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Some remarks on the square graph of the hypercube

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Abstract

Let $\Gamma = (V, E)$ be a graph. The square graph Γ^2 of the graph Γ is the graph with the vertex set $V(\Gamma^2) = V$ in which two vertices are adjacent if and only if their distance in Γ is at most two. The square graph of the hypercube Q_n has some interesting properties. For instance, it is highly symmetric and panconnected. In this paper, we investigate some algebraic properties of the graph Q_n^2. In particular, we show that the graph Q_n^2 is distance-transitive. We show that the graph Q_n^2 is an imprimitive distance-transitive graph if and only if n is an odd integer. Also, we determine the spectrum of the graph Q_n^2. Finally, we show that when $n > 2$ is an even integer, then Q_n^2 is an automorphic graph, that is, Q_n^2 is a distance-transitive primitive graph which is not a complete or a line graph.

Keywords: Square of a graph, distance-transitive graph, hypercube, automorphism group, Johnson graph, automorphic graph.

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1 Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminology and notation not defined here, we follow [1, 3, 5, 6, 9].

Let $\Gamma = (V, E)$ be a graph. The *square graph* Γ^2 of the graph Γ is the (simple) graph with vertex set V in which two vertices are adjacent if and only if their distance in Γ is at most two. It is easy to see that $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Gamma^2)$, where $\mathrm{Aut}(\Gamma)$ denotes the automorphism group of the graph Γ. Thus, if the graph Γ is a vertex-transitive graph, then Γ^2 is a vertex-transitive graph. A graph Γ of order $n > 2$ is *Hamilton-connected* if for any pair of distinct vertices u and v, there is a Hamilton u-v path, namely, there is a u-v path

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of length $n-1$. It is clear that if a graph Γ is Hamilton-connected then it is Hamiltonian. A graph Γ of order $n > 2$ is panconnected if for every two vertices u and v, there is a u-v path of length l for every integer l with $d(u,v) \leq l \leq n-1$. Note that if a graph Γ is panconnected, then it is Hamilton-connected. It is a well known fact that when a graph Γ is 2-connected, then its square Γ^2 is panconnected [4, 7]. Using this fact, and an algebraic property of Johnson graphs, recently it has been proved that the Johnson graphs are panconnected [10].

Let $n \geq 2$ be an integer. The hypercube Q_n is the graph whose vertex-set is $\{0,1\}^n$, where two n-tuples are adjacent if they differ in precisely one coordinate. This graph has been studied from various aspects by many authors. Some recent works concerning some algebraic aspects of this graph include [14, 17, 24, 28]. It is a well known fact that the graph Q_n is a distance-transitive graph [1, 3], and hence it is edge-transitive. Now, using a well known result due to Watkins [27], it follows that the connectivity of Q_n is maximal, that is, n. Like the hypercube Q_n, its square, namely, the graph Q_n^2 has some interesting properties. For instance, when $n \geq 2$, then Q_n is 2-connected. Now using a known result due to Chartrand and Fleischner [4, 7], it follows that Q_n^2 is a panconnected graph. Also, since Q_n is vertex-transitive, the graph Q_n^2 is vertex-transitive, as well. Hence Q_n^2 is a regular graph and it is easy to check that its valency is $n+\binom{n}{2}=\binom{n+1}{2}$. If $n=2$, then Q_n^2 is the complete graph K_4. When $n=3$, then Q_n^2 is a 6-regular graph with 8 vertices. This graph is isomorphic with a graph known as the *coktail-party* graph $CP(4)$ [1]. It can be shown that when $n=4$, then the graph Q_n^2 is a 10-regular graph with 16 vertices, which is isomorphic to the complement of the graph known as the *Clebsch* graph [9].

In this paper, we determine the automorphism group of the graph Q_n^2. Then we show that Q_n^2 is a distance-transitive graph. This implies that the connectivity of the graph Q_n^2 is maximal, namely, its valency $\binom{n+1}{2}$. Also, we will see that the graph Q_n^2 is an imprimitive distance-transitive graph if and only if n is an odd integer. A graph Γ is called an *automorphic* graph, when it is a distance-transitive primitive graph which is not a complete or a line graph [1]. In the last step of the paper, we show that the graph Q_n^2 is an automorphic graph if and only if n is an even integer.

2 Preliminaries

The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called *isomorphic*, if there is a bijection $\alpha\colon V_1 \longrightarrow V_2$ such that $\{a,b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. In such a case the bijection α is called an *isomorphism*. An *automorphism* of a graph Γ is an isomorphism of Γ with itself. The set of automorphisms of Γ with the operation of composition of functions is a group called the *automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$.

The group of all permutations of a set V is denoted by $\mathrm{Sym}(V)$ or just $\mathrm{Sym}(n)$ when $|V| = n$. A *permutation group* G on V is a subgroup of $\mathrm{Sym}(V)$. In this case we say that G *acts* on V. If G acts on V we say that G is *transitive* on V (or G acts *transitively* on V) if given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$. If Γ is a graph with vertex-set V then we can view each automorphism of Γ as a permutation on V and so $\mathrm{Aut}(\Gamma) = G$ is a permutation group on V.

A graph Γ is called *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. We say that Γ is *edge-transitive* if the group $\mathrm{Aut}(\Gamma)$ acts transitively on the edge set E, namely, for any $\{x,y\}, \{v,w\} \in E(\Gamma)$, there is some π in $\mathrm{Aut}(\Gamma)$, such that $\pi(\{x,y\}) = \{v,w\}$. We say that Γ is *symmetric* (or *arc-transitive* if for all vertices u, v, x, y of Γ such that u and v are adjacent, and also, x and y are adjacent, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$. We say that Γ is *distance-transitive* if for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, where $d(u, v)$ denotes the distance between the vertices u and v in Γ, there is an automorphism π in $\mathrm{Aut}(\Gamma)$ such that $\pi(u) = x$ and $\pi(v) = y$.

A vertex cut of the graph Γ is a subset U of V such that the subgraph $\Gamma - U$ induced by the set $V - U$ is either trivial or not connected. The *connectivity* $\kappa(\Gamma)$ of a nontrivial connected graph Γ is the minimum cardinality of all vertex cuts of Γ. If we denote by $\delta(\Gamma)$ the minimum degree of Γ, then $\kappa(\Gamma) \leq \delta(\Gamma)$. A graph Γ is called k-*connected* (for $k \in \mathbb{N}$) if $|V(\Gamma)| > k$ and $\Gamma - X$ is connected for every subset $X \subset V(\Gamma)$ with $|X| < k$. It is trivial that if a positive integer m is such that $m \leq \kappa(\Gamma)$, then Γ is an m-connected graph. We have the following fact.

