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On fat Hoffman graphs with smallest eigenvalue at least -3

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Abstract

We investigate fat Hoffman graphs with smallest eigenvalue at least -3, using their special graphs. We show that the special graph $S(\mathfrak{H})$ of an indecomposable fat Hoffman graph $\mathfrak{H}$ is represented by the standard lattice or an irreducible root lattice. Moreover, we show that if the special graph admits an integral representation, that is, the lattice spanned by it is not an exceptional root lattice, then the special graph $S^{-}(\mathfrak{H})$ is isomorphic to one of the Dynkin graphs A_n, D_n, or extended Dynkin graphs $\tilde{A}_n$ or $\tilde{D}_n$.

Keywords: Hoffman graph, line graph, graph eigenvalue, special graph, root system.

Math. Subj. Class.: 05C50, 05C76

1 Introduction

Throughout this paper, a graph will mean an undirected graph without loops or multiple edges.

Hoffman graphs were introduced by Woo and Neumaier [5] to extend the results of Hoffman [3]. Hoffman proved what we would call Hoffman's limit theorem (Theorem 2.14) which asserts that, in the language of Hoffman graphs, the smallest eigenvalue of a fat Hoffman graph is a limit of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree. Woo and Neumaier [5] gave a complete list of fat indecomposable Hoffman graphs with smallest eigenvalue at least $-1-\sqrt{2}$. From their list, we

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find that only $-1, -2$ and $-1-\sqrt{2}$ appear as the smallest eigenvalues. This implies, in particular, that $-1, -2$ and $-1-\sqrt{2}$ are limit points of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree. It turns out that there are no others between -1 and $-1-\sqrt{2}$. More precisely, for a negative real number λ, consider the sequences

$$\eta_k^{(\lambda)} = \inf\{\lambda_{\min}(\Gamma) \mid \min\deg\Gamma \geq k,\ \lambda_{\min}(\Gamma) > \lambda\} \quad (k = 1, 2, \dots),$$
$$\theta_k^{(\lambda)} = \sup\{\lambda_{\min}(\Gamma) \mid \min\deg\Gamma \geq k,\ \lambda_{\min}(\Gamma) < \lambda\} \quad (k = 1, 2, \dots),$$

where Γ runs through graphs satisfying the conditions specified above, namely, Γ has minimum degree at least k and Γ has smallest eigenvalue greater than (or less than) λ. Then Hoffman [3] has shown that

$$\lim_{k\to\infty} \eta_k^{(-2)} = -1, \qquad \lim_{k\to\infty} \theta_k^{(-1)} = -2,$$
$$\lim_{k\to\infty} \eta_k^{(-1-\sqrt{2})} = -2, \qquad \lim_{k\to\infty} \theta_k^{(-2)} = -1-\sqrt{2}.$$

In other words, real numbers in $(-2, -1)$ and $(-1-\sqrt{2}, -2)$ are ignorable if our concern is the smallest eigenvalues of graphs with large minimum degree. Woo and Neumaier [5] went on to prove that

$$\lim_{k\to\infty} \eta_k^{(\alpha)} = -1-\sqrt{2},$$

where $\alpha \approx -2.4812$ is a zero of the cubic polynomial $x^3 + 2x^2 - 2x - 2$. Recently, Yu [6] has shown that analogous results for regular graphs also hold.

Woo and Neumaier [5, Open Problem 4] raised the problem of classifying fat Hoffman graphs with smallest eigenvalue at least -3. They also proposed a generalization of a concept of a line graph based on a family of isomorphism classes of Hoffman graphs. This enables one to define generalized line graphs in a very simple manner. As we shall see in Proposition 3.2, the knowledge of μ-saturated indecomposable fat Hoffman graphs gives a description of all fat Hoffman graphs with smallest eigenvalue at least μ. For $\mu = -3$, this in turn should give some information on the limit points of the smallest eigenvalues of a sequence of ordinary graphs with increasing minimum degree. Also, using the generalized concept of line graphs, we can expect to give a description of all graphs with smallest eigenvalue at least -3 and sufficiently large minimum degree. Thus our ultimate goal is to classify (-3)-saturated indecomposable fat Hoffman graphs, as proposed by Woo and Neumaier [5].

The purpose of this paper is to begin the first step of this classification, by determining their special graphs for such Hoffman graphs having an integral reduced representation. One of our main result is that the special graph $\mathcal{S}^-(\mathfrak{H})$ of such a Hoffman graph $\mathfrak{H}$ is isomorphic to one of the Dynkin graphs A_n, D_n, or extended Dynkin graphs $\tilde{\mathsf{A}}_n$ or $\tilde{\mathsf{D}}_n$. We also show that, even when the Hoffman graph $\mathfrak{H}$ does not admit an integral representation, its special graph $\mathcal{S}(\mathfrak{H})$ can be represented by one of the exceptional root lattices E_n ($n = 6, 7, 8$). This might mean that the rest of the work can be completed by a computer as in the classification of maximal exceptional graphs (see [1]). Indeed, if one attaches a fat neighbor to every slim vertex of an ordinary maximal exceptional graph, then we obtain a (-3)-indecomposable fat Hoffman graph. However, maximal graphs among (-3)-indecomposable fat Hoffman graphs represented in E_8 are usually much larger (see

Example 3.8 and the comment following it), so the problem is not as trivial as it looks. We plan to discuss in the subsequent papers the determination of these special graphs and the corresponding Hoffman graphs.

2 Hoffman graphs and their smallest eigenvalues

2.1 Basic theory of Hoffman graphs

In this subsection we discuss the basic theory of Hoffman graphs. Hoffman graphs were introduced by Woo and Neumaier [5], and most of this section is due to them. Since the sums of Hoffman graphs appear only implicitly in [5] and later formulated by Taniguchi [4], we will give proof of the results that use sums for the convenience of the reader.

