On the chromatic index
of generalized truncations

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Abstract

We examine the chromatic index of generalized truncations of graphs and multigraphs. The insertion graphs considered are complete graphs, cycles, regular graphs and forests.

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1 Introduction

A broad definition of generalized truncations of graphs was introduced in [1]. We give this definition now for completeness, but first a brief word about terminology is in order. The term *multigraph* is used if multiple edges are allowed. Thus, a graph does not have multiple edges. We use $V(X)$ to denote the set of vertices of a multigraph X and $E(X)$ to denote the set of edges. The *order* of X is $|V(X)|$ and the *size* of X is $|E(X)|$. Finally, the *valency* of a vertex u, denoted $\mathrm{val}(u)$, is the number of edges incident with u. The term *k-valent* multigraph is used for regular multigraphs of valency k. Throughout the paper, $\Delta(X)$ denotes the maximum valency of the multigraph X and Δ often is used when the graph X involved is apparent.

Given a multigraph X, a *generalized truncation* of X is obtained as follows via a two-step operation. The first step is the *excision step*. Let M denote an auxiliary matching (no two edges have a vertex in common) of size $|E(X)|$. Let $F\colon E(X) \to M$ be a bijective function and for $[u, v] \in E(X)$, label the ends of the edge $F([u, v])$ with u and v. Let

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M_F denote the vertex-labelled matching thus obtained. So M_F represents the edges of X completely disassembled.

The second step is the *assemblage step*. For each $v \in V(X)$, the set of vertices of M_F labelled with v is called the *cluster at* v and is denoted $\mathrm{cl}(v)$. Insert an arbitrary graph on $\mathrm{cl}(v)$. The inserted graph on $\mathrm{cl}(v)$ is called the *constituent graph at* v and is denoted $\mathrm{con}(v)$. The resulting graph

$$M_F \cup_{v \in V(X)} \mathrm{con}(v)$$

is a *generalized truncation* of X. We usually think of the labels on the vertices of M_F as being removed following the assemblage stage, but there are many times when the labels are useful in the exposition. We use $\mathrm{TR}(X)$ to denote a generalized truncation of the multigraph X.

Truncations arise via action involving the edges incident with a vertex. Consequently, isolated vertices are useless and we make the important convention that the multigraphs from which we are forming generalized truncations do not have isolated vertices. This will not be mentioned in the subsequent material, but is required for the validity of a few statements.

Recall that a *proper edge coloring* of a multigraph X is a coloring of the edges so that adjacent edges do not have the same color. The *chromatic index* of X, denoted $\chi'(X)$, is the fewest number of colors for which a proper edge coloring exists. We now state Vizing's well-known theorem [4] which plays a fundamental role in studying chromatic index.

Theorem 1.1. *If X is a graph, then its chromatic index is either $\Delta(X)$ or $\Delta(X) + 1$.*

The preceding theorem leads to a classification of graphs as follows. A graph is *class* I if its chromatic index is equal to its maximum valency Δ and is *class* II otherwise. There is a generalization for multigraphs and to ease subsequent exposition we shall say a multigraph is class I if its chromatic index equals its maximum valency.

The following result was proved in [1]. A generalized truncation $\mathrm{TR}(X)$ is called *complete* when every constituent is a complete subgraph.

Theorem 1.2. *If X is a class* I *graph, then its complete truncation also is class* I. *If X is a class* II *graph and its maximum valency is even, then its complete truncation is class* I.

The purpose of this paper is to extend the preceding result and explore edge colorings of generalized truncations in more detail.

2   Some useful results

In the following material we shall be considering the relationship between multigraphs and their generalized truncations that are class I. Doing so requires the use of some well-known edge coloring results which we now give for completeness.