Theorem 2.1 ([27]). *If a connected graph Γ is edge-transitive, then $\kappa(\Gamma) = \delta(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of vertices of Γ.*

Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, ..., n\}$. The *Johnson graph* $J(n, k)$ is defined as the graph whose vertex set is $V = \{v \mid v \subseteq [n], |v| = k\}$ and two vertices v,w are adjacent if and only if $|v \cap w| = k - 1$. The class of Johnson graphs is a well known class of distance-transitive graphs [3]. It is an easy task to show that the set of mappings $H = \{f_\theta \mid \theta \in \mathrm{Sym}([n])\}$, $f_\theta(\{x_1, ..., x_k\}) = \{\theta(x_1), ..., \theta(x_k)\}$, is a subgroup of $\mathrm{Aut}(J(n, k))$ [9]. It has been shown that $\mathrm{Aut}(J(n, k)) \cong \mathrm{Sym}([n])$ if $n \neq 2k$, and $\mathrm{Aut}(J(n, k)) \cong \mathrm{Sym}([n]) \times \mathbb{Z}_2$, if $n = 2k$, where $\mathbb{Z}_2$ is the cyclic group of order 2 [3, 13, 18].

Although in most situations it is difficult to determine the automorphism group of a graph Γ and how it acts on its vertex and edge sets, there are various papers in the literature, and some of the recent works include [8, 11, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 26, 28].

Let G be any abstract finite group with identity 1, and suppose Ω is a subset of G, with the properties:

(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega$,

(ii) $1 \notin \Omega$.

The *Cayley graph* $\Gamma = \mathrm{Cay}(G; \Omega)$ is the (simple) graph whose vertex-set and edge-set are defined as follows:

$V(\Gamma) = G$, $E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$.

It can be shown that the Cayley graph $\Gamma = \mathrm{Cay}(G; \Omega)$ is connected if and only if the set Ω is a generating set in the group G [1].

The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that:

(i) $N \trianglelefteq G$ (N is a normal subgroup of G)

(ii) $NQ = G$; and

(iii) $N \cap Q = 1$.

3 Main results

The hypercube Q_n is the graph whose vertex set is $\{0,1\}^n$, where two n-tuples are adjacent if they differ in precisely one coordinate. It is easy to show that $Q_n = \mathrm{Cay}(\mathbb{Z}_2^n; S)$, where $\mathbb{Z}_2$ is the cyclic group of order 2, and $S = \{e_i \mid 1 \leq i \leq n\}$, where $e_i = (0, ..., 0, 1, 0, ..., 0)$, with 1 at the ith position. It is easy to show that the set $H = \{f_\theta | \theta \in \mathrm{Sym}([n])\}$, $f_\theta(x_1, ..., x_n) = (x_{\theta(1)}, ..., x_{\theta(n)})$ is a subgroup of the group $\mathrm{Aut}(Q_n)$. It is clear that $H \cong \mathrm{Sym}([n])$. We know that in every Cayley graph $\Gamma = \mathrm{Cay}(G; S)$, the group $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic with the group G. In fact, if $x \in \mathbb{Z}_2^n$, and we define the mapping $f_x(v) = x + v$, for every $v \in V(Q_n)$, then f_x is an automorphism of the hypercube Q_n. Hence $\mathbb{Z}_2^n$ is (isomorphic with) a subgroup of $\mathrm{Aut}(Q_n)$. It has been proved that $\mathrm{Aut}(Q_n) = \langle \mathbb{Z}_2^n, \mathrm{Sym}([n])\rangle \cong \mathbb{Z}_2^n \rtimes \mathrm{Sym}([n])$ [14]. It is clear that when Γ is a graph then $\mathrm{Aut}(\Gamma)$ is a subgroup of $\mathrm{Aut}(\Gamma^2)$. Thus we have $\mathrm{Aut}(Q_n) \leq \mathrm{Aut}(Q_n^2)$. In the sequel, we wish to show that the graph Q_n^2 is a distance-transitive graph, and for doing this we need the automorphism group of Q_n^2. When $n = 3$, then Q_n^2 is isomorphic with the coktail-party graph $CP(4)$. The complement of this graph is a disjoint union of 4 copies of K_2. Thus $\mathrm{Aut}(Q_3^2) \cong \mathrm{Sym}([2])\ wr_I\ \mathrm{Sym}([4])$, where $I = \{1, 2, 3, 4\}$ [3, 22] (for an acquaintance with the notion of wreath product of groups see [6]). Now it can be checked that this graph is a distance-transitive graph. Hence, in the sequel we assume that $n > 4$. It is easy to see that for the graph Q_n^2 we have, $Q_n^2 = \mathrm{Cay}(\mathbb{Z}_2^n; T)$, $T = S \cup S_1$, where $S_1 = \{e_i + e_j | \ i, j \in [n], i \neq j\}$. Let $A = \mathrm{Aut}(Q_n^2)$ and A_0 be the stabilizer subgroup of the vertex $v = 0$ in A. Since Q_n^2 is a vertex-transitive graph, then from the orbit-stabilizer theorem we have $|A| = |A_0||V(Q_n^2)| = 2^n|A_0|$. The following lemma determines an upper bound for $|A_0|$.

Lemma 3.1. *Let $n > 4$ and $A = \mathrm{Aut}(Q_n^2)$. Let A_0 be the stabilizer subgroup of the vertex $v = 0$. Then $|A_0| \leq (n+1)!$.*

Proof. Let $\Gamma = Q_n^2$. We know that $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^n; T)$, $T = S \cup S_1$, where $S = \{e_i \mid 1 \leq i \leq n\}$ and $S_1 = \{e_i + e_j | \ i, j \in [n], i \neq j\}$. Let $f \in A_0$. Then $f(T) = T$. Let G be the subgraph of Γ which is induced by the subset T. Let $h = f|_T$ be the restriction of the mapping f to the subset T. It is clear that h is an automorphism of the graph G. It is easy to see that the mapping $\Phi\colon A_0 \to \mathrm{Aut}(G)$, which is defined by the rule $\Phi(g) = g|_T$, is a group homomorphism. Thus we have $\frac{A_0}{\ker(\Phi)} \cong \mathrm{im}(\Phi)$, and hence we have $|A_0| = |\ker(\Phi)||\mathrm{im}(\Phi)|$. Since $im(\Phi)$ is a subgroup of $\mathrm{Aut}(G)$, then we have $|A_0| \leq |\ker(\Phi)||\mathrm{Aut}(G)|$. If we show that $|\mathrm{Aut}(G)| \leq (n+1)!$ and $\ker(\Phi) = \{1\}$, then the lemma is proved. Hence in the rest of the proof we show that:

(i) $|\mathrm{Aut}(G)| \leq (n+1)!$,

(ii) $\ker(\Phi) = \{1\}$.

(i) We give two proofs for proving this claim. The first is more elementary than the second, but we need some parts of it in the proof of (ii). The second is based on the automorphism group of the Johnson graph $J(n, k)$.