Definition 2.1. A *Hoffman graph* $\mathfrak{H}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labeling map $\mu : V \to \{f, s\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) vertices with label f are pairwise non-adjacent.

We call a vertex with label s a *slim vertex*, and a vertex with label f a *fat vertex*. We denote by $V_s = V_s(\mathfrak{H})$ (resp. $V_f(\mathfrak{H})$) the set of slim (resp. fat) vertices of $\mathfrak{H}$. The subgraph of a Hoffman graph $\mathfrak{H}$ induced on $V_s(\mathfrak{H})$ is called the *slim subgraph* of $\mathfrak{H}$. If every slim vertex of a Hoffman graph $\mathfrak{H}$ has a fat neighbor, then we call $\mathfrak{H}$ *fat*.

For a vertex x of $\mathfrak{H}$ we define $N^f(x) = N^f_{\mathfrak{H}}(x)$ (resp. $N^s(x) = N^s_{\mathfrak{H}}(x)$) the set of fat (resp. slim) neighbors of x in $\mathfrak{H}$. The set of all neighbors of x is denoted by $N(x) = N_{\mathfrak{H}}(x)$, that is $N(x) = N^f(x) \cup N^s(x)$. In a similar fashion, for vertices x and y we define $N^f(x, y) = N^f_{\mathfrak{H}}(x, y)$ to be the set of common fat neighbors of x and y.

Definition 2.2. A Hoffman graph $\mathfrak{H}_1 = (H_1, \mu_1)$ is called an (induced) *Hoffman subgraph* of another Hoffman graph $\mathfrak{H} = (H, \mu)$, if H_1 is an (induced) subgraph of H and $\mu(x) = \mu_1(x)$ for all vertices x of $\mathfrak{H}_1$. Unless stated otherwise, by a subgraph we mean an induced Hoffman subgraph. For a subset S of $V_s(\mathfrak{H})$, we denote by $\langle\langle S \rangle\rangle_{\mathfrak{H}}$ the subgraph of $\mathfrak{H}$ induced on the set of vertices

$$S \cup \left(\bigcup_{x \in S} N^f_{\mathfrak{H}}(x) \right).$$

Definition 2.3. Two Hoffman graphs $\mathfrak{H} = (H, \mu)$ and $\mathfrak{H}' = (H', \mu')$ are called *isomorphic* if there exists an isomorphism from H to H' which preserves the labeling.

An ordinary graph H without labeling can be regarded as a Hoffman graph $\mathfrak{H} = (H, \mu)$ without any fat vertex, i.e., $\mu(x) = s$ for all vertices x. Such a graph is called a *slim graph*.

Definition 2.4. For a Hoffman graph $\mathfrak{H}$, let A be its adjacency matrix,

$$A = \begin{pmatrix} A_s & C \\ C^T & O \end{pmatrix}$$

in a labeling in which the fat vertices come last. *Eigenvalues* of $\mathfrak{H}$ are the eigenvalues of the real symmetric matrix $B(\mathfrak{H}) = A_s - CC^T$. Let $\lambda_{\min}(\mathfrak{H})$ denote the smallest eigenvalue of $\mathfrak{H}$.

Definition 2.5 ([5])**.** For a Hoffman graph $\mathfrak{H}$ and a positive integer n, a mapping $\phi : V(\mathfrak{H}) \to \mathbf{R}^n$ such that

$$(\phi(x), \phi(y)) = \begin{cases} m & \text{if } x = y \in V_s(\mathfrak{H}), \\ 1 & \text{if } x = y \in V_f(\mathfrak{H}), \\ 1 & \text{if } x \text{ and } y \text{ are adjacent in } \mathfrak{H}, \\ 0 & \text{otherwise,} \end{cases}$$

is called a *representation of norm* m. We denote by $\Lambda(\mathfrak{H}, m)$ the lattice generated by $\{\phi(x) \mid x \in V(\mathfrak{H})\}$. Note that the isomorphism class of $\Lambda(\mathfrak{H}, m)$ depends only on m, and is independent of ϕ, justifying the notation.

Definition 2.6. For a Hoffman graph $\mathfrak{H}$ and a positive integer n, a mapping $\psi : V_s(\mathfrak{H}) \to \mathbf{R}^n$ such that

$$(\psi(x), \psi(y)) = \begin{cases} m - |N^f_{\mathfrak{H}}(x)| & \text{if } x = y, \\ 1 - |N^f_{\mathfrak{H}}(x, y)| & \text{if } x \text{ and } y \text{ are adjacent,} \\ -|N^f_{\mathfrak{H}}(x, y)| & \text{otherwise.} \end{cases}$$

is called a *reduced representation of norm* m. We denote by $\Lambda^{\mathrm{red}}(\mathfrak{H}, m)$ the lattice generated by $\{\psi(x) \mid x \in V_s(\mathfrak{H})\}$. Note that the isomorphism class of $\Lambda^{\mathrm{red}}(\mathfrak{H}, m)$ depends only on m, and is independent of ψ, justifying the notation.

While it is clear that a representation of norm $m > 1$ is an injective mapping, a reduced representation of norm m may not be. See Section 4 for more details.