Let σ be a permutation of the vertex set of the complete graph K_n of order n. If Y is a subgraph of K_n, then $\sigma(Y)$ denotes the subgraph of K_n whose edge set is $\{[\sigma(u), \sigma(v)] : [u, v] \in E(Y)\}$. Let n be even and ρ be the permutation defined by $\rho = (u_0)(u_1 \ u_2 \ u_3 \ \cdots \ u_{n-1})$. So ρ fixes one vertex and cyclically rotates the remaining vertices. Let Y be the perfect matching of K_n consisting of the edges

$$[u_0, u_1], [u_2, u_{n-1}], [u_3, u_{n-2}], \ldots, [u_{n/2}, u_{(n+2)/2}].$$
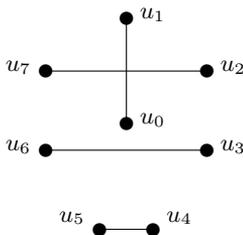
Figure 1: Canonical edge coloring.

We then obtain a proper edge coloring of K_n by letting the color classes be $Y, \rho(Y), \rho^2(Y), \ldots, \rho^{n-2}(Y)$. We call this proper edge coloring the *canonical edge coloring* of K_n. Figure 1 shows Y for K_8. We obtain a proper edge coloring of K_n with n colors, n odd, by starting with the canonical edge coloring of K_{n+1} and then removing the central vertex u_0 and the edges incident with it. Note that each vertex is missing an edge of one color and the set of missing colors has cardinality n. This fact is true for any proper edge coloring of K_n, n odd.

Because we use the preceding canonical edge coloring for both even and odd orders, we describe it by referring to the edge of length 1 as the *anchor*. When n is even, the anchor is the edge $[u_{n/2}, u_{1+n/2}]$. When n is odd, the anchor is the edge $[u_{(n+1)/2}, u_{(n+3)/2}]$. To obtain proper edge colorings of complete graphs, we cyclically rotate the canonical edge coloring which, of course, cyclically rotates the anchor. So we may determine a color class by specifying the anchor.

Lemma 2.1. *Every proper edge coloring of K_n with n colors, n odd, has the property that there is one color missing on the edges incident with a given vertex, and the set of missing colors at the vertices has cardinality n.*

Proof. There are at most $(n-1)/2$ edges of a fixed color in a proper edge coloring of K_n because n is odd. Thus, there is no proper edge coloring using just $n-1$ colors because $(n-1)^2/2 < \binom{n}{2}$. There is a proper edge coloring using n colors by Vizing's Theorem. Hence, each color class contains precisely $(n-1)/2$ edges from which the conclusion follows. $\qquad\square$

We use a result about list chromatic index so we discuss it briefly here and present the result. Given a graph X and for each edge $[u, v]$ a list $L_{[u,v]}$ of colors we may use for the edge $[u, v]$, a *proper list edge coloring* is a proper edge coloring of X so that the color on each edge $[u, v]$ belongs to $L_{[u,v]}$. The *list chromatic index* of X is the smallest N such that if every list $L_{[u,v]}$ has cardinality N, then X admits a proper list edge coloring. We denote this value by $\chi'_L(X)$. The following important result was proved by Häggkvist and Jansen in [2].

Theorem 2.2. *The complete graph K_n satisfies $\chi'_L(K_n) \le n$.*

3 A general result

We are interested in determining which generalized truncations of a given multigraph X are class I. Theorem 3.1 below provides a general answer and in subsequent sections we use the theorem to describe class I generalized truncations of specific types. Some definitions and notation are required before stating the theorem.

Given a generalized truncation $\mathrm{TR}(X)$, for convenience we call the subgraph induced by the edges of $\mathrm{con}(v)$ together with the edges of M_F incident with the vertices of $\mathrm{con}(v)$ the *sun centered at* $\mathrm{con}(v)$. Given that $\mathrm{con}(v)$ is regular of valency $d-1$, when is the sun centered at $\mathrm{con}(v)$ class I? If it is class I, then there is a proper edge coloring using d colors. We use a vector of length d to describe the number of edges of M_F of the various colors in the sun centered at $\mathrm{con}(v)$.

Given a vertex $v \in X$ of valency r, we say that a vector $(x_1, x_2, \ldots, x_d)$ is *admissible* if $\sum_i x_i = r$, the edges of M_F are colored according to the vector, that is, x_i edges have color i, and there is a $(d-1)$-regular graph on $\mathrm{con}(v)$ such that the sun centered at $\mathrm{con}(v)$ is class I. We say the vector is *totally inadmissible* if there is no $(d-1)$-regular graph on $\mathrm{con}(v)$ which makes the sun class I.