Proof 1 of (i). Consider the graph G. In $T = V(G)$, consider the subgraphs induced by the subsets $C_0 = S = \{e_i | \ 1 \leq i \leq n\}$, $C_i = \{e_i, e_i + e_j | \ 1 \leq j \leq n, i \neq j\}$, $1 \leq i \leq n$ (we also denote by C_i the subgraph induced by the set C_i). It is clear that C_0 is an n-clique in the graph G. Note that if $e_i + e_r$ and $e_i + e_s$ are two elements of C_i, then we have

$(e_i + e_r) - (e_i + e_s) = e_r + e_s \in T$. Hence each C_i is also an n-clique in the graph G. It can be shown that each C_i, $0 \leq i \leq n$ is a maximal n-clique in G. It is clear that if $i \neq 0$, then $C_0 \cap C_i = \{e_i\}$. Moreover, if $i, j \in \{1, ..., n\}$ and $i \neq j$, then $C_i \cap C_j = \{e_i + e_j\}$. Let M be a maximal n-clique in the graph G. It is not hard to show that $M = C_j$ for some $j \in \{0, 1, ..., n\}$. If a is an automorphism of the graph G, then $a(C_j)$ is a maximal n-clique in the graph G. Hence the natural action of a on the set $X = \{C_0, C_1, ..., C_n\}$ is a permutation on X. Let G_1 be the graph with the vertex set X in which two vertices v and w are adjacent if and only if $v \cap w \neq \emptyset$. Now, it is clear that $G_1 \cong K_{n+1}$, the complete graph on $n + 1$ vertices, and hence $\mathrm{Aut}(G_1) \cong \mathrm{Sym}(X)$. Let $a \in \mathrm{Aut}(G)$ be such that $a(C_j) = C_j$, for each $j \in \{0, 1, ..., n\}$. Noting that $C_0 \cap C_i = \{e_i\}$, $i \neq 0$, we deduce that $a(x) = x$ for every $x \in C_0$. Note that the vertex $e_i + e_j$ is the unique common neighbor of vertices e_i and e_j in the graph G which is not in C_0. This implies that $a(e_i + e_j) = e_i + e_j$. Therefore we have $a(v) = v$ for every $v \in T$. Now it is easy to see that the mapping $\pi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G_1)$ defined by the rule $\pi(a) = f_a$, where $f_a(C_i) = a(C_i)$ for every $C_i \in X$, is an injection and therefore we have $(n + 1)! \geq |\mathrm{Aut}(G)|$. □

Proof 2 *of* (i). Consider the graph G. We show that this graph is isomorphic with the Johnson graph $J(n + 1, 2)$. We define the mapping

$$f\colon V(G) \to V(J(n+1, 2)),$$

by the rule:

$$f(v) = \begin{cases} \{i, n+1\}, if\ v = e_i \\ \{i, j\}, \quad if\ \ v = e_i + e_j \end{cases}$$

It is clear that f is a bijection. Let $\{v, w\}$ be an edge in the graph G. Then we have three possibilities:
(1) $\{v, w\} = \{e_i, e_j\}$, (2) $\{v, w\} = \{e_i, e_i + e_k\}$, (3) $\{v, w\} = \{e_i + e_k, e_i + e_j\}$.
Now, we have (1) $f(\{v, w\}) = \{\{i, n + 1\}, \{j, n + 1\}\}$, (2) $f(\{v, w\}) = \{\{i, n + 1\}, \{i, k\}\}$, (3) $f(\{v, w\}) = \{\{i, k\}, \{j, k\}\}$. It follows that f is a graph isomorphism. Hence, $\mathrm{Aut}(G) \cong \mathrm{Aut}(J(n+1, 2))$. Since $\mathrm{Aut}(J(n+1, 2)) \cong \mathrm{Sym}([n+1])$ [3, 13, 18], then we have $\mathrm{Aut}(G) \cong \mathrm{Sym}([n + 1])$. □

(ii) we now show that $\ker(\Phi) = \{1\}$. Let $f \in \ker(\Phi)$. Then $f(0) = 0$ and $h = f|_T$ is the identity automorphism of the graph G. Hence $f(x) = x$ for every $x \in T$. Note that when $x \in T$, then $w(x) \in \{1, 2\}$, where $w(x)$ is the weight of x, that is, the number of 1s in the n-tuple x. Let $x \in V(\Gamma)$ and $w(x) = m$. We show by induction on m, that $f(x) = x$. It is clear that when $m = 0, 1, 2$, then the claim is true. Let the claim be true when $w(x) \leq m$, $m \geq 2$. We show that if $w(x) = m + 1$, then $f(x) = x$. Let $y = e_{i_1} + ... + e_{i_m} + e_{i_{m+1}}$ be a vertex of weight $m + 1$. Let $v = y + e_{i_m} + e_{i_{m+1}}$. Since $W(v) = m - 1$, thus $f(v) = v$. Let N be the subgraph of Γ which is induced by the set $N(v)$. Since Γ is vertex-transitive, then $G \cong N$. Also, since $f(v) = v$, then the restriction of f to $N(v)$ is an automorphism of the graph N. In $N(v)$ we define the subsets $M_0 = \{v + e_i | \ 1 \leq i \leq n\}$, $M_i = \{v + e_i, v + e_i + e_j | \ 1 \leq j \leq n, j \neq i\}$, $1 \leq i \leq n$. It can be check that the subgraph induced by each M_i is a maximal n-clique in the graph N. Also, $M_0 \cap M_i = \{v + e_i\}$. Moreover, $v + e_i + e_j$ is the unique common neighbor of the vertices $v + e_i$ and $v + e_j$ in the graph N which is not in M_0. If $x \in M_0$, then $f(x) = x$, because $w(x) \leq m$. This implies that $f(M_i) = M_i$. Now, by an argument similar to what is done in Proof 1, we can see that $f(x) = x$ for every $x \in N(v)$. Since $y \in N(v)$, we have $f(y) = y$. We now conclude that f is the identity automorphism of Γ. Hence we have $\ker(\Phi) = \{1\}$. □

Theorem 3.2. *Let $n > 4$ and $\Gamma = Q_n^2$ be the square of the hypercube Q_n. Then we have* $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_2^n \rtimes \mathrm{Sym}([n+1])$.

Proof. Let A_0 be the stabilizer subgroup of the vertex $v = 0$ in the group $\mathrm{Aut}(\Gamma)$. We know from Lemma 3.1, that $|A_0| \leq (n+1)!$. Let T and $X = \{C_0, ..., C_n\}$ be the sets which are defined in the proof of Lemma 3.1. Note that $\mathbb{Z}_2^n$ is a vector space over the field $\mathbb{Z}_2$ and C_i, $0 \leq i \leq n$, is a basis for this vector space. Let $f_i \colon C_0 \to C_i$ be a bijection. We can linearly extend f_i to an automorphism $e(f_i)$ of the group $\mathbb{Z}_2^n$. It is clear that $e(f_i) \in A_0$. We know that every automorphism of the group $\mathbb{Z}_2^n$ which fixes the set T is an automorphism of the graph Γ. We can see that when $x, y \in C_i$ and $x \neq y$ then $x + y \in T$. Thus we have $e(f_i)(e_r + e_s) = e(f_i)(e_r) + e(f_i)(e_s) \in T$. Hence we have $e(f_i)(T) = T$. Since the number of permutations f_i is $n!$, hence the number of automorphisms of $e(f_i)$ is $n!$. Note that when $i \neq j$, then $e(f_i) \neq e(f_j)$. Now since $0 \leq i \leq n$, then we have at least $(n+1)(n!) = (n+1)!$ distinct automorphisms in the group A_0. Thus by Lemma 3.1, we have $|A_0| = (n+1)!$. We saw, in the proof of Lemma 3.1, that A_0 is isomorphic with a subgroup of $\mathrm{Sym}([n+1])$. Hence we deduce that $A_0 \cong \mathrm{Sym}([n+1])$.