Lemma 2.7. *Let $\mathfrak{H}$ be a Hoffman graph having a representation of norm m. Then $\mathfrak{H}$ has a reduced representation of norm m, and $\Lambda(\mathfrak{H}, m)$ is isomorphic to $\Lambda^{\mathrm{red}}(\mathfrak{H}, m) \oplus \mathbb{Z}^{|V_f(\mathfrak{H})|}$ as a lattice.*

Proof. Let $\phi : V(\mathfrak{H}) \to \mathbf{R}^n$ be a representation of norm m. Let U be the subspace of $\mathbf{R}^n$ generated by $\{\phi(x) \mid x \in V_f(\mathfrak{H})\}$. Let ρ, $\rho^\perp$ denote the orthogonal projections from $\mathbf{R}^n$ onto U, $U^\perp$, respectively. Then $\rho^\perp \circ \phi$ is a reduced representation of norm m, $\rho^\perp(\Lambda(\mathfrak{H}, m)) \cong \Lambda^{\mathrm{red}}(\mathfrak{H}, m)$, and $\rho(\Lambda(\mathfrak{H}, m)) \cong \mathbb{Z}^{|V_f(\mathfrak{H})|}$. □

Theorem 2.8. *For a Hoffman graph $\mathfrak{H}$, the following conditions are equivalent:*

(i) *$\mathfrak{H}$ has a representation of norm m,*

(ii) *$\mathfrak{H}$ has a reduced representation of norm m,*

(iii) $\lambda_{\min}(\mathfrak{H}) \geq -m$.

Proof. From Lemma 2.7, we see that (i) implies (ii).

Let ψ be a reduced representation of $\mathfrak{H}$ of norm m. Then the matrix $B(\mathfrak{H}) + mI$ is the Gram matrix of the image of ψ, and hence positive semidefinite. This implies that $B(\mathfrak{H})$ has smallest eigenvalue at least $-m$ and hence $\lambda_{\min}(\mathfrak{H}) \geq -m$. This proves (ii) $\Longrightarrow$ (iii).

The proof of equivalence of (i) and (iii) can be found in [5, Theorem 3.2]. □

From Theorem 2.8, we obtain the following lemma:

Lemma 2.9. *If $\mathfrak{G}$ is a subgraph of a Hoffman graph $\mathfrak{H}$, then $\lambda_{\min}(\mathfrak{G}) \geq \lambda_{\min}(\mathfrak{H})$ holds.*

Proof. Let $m := -\lambda_{\min}(\mathfrak{H})$. Then $\mathfrak{H}$ has a representation ϕ of norm m by Theorem 2.8. Restricting ϕ to the vertices of $\mathfrak{G}$ we obtain a representation of norm m of $\mathfrak{G}$, which implies $\lambda_{\min}(\mathfrak{G}) \geq -m$ by Theorem 2.8. $\square$

In particular, if Γ is the slim subgraph of $\mathfrak{H}$, then $\lambda_{\min}(\Gamma) \geq \lambda_{\min}(\mathfrak{H})$.

Under a certain condition, equality holds in Lemma 2.9. To state this condition we need to introduce decompositions of Hoffman graphs. This was formulated first by the third author [4], although it was already implicit in [5].

Definition 2.10. Let $\mathfrak{H}$ be a Hoffman graph, and let $\mathfrak{H}^1$ and $\mathfrak{H}^2$ be two non-empty induced Hoffman subgraphs of $\mathfrak{H}$. The Hoffman graph $\mathfrak{H}$ is said to be the *sum* of $\mathfrak{H}^1$ and $\mathfrak{H}^2$, written as $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$, if the following conditions are satisfied:

(i) $V(\mathfrak{H}) = V(\mathfrak{H}^1) \cup V(\mathfrak{H}^2)$;

(ii) $\{V_s(\mathfrak{H}^1), V_s(\mathfrak{H}^2)\}$ is a partition of $V_s(\mathfrak{H})$;

(iii) if $x \in V_s(\mathfrak{H}^i)$, $y \in V_f(\mathfrak{H})$ and $x \sim y$, then $y \in V_f(\mathfrak{H}^i)$;

(iv) if $x \in V_s(\mathfrak{H}^1)$, $y \in V_s(\mathfrak{H}^2)$, then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.

If $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ for some non-empty subgraphs $\mathfrak{H}^1$ and $\mathfrak{H}^2$, then we call $\mathfrak{H}$ *decomposable*. Otherwise $\mathfrak{H}$ is called *indecomposable*. Clearly, a disconnected Hoffman graph is decomposable.

It follows easily that the above-defined sum is associative and so that the sum

$$\mathfrak{H} = \biguplus_{i=1}^{n} \mathfrak{H}^i$$

is well-defined. The main reason for defining the sum of Hoffman graphs is the following lemma.

Lemma 2.11. *Let $\mathfrak{H}$ be a Hoffman graph and let $\mathfrak{H}^1$ and $\mathfrak{H}^2$ be two (non-empty) induced Hoffman subgraphs of $\mathfrak{H}$ satisfying* (i)–(iii) *of Definition 2.10. Let ψ be a reduced representation of $\mathfrak{H}$ of norm m. Then the following are equivalent.*

(i) $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$,

(ii) $(\psi(x), \psi(y)) = 0$ *for any* $x \in V_s(\mathfrak{H}^1)$ *and* $y \in V_s(\mathfrak{H}^2)$.

Proof. This follows easily from the definitions of $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ and a reduced representation of norm m. $\square$

Lemma 2.12. *If $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$, then*

$$\lambda_{\min}(\mathfrak{H}) = \min\{\lambda_{\min}(\mathfrak{H}^1), \lambda_{\min}(\mathfrak{H}^2)\}.$$

Proof. Let $m = -\min\{\lambda_{\min}(\mathfrak{H}^1), \lambda_{\min}(\mathfrak{H}^2)\}$. In view of Lemma 2.9 we only need to show that $\lambda_{\min}(\mathfrak{H}) \geq -m$. By Theorem 2.8, $\mathfrak{H}^i$ has a reduced representation $\psi^i : V(\mathfrak{H}^i) \to \mathbf{R}^{n_i}$ of norm m, for each $i = 1, 2$. Define $\psi : V(\mathfrak{H}) \to \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}$ by $\psi(x) = \psi^1(x) \oplus 0$ if $x \in V(\mathfrak{H}^1)$, $\psi(x) = 0 \oplus \psi^2(x)$ otherwise. It is easy to check that ψ is a reduced representation of norm m, and the result then follows from Theorem 2.8. $\square$

2.2 Hoffman's limit theorem

In this subsection, we state and prove Hoffman's limit theorem (Theorem 2.14) using the concept of Hoffman graphs.