Theorem 3.1. *If* $(x_1, x_2, \ldots, x_d)$ *satisfies* $x_1 + x_2 + \cdots + x_d = r \geq d$, *then* $(x_1, x_2, \ldots, x_d)$ *is admissible if and only if all of* $x_1, x_2, \ldots, x_d$ *and* r *have the same parity and when the parity is odd,* d *also is odd. Otherwise, the vector is totally inadmissible.*

Proof. Suppose that $\mathrm{val}(v) = r$ in a multigraph X and that there is a class I sun centered at $\mathrm{con}(v)$ which is regular of valency d. Let $(x_1, x_2, \ldots, x_d)$ be the vector for the colors of M_F. For an arbitrary color $c(i)$, if the number of edges of color $c(i)$ in $\mathrm{con}(v)$ is k, then the number of edges of color $c(i)$ in M_F must be $r - 2k$. It then follows that $x_1, x_2, \ldots, x_d$ and r all have the same parity. Moreover, if r is odd, then the graph $\mathrm{con}(v)$ is regular of valency $d-1$ and order r which implies that $d-1$ is even. This completes the necessity.

Now suppose that all coordinates of $(x_1, x_2, \ldots, x_d)$ are odd, r is odd and $x_1 + x_2 + \cdots + x_d = r$. In this case we also have that d is odd. Label the r vertices of the constituent $u_1, u_2, u_3, \ldots, u_r$ so that the first x_1 vertices are incident with the x_1 edges of M_F of color $c(1)$, the next x_2 vertices are incident with the x_2 edges of M_F of color $c(2)$ and continue in this way. Also carry out subscript arithmetic modulo r on the residues $1, 2, 3, \ldots, r$.

Consider the x_i successive vertices incident with the x_i edges of M_F of color $c(i)$. Because x_i is odd, there is a central edge of M_F of color $c(i)$ and let it be incident with u_a. Letting $\alpha = (x_i - 1)/2$, the vertices incident with edges of M_F of color $c(i)$ are $u_{a-\alpha}, u_{a-\alpha+1}, \ldots, u_a, u_{a+1}, \ldots, u_{a+\alpha}$.

Consider the canonical color class whose anchor is $[u_{a+(r-1)/2}, u_{a+(r+1)/2}]$. Color the edges of this class with color $c(i)$ starting with the anchor and stopping with the edge $[u_{a-\alpha-1}, u_{a+\alpha+1}]$. This yields edges colored with $c(i)$ so that each vertex of the constituent is incident with one edge of color $c(i)$. After doing this for each of the d colors, the resulting sun centered at the constituent is class I and the constituent is regular of valency $d-1$.

This leaves us with the case that all the coordinates of $(x_1, x_2, \ldots, x_d)$ are even so that their sum r also is even. There is no restriction on the parity of d in this case. Also note that $x_i = 0$ is possible for various values of i. Without loss of generality, let $x_1, x_2, \ldots, x_a$ be the non-zero entries of the vector.

We label the vertices of the constituent a little differently using the canonical edge coloring of Figure 1 as a template. The central vertex is labelled u_0 and the others are labelled $u_1, u_2, \ldots, u_{r-1}$.

Let the x_1 edges of M_F of color $c(1)$ be incident with $u_0, u_1, \ldots, u_{x_1-1}$. Let the remaining edges of M_F be incident with vertices of the constituent as in the preceding case, that is, edges of the same color are incident with successively labelled vertices.

The edges of color $c(1)$ have a different form than the other colored edges making the completion of the coloring a little different for them. Vertices $u_1, u_2, \ldots, u_{x_1-1}$ are incident with edges of M_F of color $c(1)$. This corresponds to the color class with anchor $[u_{(x_1+r-4)/2}, u_{(x_1+r-2)/2}]$. So color the edges of that class with color $c(1)$ starting with the anchor until finishing with $[u_{r-1}, u_{x_1}]$. Every vertex of the constituent now is incident with an edge of color $c(1)$.

