{bmatrix}_{48\times 49}
$$

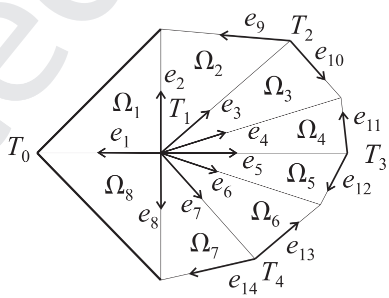

Figure 7: A cell of degree 8 at the boundary of the triangulation.

In the case $k = 8$ we choose the points on the edges as $e_1 : T_1,\ e_2 : T_1,\ e_3 : T_1,\ e_4 : T_1,\ e_5 : T_1,\ e_6 : T_1,\ e_7 : T_1,\ e_8 : T_1,\ e_9 : T_2,\ e_{10} : T_2,\ e_{11} : T_3,\ e_{12} : T_3,\ e_{13} : T_4,\ e_{14} : T_4$, and the points for the faces as $\Omega_1 : T_1,\ \Omega_2 : T_1,\ \Omega_3 : T_1,\ \Omega_4 : T_1,\ \Omega_5 : T_1,\ \Omega_6 : T_1,\ \Omega_7 : T_1,\ \Omega_8 : T_1$ (Fig. 7). Therefore, in the matrix M_{12} only 6 nonzero blocks remain: $Q_{92}^{12}, Q_{10,3}^{12}, Q_{11,4}^{12}, Q_{12,5}^{12}, Q_{13,6}^{12}, Q_{14,7}^{12}$. The matrix $\widetilde{M}(\triangle_1, \triangle)$ is

$$\begin{bmatrix}
Q_{11}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & Q_{18}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Q_{21}^{11} & Q_{22}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{32}^{11} & Q_{33}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_{43}^{11} & Q_{44}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{54}^{11} & Q_{55}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{65}^{11} & Q_{66}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{76}^{11} & Q_{77}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q_{87}^{11} & Q_{88}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & Q_{92}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{92}^{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & Q_{10,3}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{10,3}^{12} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Q_{11,4}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{11,4}^{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & Q_{12,5}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{12,5}^{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & Q_{13,6}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{13,6}^{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & Q_{14,7}^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{14,7}^{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{11}^{22} & 0 & 0 & 0 & 0 & 0 & 0 & Q_{18}^{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{21}^{22} & Q_{22}^{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{32}^{22} & Q_{33}^{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{43}^{22} & Q_{44}^{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{54}^{22} & Q_{55}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{65}^{22} & Q_{66}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{76}^{22} & Q_{77}^{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{87}^{22} & Q_{88}^{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{92}^{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{10,3}^{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{11,4}^{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{12,5}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{13,6}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_{14,7}^{22} & 0
\end{bmatrix}_{56 \times 56}.$$

The cases $k = 7$ and $k = 8$ can be considered simultaneously. Since the submatrix

$$\begin{bmatrix} Q_{11}^{22} \\ Q_{21}^{22} \end{bmatrix},$$

that belong to the block column for Ω_1 in M_{22}, is nonsingular, Gaussian eliminations on the rows transform it to the identity matrix. Then the eliminations, applied on the columns, can be used to set all the rest of the elements in the block rows for e_1 and e_2 in M_{22} to zero. This simplifies the matrix, the considered block rows and the block column for Ω_1 can be omitted. In the last block column only the nonzero block Q_{77}^{22} (Q_{88}^{22}) of dimension 2×4 remains. In order to choose minors to prove full rank of the matrix, we have to omit two of the columns in the last block column, otherwise the columns are linearly dependent. But this is not possible because of the matrix dimensions. For $k = 7$ one and for $k = 8$ none of the critical columns can be removed. Therefore the matrices $\widetilde{M}(\triangle_1, \triangle)$ for $k = 7, 8$ are not of full rank. $\square$

7 Conclusion

The problem of determining the dimension of the cubic C^1 bivariate spline space over triangulations may seem easy. But for over 40 years it remains unsolved. Various mathematical tools were applied, from numerical mathematics, algebra and graph theory. A possible way in practice is by introducing triangle subdivision. Unfortunately this modifies the triangulation and significantly increases the number of triangles, and thus influences further numerical computations. In this paper an approach by using cell reduction at the boundary of a triangulation is presented. If the triangulation is reduced to a single triangle, this proves the dimension result. Otherwise it could be combined with some other method, that would yield the result for the remaining subtriangulation. Larger interior cells remain to be analysed, since the applied technique has a very large memory and processor power requirements.

An another interesting research topic is the study of the space $S_2^1(\triangle)$ and its generalization $S_{2r}^r(\triangle)$, $r \geq 1$, where very little is known in general. In a more general setting, not much is known on very complex spline spaces on higher dimensional simplical complexes, where tools from commutative algebra show a lot of promise.

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