Lemma 2.13. *Let $\mathfrak{G}$ be a Hoffman graph whose vertex set is partitioned as $V_1 \cup V_2 \cup V_3$ such that*

(i) $V_2 \cup V_3 \subset V_s(\mathfrak{G})$,

(ii) *there are no edges between V_1 and V_3,*

(iii) *every pair of vertices $x \in V_2$ and $y \in V_3$ are adjacent,*

(iv) *V_3 is a clique.*

Let $\mathfrak{H}$ be the Hoffman graph with the set of vertices $V_1 \cup V_2$ together with a fat vertex $f \notin V(\mathfrak{G})$ adjacent to all the vertices of V_2. If $\mathfrak{G}$ has a representation of norm m, then $\mathfrak{H}$ has a representation of norm

$$m + \frac{(m-1)|V_2|}{|V_3| + m - 1}.$$

Proof. Let $\phi : V(\mathfrak{G}) \to \mathbf{R}^d$ be a representation of norm m, and let

$$P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

be the $|V(\mathfrak{G})| \times d$ matrix whose rows are the images of $V(\mathfrak{G}) = V_1 \cup V_2 \cup V_3$ under ϕ. Set

$$\begin{aligned} \boldsymbol{u} &= \frac{1}{\sqrt{|V_3|(|V_3| + m - 1)}} \sum_{x \in V_3} \phi(x), \\ \epsilon_1 &= 1 - \sqrt{\frac{|V_3|}{|V_3| + m - 1}}, \\ \epsilon_2 &= \sqrt{\frac{m-1}{|V_3| + m - 1}}. \end{aligned}$$

Let $\boldsymbol{j}$ denote the row vector of length $|V_2|$ all of whose entries are 1. Then

$$\boldsymbol{u}\boldsymbol{u}^T = 1, \tag{2.1}$$

$$P_1 \boldsymbol{u}^T = 0, \tag{2.2}$$

$$P_2 \boldsymbol{u}^T = (1 - \epsilon_1)\boldsymbol{j}^T, \tag{2.3}$$

$$\epsilon_2^2 = 2\epsilon_1 - \epsilon_1^2. \tag{2.4}$$

Fix an orientation of the complete digraph on V_2, and let B be the $|V_2| \times \binom{|V_2|}{2}$ matrix defined by

$$B_{\alpha,(\beta,\gamma)} = \delta_{\alpha\beta} - \delta_{\alpha\gamma} \quad (\alpha, \beta, \gamma \in V_2,\ \beta \neq \gamma).$$

Then

$$BB^T = |V_2| I - J. \tag{2.5}$$

We now construct the desired representation of $\mathfrak{H}$, as the row vectors of the matrix

$$D = \begin{pmatrix} P_1 & \epsilon_2\sqrt{|V_2|}I & 0 \\ P_2 + \epsilon_1 \boldsymbol{j}^T\boldsymbol{u} & 0 & \epsilon_2 B \\ \boldsymbol{u} & 0 & 0 \end{pmatrix}.$$

Then, using (2.1)–(2.5), we find

$$\begin{aligned} DD^T &= \begin{pmatrix} P_1 & \epsilon_2\sqrt{|V_2|}I & 0 \\ P_2 + \epsilon_1 \boldsymbol{j}^T\boldsymbol{u} & 0 & \epsilon_2 B \\ \boldsymbol{u} & 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^T & P_2^T + \epsilon_1 \boldsymbol{u}^T\boldsymbol{j} & \boldsymbol{u}^T \\ \epsilon_2\sqrt{|V_2|}I & 0 & 0 \\ 0 & \epsilon_2 B^T & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_1P_1^T + \epsilon_2^2|V_2|I & P_1P_2^T & 0 \\ P_2P_1^T & P_2P_2^T + (2\epsilon_1 - \epsilon_1^2)J + \epsilon_2^2(|V_2|I - J) & \boldsymbol{j}^T \\ 0 & \boldsymbol{j} & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1P_1^T + \epsilon_2^2|V_2|I & P_1P_2^T & 0 \\ P_2P_1^T & P_2P_2^T + \epsilon_2^2|V_2|I & \boldsymbol{j}^T \\ 0 & \boldsymbol{j} & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1P_1^T & P_1P_2^T & 0 \\ P_2P_1^T & P_2P_2^T & \boldsymbol{j}^T \\ 0 & \boldsymbol{j} & 1 \end{pmatrix} + \epsilon_2^2|V_2| \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore, the row vectors of D define a representation of norm $m + \epsilon_2^2|V_2|$ of the Hoffman graph $\mathfrak{H}$. ☐

Theorem 2.14 (Hoffman)**.** *Let $\mathfrak{H}$ be a Hoffman graph, and let $f_1, \ldots, f_k \in V_f(\mathfrak{H})$. Let $\mathfrak{G}^{n_1,\ldots,n_k}$ be the Hoffman graph obtained from $\mathfrak{H}$ by replacing each f_i by a slim n_i-clique K^i, and joining all the neighbors of f_i with all the vertices of K^i by edges. Then*

$$\lambda_{\min}(\mathfrak{G}^{n_1,\ldots,n_k}) \geq \lambda_{\min}(\mathfrak{H}), \tag{2.6}$$

and

$$\lim_{n_1,\ldots,n_k\to\infty} \lambda_{\min}(\mathfrak{G}^{n_1,\ldots,n_k}) = \lambda_{\min}(\mathfrak{H}). \tag{2.7}$$