For all the other colors $c(j)$, $1 < j \leq a$, there are an even number of successive vertices, say $u_j, u_{j+1}, \ldots, u_{j+t}$, incident with edges of M_F of color $c(j)$. Then the edge $[u_{j+(t-1)/2}, u_{j+(t+1)/2}]$ is the anchor of a color class. So complete the coloring of the edges of this color class starting with $[u_{j-1}, u_{j+t+1}]$ moving away from the anchor. For $x_i = 0$, simply include an entire color class from the canonical coloring scheme using an unused anchor.

Doing the preceding yields a class I coloring of the sun centered at the constituent so that the constituent graph is regular of valency $d - 1$ and completes the proof. $\square$

We now obtain three corollaries from Theorem 3.1, but first require a couple of definitions. An edge coloring of a multigraph X is said to be *parity-balanced* if for each vertex v of X, the parity of the number of edges of each color incident with v is the same as the parity of $\mathrm{val}(v)$. A *regular truncation* is a generalized truncation for which every vertex has the same valency. A generalized truncation is said to be *semiregular* if each sun centered at a constituent $\mathrm{con}(v)$ is class I and the subgraph $\mathrm{con}(v)$ is regular. The distinction between a semiregular truncation and a regular truncation is that valencies of regularity for a semiregular truncation may differ over the constituents. Note that the source multigraph need not be regular in order to have a regular truncation.

Corollary 3.2. *Let every entry of the feasible vector $(x_1, x_2, \ldots, x_d)$ be even and $x_1 + x_2 + \cdots + x_d = r$. If d' is the number of non-zero entries of the vector, then there are class I suns centered at the appropriate constituent such that the constituents are regular of every valency from d' through $r - 1$.*

Proof. This result follows from the proof of Theorem 3.1 rather than the statement. Because the number of colors on edges of M_F is d', the scheme used in the proof of Theorem 3.1 introduces no new colors so that the valency of regularity is d'. Because r is even, we may add canonical coloring classes one at a time until reaching K_r. This completes the proof. $\square$

Corollary 3.3. *A multigraph X has a class I semiregular truncation if and only if it has a parity-balanced edge coloring.*

Proof. If X has a parity-balanced edge coloring, it follows immediately from Theorem 3.1 that it has a semiregular truncation. On the other hand, if X has a semiregular truncation, then performing the standard contraction and retention of the colors on the edges of M_F produces a parity-balanced edge coloring of X. $\square$

Corollary 3.4. *A multigraph X has a class I regular truncation of valency d if and only if it has a parity-balanced coloring and one of the following conditions holds:*

(i) *When d is odd, every vertex of odd valency has precisely d colors on its incident edges, and every vertex of even valency has valency at least $d + 1$ and at most d colors on its incident edges ; or*

(ii) *When d is even, every vertex of X has even valency at least d.*

Proof. Let X be a mutigraph and suppose it has a generalized truncation Y which is regular of valency d. Consider the case that d is even. This means that a constituent $\mathrm{con}(v)$ is regular of valency $d - 1$ and the latter is odd. Thus, the constituent has even order. This implies that every vertex of X has even valency. Clearly every valency is at least d.

When d is odd, each constituent must itself be regular of valency $d - 1$. Because $d - 1$ is even, there is no restriction on the order N of the constituent other than it must be at least d. If N is odd, then for each of the d distinct colors, there must be at least one edge of that color belonging to the edges of M_F incident with vertices of the constituent. So the corresponding vertex of X is incident with edges of precisely d different colors. When N is even there is no restriction on the number of colors on edges of M_F incident with vertices of the constituent other than it is at most d.

For the other direction, when d is even, every constituent has even order and by Corollary 3.2 it is easy to obtain regular valency of $d - 1$ on the constituent. When d is odd, the result follows from Theorem 3.1. $\qquad\square$

4   Complete truncations

Let X be a multigraph with maximum valency Δ. If Δ is odd and X has an edge coloring with Δ colors such that the edges incident with every vertex of valency Δ have distinct colors, then we say that X is *edge-feasible*. The next result characterizes graphs whose complete truncations are class I. Of course, a complete truncation is a semi-regular truncation.