We know, by the orbit-stabilizer theorem, that $|V(\Gamma)||A_0| = |\mathrm{Aut}(\Gamma)|$. Thus we have $|\mathrm{Aut}(\Gamma)| = 2^n[(n+1)!]$. For every $v \in \mathbb{Z}_2^n$, the mapping $f_v(x) = v + x$, for every $x \in \mathbb{Z}_2^n$, is an automorphism of the graph Γ. It is easy to check that $L = \{f_v |\, v \in \mathbb{Z}_2^n\}$ is a subgroup of $\mathrm{Aut}(\Gamma)$ which is isomorphic with $\mathbb{Z}_2^n$. Also it is easy to check that $L \cap A_0 = \{1\}$. Hence we have $|LA_0| = |L||A_0| = 2^n[(n+1)!] = |\mathrm{Aut}(\Gamma)|$. This implies that $\mathrm{Aut}(\Gamma) = LA_0$. Also we can see that for every $v \in \mathbb{Z}_2^n$ and every $a \in A_0$ we have $a^{-1} f_v a = f_{a^{-1}(v)}$. Thus we deduce that L is a normal subgroup of $\mathrm{Aut}(\Gamma)$. We now conclude that

$$\mathrm{Aut}(\Gamma) \cong L \rtimes A_0 \cong \mathbb{Z}_2^n \rtimes \mathrm{Sym}([n+1]).$$ □

The graph Q_n^2 has some interesting properties. In the next theorem, we show that Q_n^2 is distance-transitive.

Theorem 3.3. *Let $n \geq 4$ be an integer. Then the graph Q_n^2 is a distance-transitive graph.*

Proof. Let v and w be vertices in Q_n^2. It is easy to check that $d(x, y) = \lceil \frac{w(x+y)}{2} \rceil$. Hence we have $d(x, 0) = \lceil \frac{w(x)}{2} \rceil$. Let D be the diameter of Q_n^2. it follows from the first two sentences that $D = \lceil \frac{n}{2} \rceil$. Let A_0 be the stabilizer subgroup the vertex $v = 0$ in $\mathrm{Aut}(Q_n^2)$. Since the graph Q_n^2 is a vertex-transitive graph, it is sufficient to show that the action of A_0 on the set Γ_k is transitive, where Γ_k is the set of vertices at distance k from the vertex $v = 0$. Let x and y be two vertices in Γ_k. There are two possible cases, that is,

(i) $w(x) = w(y)$ or

(ii) $w(x) \neq w(y)$.

(i) Let $w(x) = w(y)$. We know that $w(x) \in \{2k, 2k-1\}$. Without loss of generality, we can assume that $w(x) = 2k$. Let $x = e_{i_1} + ... + e_{i_{2k}}$ and $y = e_{j_1} + ... + e_{j_{2k}}$. There are vertices $e_{x_1}, ..., e_{x_{n-2k}}$ and $e_{y_1}, ..., e_{y_{n-2k}}$ in Q_n^2 such that

$$\{e_{i_1}, ..., e_{i_{2k}}, e_{x_1}, ..., e_{x_{n-2k}}\} = C_0 = \{e_1, e_2, ..., e_n\} = \{e_{j_1}, ..., e_{j_{2k}}, e_{y_1}, ..., e_{y_{n-2k}}\}.$$

Let f be the permutation on the set C_0 which is defined by the rule, $f(e_{i_r}) = e_{j_r}, 1 \leq r \leq 2k$, and $f(e_{x_l}) = e_{y_l}, 1 \leq l \leq n-2k$. We now can see that $e(f)(x) = y$, where $e(f)$ is the linear extension of f to $\mathbb{Z}_2^n$ (see the proof of Theorem 3.2).
(ii) Let $w(x) \neq w(y)$. Without loss of generality we can assume that $w(x) = 2k-1$ and $w(y) = 2k$. Let $x = e_{i_1} + ... + e_{i_{2k-1}}$ and $y = e_{j_1} + ... + e_{j_{2k}}$. Note that $y = (e_{j_1} + e_{j_{2k}}) + (e_{j_2} + e_{j_{2k}}) + ... + (e_{j_{2k-2}} + e_{j_{2k}}) + (e_{j_{2k-1}} + e_{j_{2k}})$. There are vertices $e_{x_1}, ..., e_{x_{n-2k+1}}$ and $e_{y_1} = e_{j_{2k}}, e_{y_2}, ..., e_{y_{n-2k+1}}$ in Q_n^2 such that

$$\{e_{i_1}, ..., e_{i_{2k-1}}, e_{x_1}, ..., e_{x_{n-2k+1}}\} = C_0,$$
$$\{e_{j_1} + e_{j_{2k}}, e_{j_2} + e_{j_{2k}}, ... + e_{j_{2k-2}} + e_{j_{2k}}, e_{j_{2k-1}} + e_{j_{2k}}\} \cup \{e_{y_1}, e_{y_2} + e_{j_{2k}}, ..., e_{y_{n-2k+1}} + e_{j_{2k}}\} = C_{j_{2k}}$$

We now define the bijection g from C_0 to $C_{j_{2k}}$ by the rule $g(e_{i_r}) = e_{j_r} + e_{j_{2k}}$, and $g(e_{x_1}) = e_{y_1}, g(e_{x_i}) = e_{y_i} + e_{j_{2k}}, i \neq 1$. Let $e(g)$ be the linear extension of g to $\mathbb{Z}_2^n$. This yields that $e(g)$ is an automorphism of the graph Q_n^2 such that $e(g)(x) = y$. □

Theorem 3.3 implies many results. For instance, we now can deduce from it the following corollary, which is important in applied graph theory and interconnection networks.

Corollary 3.4. *Let $n \geq 4$ be an integer. Then the connectivity of the graph Q_n^2 is maximal, namely, $n+\binom{n}{2}$ (its valency).*

Proof. By Theorem 3.3 the graph Q_n^2 is distance-transitive, then it is edge-transitive. Thus, it follows from Theorem 2.1, that the connectivity of the graph Q_n^2 is its valency, namely, $n+\binom{n}{2}$. □

A block B, in the action of a group G on a set X, is a subset of X such that $B \cap g(B) \in \{B, \emptyset\}$, for each g in G. If G is transitive on X, then we say that the permutation group (X, G) is primitive if the only blocks are the trivial blocks, that is, those with cardinality $0, 1$ or $|X|$. In the case of an imprimitive permutation group (X, G), the set X is partitioned into a disjoint union of non-trivial blocks, which are permuted by G. We refer to this partition as a block system. A graph Γ is said to be primitive or imprimitive according to the group $\mathrm{Aut}(\Gamma)$ acting on $V(\Gamma)$ has the corresponding property. In the sequel, we need the following definition.