Proof. We prove the assertions by induction on k. First suppose $k = 1$. Let $\mu_n = -\lambda_{\min}(\mathfrak{G}^n)$. Let $\mathfrak{H}^n$ denote the Hoffman graph obtained from $\mathfrak{H}$ by attaching a slim n-clique K to the fat vertex f_1, joining all the neighbors of f_1 and all the vertices of K by edges. Then $\mathfrak{H}^n$ contains both $\mathfrak{H}$ and $\mathfrak{G}^n$ as subgraphs, and $\mathfrak{H}^n = \mathfrak{H} \uplus \mathfrak{H}'$, where $\mathfrak{H}'$ is the subgraph induced on $K \cup \{f_1\}$. Since $\lambda_{\min}(\mathfrak{H}') = -1$, Lemma 2.12 implies

$$\lambda_{\min}(\mathfrak{H}) = \lambda_{\min}(\mathfrak{H}^n) \leq \lambda_{\min}(\mathfrak{G}^n) = -\mu_n.$$

Thus (2.6) holds for $k = 1$. Since n is arbitrary and $\{-\mu_n\}_{n=1}^{\infty}$ is decreasing, we see that $\lim_{n\to\infty} \mu_n$ exists and

$$\lambda_{\min}(\mathfrak{H}) \leq -\lim_{n\to\infty} \mu_n. \tag{2.8}$$

Since $\mathfrak{G}^n$ has a representation of norm μ_n, it follows from Lemma 2.13 that $\mathfrak{H}$ has a representation of norm

$$\mu_n + \frac{(\mu_n - 1)|N_{\mathfrak{H}}(f)|}{n + \mu_n - 1}.$$

By Theorem 2.8, we have

$$\lambda_{\min}(\mathfrak{H}) \geq -\mu_n - \frac{(\mu_n - 1)|N_{\mathfrak{H}}(f)|}{n + \mu_n - 1},$$

which implies

$$\lambda_{\min}(\mathfrak{H}) \geq -\lim_{n\to\infty} \mu_n. \tag{2.9}$$

Combining (2.9) with (2.8), we conclude that (2.7) holds for $k = 1$.

Next, suppose $k \geq 2$. Let $\mathfrak{G}^{n_1,\dots,n_{k-1}}$ be the Hoffman graph obtained from $\mathfrak{H}$ by replacing each f_i ($1 \leq i \leq k-1$) by a slim n_i-clique K^i, and joining all the neighbors of f_i with all the vertices of K^i by edges. Then $\mathfrak{G}^{n_1,\dots,n_k}$ is obtained from $\mathfrak{G}^{n_1,\dots,n_{k-1}}$ by replacing f_k by a slim n_k-clique K^k, and joining all the neighbors of f_k with all the vertices of K^k by edges. Then it follows from the assertions for $k = 1$ that

$$\lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) \geq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}), \tag{2.10}$$

and

$$\lim_{n_k\to\infty} \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) = \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}).$$

This means that, for any $\epsilon > 0$, there exists N_1 such that

$$n_k \geq N_1 \implies 0 \leq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) - \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) < \epsilon.$$

By induction, we have

$$\lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) \geq \lambda_{\min}(\mathfrak{H}), \tag{2.11}$$

and

$$\lim_{n_1,\dots,n_{k-1}\to\infty} \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) = \lambda_{\min}(\mathfrak{H}). \tag{2.12}$$

Combining (2.10) with (2.11), we obtain (2.6), while (2.11) and (2.12) imply that there exists N_0 such that

$$n_1,\dots,n_{k-1} \geq N_0 \implies 0 \leq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_{k-1}}) - \lambda_{\min}(\mathfrak{H}) < \epsilon.$$

Setting $N = \max\{N_0, N_1\}$, we see that

$$n_1,\dots,n_k \geq N \implies 0 \leq \lambda_{\min}(\mathfrak{G}^{n_1,\dots,n_k}) - \lambda_{\min}(\mathfrak{H}) < 2\epsilon.$$

This establishes (2.7). □

Corollary 2.15. *Let $\mathfrak{H}$ be a Hoffman graph. Let Γ^n be the slim graph obtained from $\mathfrak{H}$ by replacing every fat vertex f of $\mathfrak{H}$ by a slim n-clique $K(f)$, and joining all the neighbors of f with all the vertices of $K(f)$ by edges. Then*

$$\lambda_{\min}(\Gamma^n) \geq \lambda_{\min}(\mathfrak{H}),$$

and

$$\lim_{n\to\infty} \lambda_{\min}(\Gamma^n) = \lambda_{\min}(\mathfrak{H}).$$

In particular, for any $\epsilon > 0$*, there exists a natural number* n *such that, every slim graph* Δ *containing* Γ^n *as an induced subgraph satisfies*

$$\lambda_{\min}(\Delta) \le \lambda_{\min}(\mathfrak{H}) + \epsilon.$$

Proof. Immediate from Theorem 2.14. □

3 Special graphs of Hoffman graphs

Definition 3.1. Let μ be a real number with $\mu \le -1$ and let $\mathfrak{H}$ be a Hoffman graph with smallest eigenvalue at least μ. Then $\mathfrak{H}$ is called *μ-saturated* if no fat vertex can be attached to $\mathfrak{H}$ in such a way that the resulting graph has smallest eigenvalue at least μ.