Theorem 4.1. *The complete truncation of a multigraph X is class I if and only if either the maximum valency Δ of X is even, or Δ is odd and X is edge-feasible.*

Proof. This theorem is an improvement on Theorem 1.2 as the latter does not include multigraphs as part of the hypotheses. Because a complete truncation is a semiregular truncation, Corollary 3.3 implies that the coloring of the edges in M_F must correspond to a parity-balanced coloring of X. When Δ is even, coloring all the edges of M_F with a single color corresponds to a parity-balanced coloring of X. Any constituent of order Δ can be properly edge-colored with $\Delta - 1$ colors because Δ is even. Any constituent of order less than Δ can be properly edge-colored with at most $\Delta - 1$ colors. Hence, the complete truncation of X is class I.

For the remainder of the proof we assume that Δ is odd. First suppose that the complete truncation $\mathrm{TR}(X)$ is class I. If $u \in V(X)$ has valency Δ, then $\mathrm{con}(u) = K_\Delta$ the complete graph of order Δ. We conclude that the edges of M_F incident with $\mathrm{con}(u)$ all have different colors by Lemma 2.1. So if we contract each constituent to a single vertex, delete all the loops formed and keep the colors of the edges of M_F, we obtain an edge coloring of X. Clearly, all the edges incident to any vertex of valency Δ in X have distinct colors. Thus, X is edge-feasible.

Now let X be edge-feasible and choose an edge coloring of X which is edge-feasible. Let $\mathrm{TR}(X)$ be the complete truncation of X. Color the edges of M_F with the same color they had in the edge coloring of X. These are the edges between the constituents all of which are complete subgraphs. At this point we have used Δ colors. It suffices to show that we can color the edges of the constituents without introducing any new colors so that the resulting edge coloring of $\mathrm{TR}(X)$ is proper.

Any constituent $\mathrm{con}(u)$ of order Δ corresponds to a vertex u of X of valency Δ. This implies that the edges of M_F incident with the vertices of $\mathrm{con}(u)$ all have distinct colors because the coloring of X was edge-feasible. From Lemma 2.1 it is clear that we may color the edges of $\mathrm{con}(u)$ with Δ colors so that the missing color at each vertex is the color of the edge of M_F at the vertex. This coloring of the edges does not violate the definition of a proper edge coloring.

Now consider any constituent $\mathrm{con}(u)$ of order $r \leq \Delta - 2$. Because each edge of $\mathrm{con}(u)$ has one edge of M_F at each of its end vertices, the number of possible colors for the edge that do not violate the proper edge coloring condition is $\Delta - 2$. So each edge has a list of $\Delta - 2$ possible colors, and Theorem 2.2 implies that we may color the edges of $\mathrm{con}(u)$ without violating the proper edge coloring condition because $r \leq \Delta - 2$.

This leaves us with the case that $\mathrm{con}(u)$ has order $\Delta - 1$ and this is the most complicated case. The first observation we make is that the coloring pattern of the edges of M_F incident with the vertices of $\mathrm{con}(u)$ can vary all the way from having $\Delta - 1$ distinct colors to every color being the same. So we introduce a sequence describing the color pattern. Let $(s_1, s_2, \ldots, s_t)$ satisfy $s_1 \leq s_2 \leq \cdots \leq s_t$ and $s_1 + s_2 + \cdots + s_t = \Delta - 1$. The sequence means there are t distinct colors on the edges of M_F incident with vertices of $\mathrm{con}(u)$ and s_i such edges have the same color $c(i)$ for $i = 1, 2, \ldots, t$.

We need to show that no matter what the color sequence is we may color the edges of $\mathrm{con}(u)$ so that the sun centered at $\mathrm{con}(u)$ is properly edge colored with Δ colors. As a first step, label the vertices of $\mathrm{con}(u)$ with $v_0, v_1, \ldots, v_a$, where $a = \Delta - 2$. Let the vertices $v_1, v_2, \ldots, v_{s_1}$ be incident with the s_1 edges of M_F of color $c(1)$. Let the next s_2 vertices be incident with the s_2 edges of M_F of color $c(2)$. Continue labelling the vertices in the obvious manner and let the edge of M_F incident with v_0 have color $c(t)$.