Definition 3.5. A graph $\Gamma = (V, E)$ of diameter D is said to be *antipodal* if for any $u, v, w \in V$ such that $d(u, v) = d(u, w) = D$, then we have $d(v, w) = D$ or $v = w$.

Let $\Gamma_i(x)$ denote the set of vertices of Γ at distance i from the vertex x. Let Γ be a distance-transitive graph. From Definition 3.5 it follows that if $\Gamma_D(x)$ is a singleton set, then the graph Γ is antipodal. It is easy to see that the hypercube Q_n is antipodal, since every vertex u has a unique vertex at maximum distance from it. Note that this graph is at the same time bipartite. We have the following fact [1].

Proposition 3.6. *A distance-transitive graph Γ of diameter D has a block $X = \{u\} \cup \Gamma_D(u)$ if and only if Γ is antipodal, where $\Gamma_D(u)$ is the set of vertices of Γ at distance D from the vertex u.*

Also, we have the following important fact [1].

Theorem 3.7. *An imprimitive distance-transitive graph is either bipartite or antipodal. (Both possibilities can occur in the same graph.)*

We now can state and prove the following fact concerning the square of the hypercube Q_n.

Corollary 3.8. *Let $n \geq 4$ be an integer. Then, the square of the hypercube Q_n, namely, the graph Q_n^2, is an imprimitive distance-transitive graph if and only if n is an odd integer.*

Proof. We know from Theorem 3.3, that the graph $\Gamma = Q_n^2$ is a distance-transitive graph. Let $n = 2k$ be an even integer. If D denotes the diameter of Q_n^2, then $D = k$. Let $C_0 = \{e_1, ..., e_n\}$ be the standard basis of the hypercube Q_n. Let $w = e_1 + e_2 + ... + e_n$ and $B_1 = \{w + e_i \mid 1 \leq i \leq n\}$. Consider the vertex $u = 0$. It is easy to show that $\Gamma_D(u) = \{w\} \cup B_1$. Two vertices w and $w + e_1$ are in $\Gamma_D(u)$, but they are not at distance $k = D$ from each other, since they are adjacent and $k > 1$. Thus, when n is an even integer, then the graph Q_n^2 is not antipodal. Since the girth of Q_n^2 is 3, then this graph is not bipartite. Now, Theorem 3.7 implies that the graph $\Gamma = Q_n^2$ is not imprimitive.

Now assume that $n = 2k + 1$ is an odd integer. It is easy to see that $D = k + 1$ and $\Gamma_D(0) = \{w\}$. Therefore by Proposition 3.6, Γ is antipodal, and hence has the set $\{0, w\}$ as a block. We now conclude that, when n is an odd integer, then Q_n^2 is an imprimitive graph. □

Let $\Gamma = (V, E)$ be a simple connected graph with diameter D. A *distance-regular* graph $\Gamma = (V, E)$, with diameter D, is a regular connected graph of valency k with the following property. There are positive integers

$$b_0 = k, b_1, ..., b_{D-1}; c_1 = 1, c_2, ..., c_D,$$

such that for each pair (u, v) of vertices satisfying $u \in \Gamma_i(v)$, we have

(1) the number of vertices in $\Gamma_{i-1}(v)$ adjacent to u is c_i, $1 \leq i \leq D$.

(2) the number of vertices in $\Gamma_{i+1}(v)$ adjacent to u is b_i, $0 \leq i \leq D - 1$.

The intersection array of Γ is $i(\Gamma) = \{k, b_1, ..., b_{D-1}; 1, c_2, ..., c_d\}$.

It is easy to show that if Γ is a distance-transitive graph, then it is distance-regular [1]. Hence, the hypercube $Q_n,\ n > 2$ is a distance-regular graph. We can verify by an easy argument that the intersection array of Q_n is

$$\{n, n-1, n-2, ..., 1; 1, 2, 3, ..., n\}.$$

In other words, for hypercube Q_n, we have $b_i = n - i$, $c_i = i$, $1 \leq i \leq n - 1$, and $b_0 = n$, $c_n = n$. In the following theorem, we determine the intersection array of the square of the hypercube Q_n [1].

Theorem 3.9. *Let $n > 3$ be an integer and $\Gamma = Q_n^2$ be the square of the hypercube Q_n. Let D denote the diameter of Q_n^2. Then for the intersection array of this graph we have $b_0 = \binom{n+1}{2}$, $b_i = \binom{n-2i+1}{2}$, $c_i = \binom{2i}{2}$, $1 \leq i \leq D - 1$. Also, $c_D = \binom{n+1}{2}$, when n is an odd integer and $c_D = \binom{n}{2}$ when n is an even integer.*

Proof. Since Q_n^2 is a regular graph of valency $\binom{n+1}{2}$, thus we have b_0=$\binom{n+1}{2}$. Let u be a vertex in Q_n^2 at distance i from the vertex $v = 0$. It is easy to check that $w(u) = 2i$ or $w(u) = 2i - 1$. This implies that that the diameter of the graph Q_n^2 is $D = \lceil \frac{n}{2} \rceil$.

Let u be a vertex in Q_n^2 at distance $i \geq 1$ from the vertex $v = 0$, such that $i \neq D$. There are two cases, that is, $w(u) = 2i$, or $w(u) = 2i - 1$. Without lose of generality we can assume that $w(u) = 2i$. Hence u is of the form $u = e_{j_1} + e_{j_2} + ... + e_{j_{2i}}$. Now it is easy to show that if x is a vertex of Q_n^2 adjacent to u and at distance $i - 1$ from the vertex $v = 0$, then x must be of the form $x = u + e_k + e_l$, where $e_k, e_l \in \{e_{j_1}, e_{j_2}, ..., e_{j_{2i}}\}$. It is clear that the number of such xs is equal to $\binom{2i}{2}$. Moreover, If x is a vertex of Q_n^2 adjacent to u and at distance $i + 1$ from the vertex $v = 0$, then x must be of the forms $x = u + e_k$ or $x = u + e_k + e_l$, where $e_k, e_l \in \{e_1, e_2, ..., e_n\} - \{e_{j_1}, e_{j_2}, ..., e_{j_{2i}}\}$. It is clear that the number of such xs is equal to $\binom{n-2i}{1}$+$\binom{n-2i}{2}$=$\binom{n-2i+1}{2}$. We now deduce that when $1 \leq i \leq D - 1$, then c_i=$\binom{2i}{2}$, and b_i=$\binom{n-2i+1}{2}$.