Proposition 3.2. *Let μ be a real number, and let $\mathscr{H}$ be a family of indecomposable fat Hoffman graphs with smallest eigenvalue at least μ. The following statements are equivalent:*

(i) *every fat Hoffman graph with smallest eigenvalue at least μ is a subgraph of a graph $\mathfrak{H} = \biguplus_{i=1}^{n} \mathfrak{H}^i$ such that $\mathfrak{H}^i$ is a member of $\mathscr{H}$ for all $i = 1, \dots, n$.*

(ii) *every μ-saturated indecomposable fat Hoffman graph is isomorphic to a subgraph of a member of $\mathscr{H}$.*

Proof. First suppose (i) holds, and let $\mathfrak{H}$ be a μ-saturated indecomposable fat Hoffman graph. Then $\mathfrak{H}$ is a fat Hoffman graph with smallest eigenvalue at least μ, hence $\mathfrak{H}$ is a subgraph of $\mathfrak{H}' = \biguplus_{i=1}^{n} \mathfrak{H}^i$, where $\mathfrak{H}^i$ is a member of $\mathscr{H}$ for $i = 1, \dots, n$. Since $\mathfrak{H}$ is μ-saturated, it coincides with the subgraph $\langle\langle V_s(\mathfrak{H})\rangle\rangle_{\mathfrak{H}'}$ of $\mathfrak{H}'$. Since $\mathfrak{H}$ is indecomposable, this implies that $\mathfrak{H}$ is a subgraph of $\mathfrak{H}^i$ for some i.

Next suppose (ii) holds, and let $\mathfrak{H}$ be a fat Hoffman graph with smallest eigenvalue at least μ. Without loss of generality we may assume that $\mathfrak{H}$ is indecomposable and μ-saturated. Then $\mathfrak{H}$ is isomorphic to a subgraph of a member of $\mathscr{H}$, hence (i) holds. □

Definition 3.3. For a Hoffman graph $\mathfrak{H}$, we define the following three graphs $\mathcal{S}^-(\mathfrak{H})$, $\mathcal{S}^+(\mathfrak{H})$ and $\mathcal{S}(\mathfrak{H})$ as follows: For $\epsilon \in \{-, +\}$ define the *special ϵ-graph* $\mathcal{S}^\epsilon = (V_s(\mathfrak{H}), E^\epsilon)$ as follows: the set of edges E^ϵ consists of pairs $\{s_1, s_2\}$ of distinct slim vertices such that $\operatorname{sgn}(\psi(s_1), \psi(s_2)) = \epsilon$, where ψ is a reduced representation of $\mathfrak{H}$ of norm m. The graph $\mathcal{S}(\mathfrak{H}) := \mathcal{S}^+(\mathfrak{H}) \cup \mathcal{S}^-(\mathfrak{H}) = (V_s(\mathfrak{H}), E^- \cup E^+)$ is the *special graph* of $\mathfrak{H}$.

Note that the definition of the special graph $\mathcal{S}(\mathfrak{H})$ is independent of the choice of the norm m of a reduced representation ψ.

It is easy to determine whether a Hoffman graph $\mathfrak{H}$ is decomposable or not.

Lemma 3.4. *Let $\mathfrak{H}$ be a Hoffman graph. Let $V_s(\mathfrak{H}) = V_1 \cup V_2$ be a partition, and set $\mathfrak{H}^i = \langle\langle V_i\rangle\rangle_{\mathfrak{H}}$ for $i = 1, 2$. Then $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ if and only if there are no edges connecting V_1 and V_2 in $\mathcal{S}(\mathfrak{H})$. In particular, $\mathfrak{H}$ is indecomposable if and only if $\mathcal{S}(\mathfrak{H})$ is connected.*

Proof. This is immediate from Definition 2.10(iv) and Definition 3.3. □

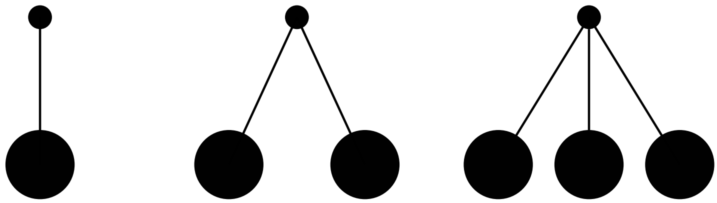

Figure 1.

For an integer $t \geq 1$, let $\mathfrak{H}^{(t)}$ be the fat Hoffman graph with one slim vertex and t fat vertices.

Lemma 3.5. *Let t be a positive integer. If $\mathfrak{H}$ is a fat Hoffman graph with $\lambda_{\min}(\mathfrak{H}) \geq -t$ containing $\mathfrak{H}^{(t)}$ as a Hoffman subgraph, then $\mathfrak{H} = \mathfrak{H}^{(t)} \uplus \mathfrak{H}'$ for some subgraph $\mathfrak{H}'$ of $\mathfrak{H}$. In particular, if $\mathfrak{H}$ is indecomposable, then $\mathfrak{H} = \mathfrak{H}^{(t)}$.*

Proof. Let x be the unique slim vertex of $\mathfrak{H}^{(t)}$. Let ψ be a reduced representation of norm t of $\mathfrak{H}$. Then $\psi(x) = 0$, hence x is an isolated vertex in $\mathcal{S}(\mathfrak{H})$. Thus $\mathfrak{H} = \mathfrak{H}^{(t)} \uplus \langle\langle V_s(\mathfrak{H}) \setminus \{x\}\rangle\rangle_{\mathfrak{H}}$ by Lemma 3.4. □

Lemma 3.6. *Let $\mathfrak{H}$ be a fat Hoffman graph with smallest eigenvalue at least -3. Let ψ be a reduced representation of norm 3 of $\mathfrak{H}$. Then for any distinct slim vertices x, y of $\mathfrak{H}$, $(\psi(x), \psi(y)) \in \{1, 0, -1\}$.*