Carry out an initial coloring of the edges of $\mathrm{con}(u)$ using the canonical edge coloring described in Section 2 with v_0 acting as the fixed vertex and $\rho = (v_0)(v_1 \; v_2 \; \cdots \; v_a)$. The strategy now is to choose the colors for the color classes in such a way that we may re-color some edges to obtain a proper edge coloring for the sun centered at $\mathrm{con}(u)$.

The first observation we make is that only $\Delta - 2$ colors are required for a canonical edge coloring of $\mathrm{con}(u)$. So we do not use the colors $c(t-1)$ or $c(t)$ for the canonical edge coloring of $\mathrm{con}(u)$. Hence, if $t = 1$ or $t = 2$, we already have an edge coloring of $\mathrm{con}(u)$ that does not violate the conditions for a proper edge coloring. Thus, we assume that $t \geq 3$.

The anchor for a color class plays an active role as follows. Suppose we require two edges of a color class so that the two edges use four successive vertices under the cyclic labelling of $\{v_1, v_2, \ldots, v_a\}$. If the anchor is $[v_i, v_{i+1}]$, then adding the edge $[v_{i-1}, v_{i+2}]$ from the same color class easily does the job. It is now easy to see how we may obtain k edges from the same color class so that they cover $2k$ successive vertices. With this in mind, we now describe an iterative process for determining the colors of certain color classes.

If s_1 is odd, then we color the color class for which $[v_{(s_1+1)/2}, v_{(s_1+3)/2}]$ is the anchor with $c(1)$. If s_1 is even, then we color the color class for which $[v_{s_1/2}, v_{(s_1+2)/2}]$ is the

anchor with $c(1)$. There are two points to observe about the preceding choices. When s_1 is odd, the edge from v_1 to v_{s_1+1} has color $c(1)$ and note that the edge of M_F incident with v_{s_1+1} has color $c(2)$. When s_1 is even, the edge from v_1 to v_{s_1} has color $c(1)$ and vertex v_{s_1} is the last vertex for which the edge of M_F incident with it has color $c(1)$.

We continue in the manner suggested by the preceding paragraph, but discuss it further to make it clearer. When s_i is odd and the first vertex incident with an edge of M_F of color $c(i)$ is v_d, then the anchor is the edge $[v_{d+(s_i-1)/2}, v_{d+(s_i+1)/2}]$ and we color this color class with $c(i)$. Note that the edge $[v_d, v_{d+s_i}]$ is in this color class, but the edge of M_F incident with v_{d+s_i} has color $c(i+1)$.

When s_i is even and the first vertex incident with an edge of M_F of color $c(i)$ is v_d, then the anchor is $[v_{d+(s_i-2)/2}, v_{d+s_i/2}]$ and we color this color class with $c(i)$. In this case, the edge from v_d to v_{d+s_i-1} is colored $c(i)$ and v_{d+s_i-1} is the last vertex incident with an edge of M_F of color $c(i)$.

The preceding procedure is carried out for the colors $c(1)$ through $c(t-2)$ at which point it stops because colors $c(t-1)$ and $c(t)$ are not used for the canonical edge coloring of $\text{con}(u)$. The edges of $\text{con}(u)$ that need to be re-colored are those that are adjacent with edges of M_F having the same color. We have seen that when s_i is odd, there is an edge in $\text{con}(u)$ of color $c(i)$ with one end vertex incident with an edge of M_F of color $c(i)$ and the other end vertex incident with an edge of M_F of color $c(i+1)$. So at the end vertex incident with an edge of M_F of color $c(i+1)$, the procedure gives another edge of color $c(i+1)$ which also needs to be re-colored.