When n is an odd integer, then the vertex $u = e_1 + e_2 + ... + e_n$ is the unique vertex of Q_n^2 at distance D from the vertex $v = 0$. Thus c_D=$\binom{n+1}{2}$, namely, the valency of u. If n is an even integer, then $\Gamma_D(0) = \{u, u + e_i | \ 1 \leq i \leq n\}$ is the set of vertices of $\Gamma = Q_n^2$ at distance D from the vertex $v = 0$. Now, by a similar argument which is done in the first section of the proof, it can be shown that c_D=$\binom{n}{2}$. □

Remark 3.10. There are distance-regular graphs $\Gamma = (V, E)$, with the property that their squares are not distance-regular. For instance, consider the cycle C_n with vertex set $\{0, 1, 2, ..., n - 1\}$. It is well known that C_n is a distance-regular graph of diameter $[\frac{n}{2}]$ with the intersection array:

$$\{2, 1, 1, ..., 1, 1; 1, 1, 1, ..., 1, 2\} \text{ when } n \text{ is an even integer and,}$$
$$\{2, 1, 1, ..., 1, 1; 1, 1, 1, ..., 1, 1\} \text{ when } n \text{ is an odd integer [1].}$$

Now, assume that $n \geq 7$. It can be shown by an easy argument that $\Gamma = C_n^2$ is not a distance-regular graph. To see this fact, let v be a vertex in C_n at distance i from the vertex 0, and $c_i(v) = |\Gamma_{i-1}(0) \cap N(v)|$. It is easy to show that $\Gamma_i(0) = \{2i, -2i, 2i-1, -2i+1\}$, and $c_i(2i) = 1$, but $c_i(2i - 1) = 2$.

Remark 3.11. Let $n, k \in \mathbb{N}$ with $k < n$, and let $[n] = \{1, ..., n\}$. Consider the Johnson graph $J(n, k)$. It is clear that the order of this graph is $\binom{n}{k}$. It is easy to check that $J(n, k) \cong J(n, n - k)$, hence we assume that $1 \leq k \leq \frac{n}{2}$. The class of Johnson graphs is one of the most well known and interesting subclass of distance-regular graphs [3]. It is easy to show that if v and w are vertices in the Johnson graph $J(n, k)$, then $d(v, w) = k - |v \cap w|$. Thus, the diameter of the Johnson graph $J(n, k)$ is k. Note that the graph $J(n, 1)$ is the complete graph K_n and hence it is distance-regular. The diameter of the graph $J(n, 2)$ is 2, hence the diameter of its square is 1. Thus the graph $J^2(n, 2)$ is the complete graph K_m, and hence it is a distance-regular graph (m=$\binom{n}{2}$). We can show that when $k = 3$, then the square of Johnson graph $\Gamma = J(n, k)$ is a distance-regular graph if and only if $n = 6$. For checking this, let $v = \{1, 2, 3\}$. Note that the diameter of the graph Γ^2 is 2 and a vertex w in Γ^2 is at distance 2 from v if and only if $|v \cap w| = 0$. Moreover w is at distance 1 from v if and only if $|v \cap w| \in \{1, 2\}$. Hence $\Gamma_1^2(v) = V(\Gamma) - \{v, v^c\}$ and $\Gamma_2^2(v) = \{v^c\}$, where v^c is the complement of the set v in the set $\{1, 2, ..., 6\}$. Thus $v^c = \{4, 5, 6\}$. Now, it is clear that $b_0(v)$=$\binom{3}{2}\binom{3}{1}$+$\binom{3}{1}\binom{3}{2}$=18. Also, for every $w \in \Gamma_1^2(v)$, $c_1(w) = 1$ and $b_1(w) = 1$, and $c_2(v^c) = |\Gamma_1^2(v)| = 18$. Thus the graph $\Gamma^2 = J^2(6, 3)$ is a distance-regular graph with intersection array $\{18, 1; 1, 18\}$. But, if $n > 6$, then the graph $\Gamma^2 = J^2(n, 3)$ is not distance-regular. In fact if $n > 6$, then for the vertex $v = \{1, 2, 3\}$, each of the vertices $u = \{1, 2, 4\}$ and $w = \{1, 4, 5\}$ is in $\Gamma_1^2(v)$. If $x \in \Gamma_2^2(v)$ is adjacent to u, then $4 \in x$, and hence $x = \{4\} \cup y$, where $y \subset v^c - \{4\}$ and $|y| = 2$. We now can deduce that $b_1(u)=\binom{n-4}{2}$. On the other hand, if $x \in \Gamma_2^2(v)$ is adjacent to w, then $4 \in x$ and $5 \notin x$, or $5 \in x$ and $4 \notin x$ or $4, 5 \in x$. Thus, $b_1(w)=2\binom{n-5}{2}+\binom{n-5}{1}=\binom{n-4}{2}+\binom{n-5}{2}$. This implies that when $n \geq 7$ then the graph $J^2(n, 3)$ cannot be distance-regular.

By a similar argument we we can show that the graph $J^2(8, 4)$ is distance-regular, but if $n > 8$, then the graph $J^2(n, 4)$ is not distance-regular.

Remark 3.12. Let $\Gamma = (V, E)$ be a graph. Γ is said to be a *strongly regular* graph with parameters (n, k, λ, μ), whenever $|V| = n$, Γ is a regular graph of valency k, every pair of adjacent vertices of Γ have λ common neighbor(s), and every pair of non-adjacent vertices of Γ have μ common neighbor(s). It is clear that the diameter of every strongly regular graph is 2. It is easy to show that if a graph Γ is a distance-regular graph of diameter 2 and order n, with intersection array $(b_0, b_1; c_1, c_2)$, then Γ is a strongly regular graph with parameters $(n, b_0, b_0 - b_1 - 1, c_2)$. We know that the diameter of the graph Q_n^2 is $\lceil \frac{n}{2} \rceil$. Now, it follows from Theorem 3.3, that Q_3^2 is a strongly regular graph with parameter $(8, 6, 4, 6)$. This graph is known as the *coktail-party* graph $CP(4)$ [1]. Also, the graph Q_4^2 is a strongly regular graph with parameter $(16, 10, 6, 6)$. We know that when a graph Γ is a strongly regular graph with parameters (n, k, λ, μ), then its complement is again a strongly regular graph with parameter $(n, n - k - 1, n - 2 - 2k + \mu, n - 2k + \lambda)$ [9]. Hence, the complement of the graph Q_4^2 is a strongly regular graph with parameter $(16, 5, 0, 2)$. This graph is known as the *Clebsch* graph [9] and it is the unique strongly regular graph with parameters $(16, 5, 0, 2)$. Figure 1 displays a version of the Clebsch graph (the complement of the graph Q_4^2) in the plane [9].

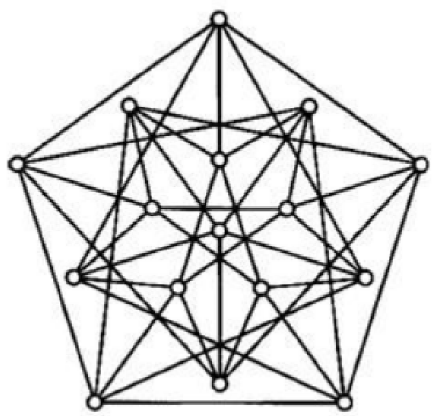

Figure 1: The Clebsch graph.