Proof. Since $\mathfrak{H}$ is fat, we have $(\psi(x), \psi(x)) \leq 2$ for all $x \in V_s(\mathfrak{H})$. Thus $|(\psi(x), \psi(y))| \leq 2$ for all $x, y \in V_s(\mathfrak{H})$ by Schwarz's inequality. Equality holds only if $\psi(x) = \pm\psi(y)$ and $(\psi(x), \psi(x)) = 2$. The latter condition implies $|N^f_{\mathfrak{H}}(x)| = 1$, hence $|N^f_{\mathfrak{H}}(x, y)| \leq 1$. Thus $(\psi(x), \psi(y)) \geq -1$, while $(\psi(x), \psi(y)) = 2$ cannot occur unless $x = y$, by Definition 2.6. Therefore, $|(\psi(x), \psi(y))| < 2$, and the result follows. □

Let $\mathfrak{H}$ be a fat Hoffman graph with smallest eigenvalue at least -3. Then by Lemma 3.6, the edge set of the special graph $\mathcal{S}^{\epsilon}(\mathfrak{H})$ is

$$\{\{x, y\} \mid x, y \in V_s(\mathfrak{H}),\ (\psi(x), \psi(y)) = \epsilon 1\},$$

for $\epsilon \in \{+, -\}$.

Theorem 3.7. *Let $\mathfrak{H}$ be a fat indecomposable Hoffman graph with smallest eigenvalue at least -3. Then every slim vertex has at most three fat neighbors. Moreover, the following statements hold:*

(i) *If some slim vertex has three fat neighbors, then $\mathfrak{H} \cong \mathfrak{H}^{(3)}$.*

(ii) *If no slim vertex has three fat neighbors and some slim vertex has exactly two fat neighbors, then $\Lambda^{\text{red}}(\mathfrak{H}, 3) \simeq \mathbb{Z}^n$ for some positive integer n.*

(iii) *If every slim vertex has a unique fat neighbor, then $\Lambda^{\text{red}}(\mathfrak{H}, 3)$ is an irreducible root lattice.*

Proof. As the smallest eigenvalue is at least -3, every slim vertex has at most three fat neighbors.

If $|N^f_{\mathfrak{H}}(x)| = 3$ for some slim vertex x of $\mathfrak{H}$, then $\mathfrak{H}$ contains $\mathfrak{H}^{(3)}$ as a subgraph. Thus $\mathfrak{H} = \mathfrak{H}^{(3)}$ by Lemma 3.5, and (i) holds. Hence we may assume that $|N^f_{\mathfrak{H}}(x)| \le 2$ for all $x \in V_s(\mathfrak{H})$. Then for each $x \in V_s(\mathfrak{H})$ we have $\|\psi(x)\|^2 = 1$ (resp. 2) if and only if $|N^f_{\mathfrak{H}}(x)| = 2$ (resp. $|N^f_{\mathfrak{H}}(x)| = 1$). Suppose that $\Lambda^{\text{red}}(\mathfrak{H}, 3)$ contains m linearly independent vectors of norm 1. We claim that $\Lambda^{\text{red}}(\mathfrak{H}, 3)$ can be written as an orthogonal direct sum $\mathbb{Z}^m \oplus \Lambda'$, where Λ' is a lattice containing no vectors of norm 1. Indeed, if x is a slim vertex such that $\|\psi(x)\|^2 = 2$ and $\psi(x) \notin \mathbb{Z}^m$, then $\psi(x)$ is orthogonal to $\mathbb{Z}^m$. This implies $\Lambda^{\text{red}}(\mathfrak{H}, 3) = \mathbb{Z}^m \oplus \Lambda'$ and $\psi(V_s(\mathfrak{H})) \subset \mathbb{Z}^m \cup \Lambda'$.

If $m > 0$ and $\Lambda' \neq 0$, then the special graph $\mathcal{S}(\mathfrak{H})$ is disconnected. This contradicts the indecomposability of $\mathfrak{H}$ by Lemma 3.4. Therefore, we have either $m = 0$ or $\Lambda' = 0$. In the latter case, (ii) holds. In the former case, $\Lambda^{\text{red}}(\mathfrak{H}, 3) = \Lambda'$ is generated by vectors of norm 2, hence it is a root lattice. Again by the assumption and Lemma 3.4, (iii) holds. □

We shall see some examples for the case (ii) of Theorem 3.7 in the next section. As for the case (iii), $\Lambda^{\text{red}}(\mathfrak{H}, 3)$ is an irreducible root lattice of type A_n, D_n or E_n. If $\Lambda^{\text{red}}(\mathfrak{H}, 3)$ is not an irreducible root lattice of type E_n, then it can be imbedded into the standard lattice, hence the results of the next section applies. On the other hand, if $\Lambda^{\text{red}}(\mathfrak{H}, 3)$ is an irreducible root lattice of type E_n, then it is contained in the irreducible root lattice of type E_8, and hence there are only finitely many possibilities. For example, Let Γ be any ordinary graph with smallest eigenvalue at least -2 (see [1] for a description of such graphs). Attaching a fat neighbor to each vertex of Γ gives a fat Hoffman graph with smallest eigenvalue at least -3. However, this Hoffman graph may not be maximal among fat Hoffman graphs with smallest eigenvalue at least -3. Therefore, we aim to classify fat Hoffman graphs with smallest eigenvalue at least -3 which are maximal in E_8. This may seem a computer enumeration problem, but it is harder than it looks.