Thus, the edges that need to be recolored are isolated and possibly paths. Luckily we have two colors to use for the re-coloring and we arbitrarily re-color isolated edges with either $c(t-1)$ or $c(t)$, and alternately re-color the edges of the paths making certain that if one of the paths terminates with a vertex whose incident edge from M_F has color $c(t-1)$, we color the edge of the path terminating there with $c(t)$. This removes all potential color conflicts for $\text{con}(u)$ and completes the proof of the theorem. $\square$

Corollary 4.2. *Let X be a class I graph and $\text{TR}(X)$ a generalized truncation. If $\Delta(X) = \Delta(\text{TR}(X))$, then $\text{TR}(X)$ is class I.*

Proof. Let $\text{TR}(X)$ be a generalized truncation of X satisfying $\Delta(\text{TR}(X)) = \Delta(X)$. Then a proper edge coloring of $\text{TR}(X)$ requires at least Δ colors. We know the complete truncation Y of X is class I by Theorem 4.1, that is, it has a proper edge coloring using Δ colors. Remove any edges of Y not belonging to $\text{TR}(X)$ and we are left with a proper edge coloring of $\text{TR}(X)$ using Δ colors. So $\text{TR}(X)$ is class I. $\square$

It is well known that both the Petersen graph and its complete truncation are class II graphs. The extension to a regular multigraph is an immediate corollary of Theorem 4.1.

Corollary 4.3. *A regular multigraph of odd valency is class I if and only if its complete truncation is class I.*

5   Cyclic truncations

Complete truncations have been used by various authors, and there is another generalized truncation that has been employed frequently. Namely, the generalized truncation obtained by letting each constituent graph be a cycle. We shall call these *cyclic truncations*. Probably the best known example of this is the cube-connected cycles graph first introduced in [3].

Of course, the ancient Greeks studied truncations of Platonic and Archimedian solids, and these resulted in cyclic truncations.

When moving from a multigraph X to a generalized truncation Y, we frequently wish Y to inherit properties of X. This can be a problem for cyclic truncations. For example, poorly chosen cyclic truncations can play havoc with automorphisms.

We are interested in determining conditions for which a cyclic truncation is class I. Of course, a cyclic truncation is a regular truncation so that we may use Theorem 3.1. We use the same vector describing the numbers of colors used on the edges of M_F belonging to a sun centered at a constituent. In this case the vector has length 3 as we are looking at cyclic truncations all of which are 3-valent. There is a new term not used before, namely, a vector (x_1, x_2, x_3) is *universal* if the sun centered at the corresponding constituent is class I for every cycle on the vertices of the constituent.

Corollary 5.1. *If (x_1, x_2, x_3) satisfies $x_1 + x_2 + x_3 = d \geq 3$, then (x_1, x_2, x_3) is admissible if and only x_1, x_2, x_3 and d all have the same parity. Otherwise, the vector is totally inadmissible. Moreover, if just one of x_1, x_2 and x_3 is non-zero, then (x_1, x_2, x_3) is universal when d is even.*

Proof. The portion of the statement prior to the sentence about universality is simply Theorem 3.1 restricted to $d = 3$. When the vector has the form $(x_1, 0, 0)$ and $x_1 = d$ is even, this corresponds to all the edges of M_F incident with the vertices of the constituent having the same color. Clearly, no matter which even length cycle we form on the vertices of the constituent, the resulting sun is class I. $\square$

Corollary 5.1 gives us an easy way to construct a class I cyclic truncation of a multigraph X based on parity conditions for the colors on the edges of M_F. Thus, we need to concentrate on coloring the edges in the source multigraph.

Corollary 5.2. *If every vertex of the multigraph X has even valency, then every cyclic truncation of X is class I.*

Proof. This follows from Corollary 5.1 by coloring all edges of M_F with a single color. $\square$

Corollary 5.3. *If X is a regular multigraph of odd valency $d > 1$ and is class I, then there are class I cyclic truncations of X.*

Proof. Let X be a class I regular multigraph of odd valency $d \geq 3$. Choose a proper edge coloring of X using d colors. In forming a cyclic truncation Y of X, color the edges of M_F according to the colors $c(1), c(2), \ldots, c(d)$ the edges have in the proper edge coloring of X. Now change the color of any edge of colors $c(4), c(5), \ldots, c(d)$ to $c(3)$. The vector for each constituent becomes $(1, 1, d - 2)$ all of which are odd. We now may find a cycle for each constituent such that the cyclic truncation is class I by Corollary 5.1. $\square$