4 The spectrum of the square of the hypercube

The square of the hypercube Q_n has some further interesting algebraic properties. For obtaining some of those properties, we need the spectrum of this graph. The spectrum of Q_n is known [1], however we are not aware of a paper showing the spectrum of Q_n^2. Here we compute by means of an algebraic and self-contained method the spectrum of Q_n^2.

Let $\Gamma = (V, E)$ be a graph with the vertex set $\{v_1, \cdots, v_n\}$. Then the adjacency matrix of Γ is an $n \times n$ matrix $A = (a_{ij})$, in which columns and rows are labeled by V and a_{ij} is defined as follow:

$$a_{ij} = A(v_i, v_j) = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

If $Ax = \lambda x, x \neq 0$, then λ is an eigenvalue of A, and x is an eigenvector of A corresponding to λ [9]. Let $\lambda_1, \cdots, \lambda_r$ be eigenvalues of A with multiplicities $m_1, \cdots, m_r$, respectively. The spectrum of the graph Γ is defined as

$$Spec(\Gamma) = \begin{Bmatrix} \lambda_1, & \lambda_2, & \cdots, & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{Bmatrix}.$$

When we work with graphs there is an additional refinement. We can suppose that an eigenvector is a real function f on the vertices. Then if at any vertex v you sum up the values of f on its neighboring vertices, you should get λ times the values of f at v. Formally,

$$\sum_{w \in N(v)} f(w) = \lambda f(v).$$

Let G be a finite abelian group (written additively) of order $|G|$ with identity element $0{=}0_G$. A character χ of G is a homomorphism from G into the multiplicative group U of complex numbers of absolute value 1, that is, a mapping from G into U with $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. If G is a finite abelian group, then there are integers $n_1, \cdots, n_k$, such that $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. Let $S = \{s_1, \cdots, s_n\}$ be a non-empty subset of G such that $0 \notin S$ and $S = -S$. Let $\Gamma = \mathrm{Cay}(G; S)$. Assume $f \colon G \longrightarrow \mathbb{C}^*$ is a character where $\mathbb{C}^*$ is the multiplicative group of the complex numbers. If $\omega_{ij} = e^{\frac{2\pi i j}{n_i}}$, $0 \leq i \leq k$, $1 \leq j \leq n_i$, is an n_ith root of unitary, then f is of the form $f = f_{(\omega_1, \cdots, \omega_k)}$, where $f_{(\omega_1, \cdots, \omega_k)}(x_1, \cdots, x_k) = \omega_1^{x_1} \omega_2^{x_2} \cdots \omega_k^{x_k}$, for each $(x_1, x_2, ..., x_k) \in G$ [12].

If v is a vertex of Γ, then we know that $N(v) = \{v + s_1, \cdots, v + s_n\}$ is the set of vertices that are adjacent to v. We now have

$$\sum_{w \in N(v)} f(w) = \sum_{i=1}^{n} f(v + s_i) = \sum_{i=1}^{n} f(v) f(s_i) = f(v)(\sum_{i=1}^{n} f(s_i)).$$

Therefore, if we let $\lambda = \lambda_f = \sum_{s \in S} f(s)$ then we have $\sum_{w \in N(v)} f(w) = \lambda_f f(v)$, and hence the mapping f is an eigenvector for the Cayley graph Γ with corresponding eigenvalue $\lambda = \lambda_f = \sum_{s \in S} f(s)$.

Theorem 4.1. *Let $n > 3$ be an integer and Q_n^2 be the square of the hypercube Q_n. Then each of the eigenvalues of Q_n^2 is of the form,*

$$\lambda_i = \frac{1}{2}n(n+1) - 2i(n+1) + 2i^2,$$

for $0 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$. Moreover, the multiplicity of λ_0 is 1, the multiplicity of λ_i is $m(\lambda_i)=\binom{n}{i}+\binom{n}{n+1-i}$, for $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$, when n is an even integer, and $m(\lambda_i)=\binom{n}{i}+\binom{n}{n+1-i}$ for $1 \leq i < \lfloor \frac{n+1}{2} \rfloor$, when n is an odd integer, with $m(\lambda_j)=\binom{n}{j}$ for $j = \lfloor \frac{n+1}{2} \rfloor$.

Proof. According to what is stated before this theorem, every eigenvector of the graph $\Gamma = Q_n^2 = \mathrm{Cay}(\mathbb{Z}_2^n; S)$ is of the form $f = f_{(\omega_1,\cdots,\omega_n)}$, where each ω_i, $1 \leq i \leq n$, is a complex number such that $\omega_i^2 = 1$, namely, $\omega_i \in \{1, -1\}$. We now have

$$\begin{aligned} \lambda_f &= \sum_{w \in S} f(w) = \sum_{i=1}^{n} f(e_i) + \sum_{i,j=1,\ i\neq j}^{n} f(e_i + e_j) \\ &= \sum_{i=1}^{n} f(e_i) + \sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j). \end{aligned}$$

Note that for every vertex $v = (x_1, \dots, x_n)$, $x_i \in \{0, 1\}$ in Q_n^2, we have

$$f(x_1, \dots, x_n) = f_{(w_1,\dots,w_n)}(x_1 \dots, x_n) = w_1^{x_1} \dots w_n^{x_n}.$$

Note that in the computing of the value of $w_1^{x_1} \dots w_n^{x_n}$ we can ignore w_i when $w_i = 1$. Thus, for $e_k = (0, \dots, 0, 1, 0 \dots, 0)$, where 1 is the kth entry, we have;

$$\begin{aligned} f(e_k) &= f_{(w_1,\dots,w_n)}(0, \dots, 0, 1, 0, \dots, 0) \\ &= w_1^0 \dots w_k^1 w_{k+1}^0 \dots w_n^0 = \begin{cases} -1 & if \;\; w_k = -1 \\ 1 & if \;\; w_k = 1 \end{cases} \end{aligned}$$

Hence, if in the n-tuple $(w_1, \dots, w_n)$ the number of $-1s$ is i (and therefore the number of 1s is $(n - i)$), then in the sum

$$\sum_{k=1}^{n} f(e_k) = \sum_{k=1}^{n} f_{(w_1,\dots,w_n)}(0, \dots, x_k, 0, \dots, 0), \;\; x_k = 1,$$

the contribution of -1 is i and the contribution of 1 is $n - i$. Therefore, we have

$$\sum_{k=1}^{n} f(e_k) = -i + (n - i) = n - 2i.$$

On the other hand, since

$$(\sum_{k=1}^{n} f(e_k))^2 = \sum_{k=1}^{n} f(e_k)^2 + 2 \sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j),$$

therefore, we have

$$\sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j) = \frac{1}{2}((n-2i)^2 - \sum_{k=1}^{n} f(e_k)^2).$$