We now give a sufficient condition for a multigraph to have a class I cyclic truncation. A definition is required. Let X be an arbitrary multigraph and V_0, V_1, V_2 and V_3 be a partition of $V(X)$, where V_i contains the vertices whose valencies are congruent to i modulo 4. Given a submultigraph Y of X, let $X \setminus Y$ be the submultigraph obtained by removing the edges of Y from X. Moreover, let V_0', V_1', V_2' and V_3' denote the vertices of $V(X)$ whose valencies in $X \setminus Y$ are congruent to i modulo 4. A submultigraph Y of X is called an *enabling submultigraph* if it satisfies:

- if $v \in V_0$, then $v \in V_0'$;

- if $v \in V_1$, then $v \in V_2'$;

- if $v \in V_2$, then $v \in V_0'$; and

- if $v \in V_3$, then $v \in V_2'$.

The preceding conditions are not as complicated as they may look at first. The first thing to notice is that the components of $X \setminus Y$ are Eulerian. Second, for many graphs the conditions are simple. For example, if X is 3-regular, then a perfect matching is an enabling subgraph.

Corollary 5.4. *Let X be a multigraph with an enabling submultigraph Y. If every component of $X \setminus Y$ has even size, then there is a class I cyclic truncation of X.*

Proof. Because each component of $X \setminus Y$ has even size, we may alternately color the edges of an Euler tour of the component with two colors so that we finish with same number of colors on the edges incident with every vertex. Doing this for each component yields a 2-coloring so that each vertex of $X \setminus Y$ has the same number of colors incident with it.

If we now color all the edges of Y with a third color, it is easy to verify that the conditions of Corollary 5.1 are met for X. We check one possibility for illustrative purposes. If $v \in V_3$, it belongs to V_2' in $X \setminus Y$. Thus, the number of edges of Y incident with v is congruent to 1 modulo 4, that is, it is incident to an odd number of edges of the third color in X. Because it is in V_2', it is incident with an odd number of edges of each of the first two colors. Thus, the vector for v has three odd components. $\square$

Proposition 5.5. *If a trivalent graph X has a cut edge, then X is class II.*

The proof of the preceding proposition is immediate and leads to examples such as the following. Let X be the graph of order 10 obtained by taking two vertex-disjoint copies of K_5 and joining the two copies with a single edge from a vertex of one copy to a vertex of the other copy. Lemma 2.1 implies that X is class I. If we take any cyclic truncation $\mathrm{TR}(X)$ of X, then $\mathrm{TR}(X)$ is class II by Proposition 5.5 because the edge of M_F connecting the two copies still is a cut edge in $\mathrm{TR}(X)$.

6   Arboreal truncations

We now examine another generalized truncation. An *arboreal truncation* is a generalized truncation for which every constituent graph is a forest. The following result is useful for arboreal truncations.

Theorem 6.1. *Let $\mathrm{TR}(X)$ be a generalized truncation of a multigraph X. If the maximum valency of $\mathrm{TR}(X)$ is Δ and every constituent of $\mathrm{TR}(X)$ having a vertex of valency Δ is class I, then $\mathrm{TR}(X)$ is class I.*

Proof. Let $\mathrm{con}(v)$ have a vertex of valency Δ in $\mathrm{TR}(X)$. Then the maximum valency in the subgraph $\mathrm{con}(v)$ is $\Delta - 1$. Its edges may be properly edge-colored with $\Delta - 1$ colors because it is class I. Any constituent not having a vertex of valency Δ may have its edges properly colored with at most $\Delta - 1$ colors because its maximum valency is $\Delta - 2$. Then the edges of M_F may be colored with a new color yielding a proper edge coloring of $\mathrm{TR}(X)$ with Δ colors. The conclusion now follows. $\square$

Corollary 6.2. *Every arboreal truncation of a multigraph X is class I.*

Proof. This follows immediately from Theorem 6.1 because forests are class I. □

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