Now since $\sum_{k=1}^{n} f(e_k)^2 = n$, thus we have

$$\lambda_f = \sum_{i=1}^{n} f(e_i) + \sum_{i,j=1,\ i\neq j}^{n} f(e_i)f(e_j) = (n-2i) + \frac{1}{2}((n-2i)^2 - n)$$
$$= \frac{1}{2}n + \frac{1}{2}n^2 - 2ni + 2i^2 - 2i = \frac{1}{2}n(n+1) - 2i(n+1) + 2i^2.$$

Note that $f = f_{(w_1,w_2,\ldots,w_n)}$, and the number of sequences $(w_1\ldots,w_n)$ in which i entries are -1 is $\binom{n}{i}$. If we denote λ_f by λ_i, then we deduce that every eigenvalue of the graph Q_n^2 is of the form

$$\lambda_i = \frac{1}{2}n(n+1) - 2i(n+1) + 2i^2,\ 0 \leq i \leq n. \quad (**)$$

Consider the real function $f(x) = \frac{1}{2}n(n+1) - 2x(n+1) + 2x^2$. Then $\lambda_i = f(i)$, $i \in \{0,1,...,n\}$. This function reaches its minimum at $x = \frac{n+1}{2}$. Now by using some calculus, we can see that $f(x) = f(n+1-x)$. Thus, we have $\lambda_i = f(i) = f(n+1-i) = \lambda_{n+1-i}$, $1 \leq i \leq n$. Now it follows that if $n = 2k$, then the multiplicity of λ_i is $\binom{n}{i}+\binom{n}{n+1-i}$, $1 \leq i \leq k$. Note that when $n = 2k+1$, then $n+1-(k+1) = k+1$, thus $\lambda_{n+1-(k+1)} = \lambda_{k+1}$. Hence if $n = 2k+1$, then the multiplicity of λ_i is $\binom{n}{i}+\binom{n}{n+1-i}$, $1 \leq i \leq k$, and the multiplicity of λ_{k+1} is $\binom{n}{k+1}$. Note that since the graph Q_n^2 is a $\binom{n+1}{2}$-regular graph, hence the multiplicity of $\lambda_0=\binom{n+1}{2}=\frac{1}{2}(n+1)n$ is 1. □

Let $\Gamma = (V,E)$ be a graph. The line graph $L(\Gamma)$ of the graph Γ is constructed by taking the edges of Γ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in Γ have a common vertex. Note that if $e = \{v,w\}$ is an edge of Γ, then its degree in the graph $L(\Gamma)$ is $\deg(v) + \deg(w) - 2$. Concerning the eigenvalues of the line graphs, we have the following fact [1, 9].

Proposition 4.2. *If λ is an eigenvalue of a line graph $L(\Gamma)$, then $\lambda \geq -2$.*

Therefore, if $\lambda < -2$ is an eigenvalue of a graph graph Γ, then Γ is not a line graph.

A (c,d)-*biregular* graph is a bipartite graph in which each vertex in one part has degree c and each vertex in the other part has degree d [25]. It is known and easy to prove that if the line graph of the graph Γ is regular, then Γ is a regular or a (c,d)-biregular bipartite graph.

Theorem 4.3. *Let $n \geq 4$ be an integer and Q_n^2 be the square of the hypercube Q_n. Then Q_n^2 cannot be a line graph.*

Proof. Let $k = \lfloor \frac{n}{2} \rfloor$. Hence, if n is an even integer, then $n = 2k$ and if n is an odd integer then $n = 2k+1$. It follows from Theorem 4.1, that the smallest eigenvalue of the graph Q_n^2 is λ_k, when n is an even integer and λ_{k+1}, when n is an odd integer. Now consider the eigenvalue λ_k of the graph Q_n^2 in (**) (in the proof of Theorem 4.1). Therefore if n is an even integer, then we have

$$\lambda_k = k(2k+1) - 2k(2k+1) + 2k^2 = k(2k+1-4k-2+2k) = -k.$$

Moreover if $n = 2k + 1$, then we have,

$$\begin{aligned}\lambda_{k+1} &= (2k+1)(k+1) - 2(k+1)(2k+2) + 2(k+1)^2 \\ &= (k+1)(2k+1-4k-4+2k+2) = -k-1.\end{aligned}$$

We now deduce that when $n \geq 5$, then $\lambda_k \leq -3$. Now, it follows from Proposition 4.2, when $n \geq 5$, then the graph Q_n^2 can not be a line graph.

Our argument shows that if λ is an eigenvalue of the graph Q_4^2, then $\lambda \geq -2$, and hence in this way we can not say anything about our claim.
We now show that Q_4^2 is not a line graph. On the contrary, assume that Q_4^2 is a line graph. Thus, there is a graph Δ such that $Q_4^2 = L(\Delta)$. Since Q_4^2 is a regular graph, hence

(i) Δ is a regular graph, or

(ii) Δ is a biregular bipartite graph.

(i) Let $\Delta = (V, E)$ be a t-regular graph of order h. Since Q_4^2 is 10-regular, thus, $L(\Delta) = Q_4^2$ is a $2t-2 = 10$-regular graph, and hence $t = 6$. Therefore we have $16 = |E| = \frac{1}{2}6h = 3h$, which is impossible.
(ii) Let $\Delta = (A \cup B, E)$ be a (c, d)-biregular bipartite graph such that every vertex in A (B) is of degree c (d). Hence we have $16 = |E| = c|A| = d|B|$. Thus c and d must divide 16. On the other hand, if $e = \{a, b\}$ is an edge of Δ, then we must have $\deg(a) + \deg(b) - 2 = 10 = c + d - 2$. Hence we have $c + d = 12$. We now can check that $\{c, d\} = \{4, 8\}$. Without loss of generality, we can assume that $d = 8$ and $c = 4$. Hence we must have $|A| \geq 8$. Now since each vertex in A is of degree $c = 4$, then we must have, $16 = |E| = c|A| = 4|A| \geq 4 \times 8 = 32$, which is impossible.

Our argument shows that the graph Q_4^2 is also not a line graph. □

An *automorphic* graph is a distance-transitive graph whose automorphism group acts primitively on its vertices, and not a complete graph or a line graph.
Automorphic graphs are apparently very rare. For instance, there are exactly three cubic automorphic graphs [1, 2]. It is clear that for $n \geq 3$, the graph Q_n^2 is not a complete graph.
We now derive from Corollary 3.8, and Theorem 4.3, the following important result.

Corollary 4.4. *Let $n \geq 4$ be an integer. Then the square of the hypercube Q_n, that is, the graph Q_n^2, is an automorphic graph if and only if n is an even integer.*

5 Conclusion

In this paper, we proved that the square of the distance-transitive graph Q_n, that is, the graph Q_n^2, is again a distance-transitive graph (Theorem 3.3). We showed that there are important classes of distance-transitive graphs (including the cycle C_n, $n \geq 7$), such that their squares are not even distance-regular (and hence are not distance-transitive) (Remark 3.11). Also, we determined the spectrum of the graph Q_n^2 (Theorem 4.1). Moreover, we showed that when $n > 3$ is an even integer, then the graph Q_n^2 is an automorphic graph, that is, a distance-transitive primitive graph which is not a complete or a line graph (Corollary 4.4).

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