Example 3.8. Let Π denote the root system of type E_8. Fix $\alpha \in \Pi$. Then there are elements $\beta_i \in \Pi$ $(i = 1, \dots, 28)$ such that

$$\{\beta \in \Pi \mid (\alpha, \beta) = 1\} = \bigcup_{i=1}^{28} \{\beta_i, \alpha - \beta_i\}.$$

Let V denote the set of 57 roots consisting of the above set and α. Then V is a reduced representation of a fat Hoffman graph $\mathfrak{H}$ with 29 fat vertices. The fat vertices of $\mathfrak{H}$ are f_i $(i = 0, 1, \dots, 28)$, f_0 is adjacent to α, and f_i is adjacent to $\beta_i, \alpha - \beta_i$ $(i = 1, \dots, 28)$. It turns out that $\mathfrak{H}$ is maximal among fat Hoffman graphs with smallest eigenvalue at least -3. Indeed, no fat vertex can be attached, since the root lattice of type E_8 is generated by $V \setminus \{\gamma\}$ for any $\gamma \in V$, and attaching another fat neighbor to γ would mean the existence of a vector of norm 1 in the dual lattice E_8^* of E_8. Since $\mathsf{E}_8^* = \mathsf{E}_8$, there are no vectors of norm 1 in E_8^*. This is a contradiction. If a slim vertex can be attached, then it can be represented by $\delta \in \Pi$ with $(\alpha, \delta) = 0$. Then there exists $i \in \{1, \dots, 28\}$ such that $(\beta_i, \delta) = \pm 1$. Exchanging β_i with $\alpha - \beta_i$ if necessary, we may assume $(\beta_i, \delta) = -1$. This implies that β_i and δ have a common fat neighbor. Since $(\beta_i, \alpha - \beta_i) = -1$, β_i and $\alpha - \beta_i$ have a common fat neighbor. Since β_i has a unique fat neighbor, δ and $\alpha - \beta_i$ have a common fat neighbor, contradicting $(\delta, \alpha - \beta_i) = 1$.

On the other hand, it is known that there is a slim graph Γ with 36 vertices represented by the root system of type E_8 (see [1]). Attaching a fat neighbor to each vertex of Γ gives a fat Hoffman graph $\mathfrak{H}'$ with smallest eigenvalue -3 such that $\Lambda^{\mathrm{red}}(\mathfrak{H}', 3)$ is isometric to the root lattice of type E_8. The graph $\mathfrak{H}'$ is not contained in $\mathfrak{H}$, so there seems a large number of maximal fat Hoffman graphs represented in the root lattice of type E_8.

4 Integrally represented Hoffman graphs

In this section, we consider a fat (-3)-saturated Hoffman graph $\mathfrak{H}$ such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is a sublattice of the standard lattice $\mathbb{Z}^n$. Since any of the exceptional root lattices $\mathsf{E}_6, \mathsf{E}_7$ and E_8 cannot be embedded into the standard lattice (see [2]), this means that, in view of Theorem 3.7, $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isometric to $\mathbb{Z}^n$ or a root lattice of type A_n or D_n. Note that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ cannot be isometric to the lattice A_1, since this would imply that $\mathfrak{H}$ has a unique slim vertex with a unique fat neighbor, contradicting (-3)-saturatedness. The following example gives a fat (-3)-saturated graph $\mathfrak{H}$ with $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3) \cong \mathbb{Z}$.

Example 4.1. Let $\mathfrak{H}$ be the Hoffman graph with vertex set $V_s(\mathfrak{H}) \cup V_f(\mathfrak{H})$, where $V_s(\mathfrak{H}) = \mathbb{Z}/4\mathbb{Z}$, $V_f(\mathfrak{H}) = \{f_i \mid i \in \mathbb{Z}/4\mathbb{Z}\}$, and with edge set

$$\{\{0,2\},\{1,3\}\} \cup \{\{i, f_j\} \mid i = j \text{ or } j+1\}.$$

Then $\mathfrak{H}$ is a fat indecomposable (-3)-saturated Hoffman graph such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isomorphic to the standard lattice $\mathbb{Z}$. Since $\mathcal{S}^-(\mathfrak{H})$ has edge set $\{\{i, i+1\} \mid i \in \mathbb{Z}/4\mathbb{Z}\}$, $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the Dynkin graph $\tilde{\mathsf{A}}_3$. Theorem 4.9 below implies that $\mathfrak{H}$ is maximal, in the sense that no fat indecomposable (-3)-saturated Hoffman graph contains $\mathfrak{H}$.

For the remainder of this section, we let $\mathfrak{H}$ be a fat indecomposable (-3)-saturated Hoffman graph such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isomorphic to a sublattice of the standard lattice $\mathbb{Z}^n$. Let ϕ be a representation of norm 3 of $\mathfrak{H}$. Then we may assume that ϕ is a mapping from $V(\mathfrak{H})$ to $\mathbb{Z}^n \oplus \mathbb{Z}^{V_f(\mathfrak{H})}$, where its composition with the projection $\mathbb{Z}^n \oplus \mathbb{Z}^{V_f(\mathfrak{H})} \to \mathbb{Z}^n$ gives a reduced representation $\psi : V_s(\mathfrak{H}) \to \mathbb{Z}^n$. It follows from the definition of a representation of norm 3 that

$$\phi(s) = \psi(s) + \sum_{f \in N^f_{\mathfrak{H}}(s)} \mathbf{e}_f,$$

$$\psi(s) = \sum_{j=1}^{n} \psi(s)_j \mathbf{e}_j,$$

$$\psi(s)_j \in \{0, \pm 1\},$$

and

$$|\{j \mid j \in \{1,\dots,n\},\ \psi(s)_j \in \{\pm 1\}\}| = 3 - |N^f_{\mathfrak{H}}(s)| \le 2. \tag{4.1}$$

Lemma 4.2. *If $i \in \{1,\dots,n\}$ and $\psi(s)_i \neq 0$ for some $s \in V_s(\mathfrak{H})$, then there exist $s_1, s_2 \in V_s(\mathfrak{H})$ such that $\psi(s_1)_i = -\psi(s_2)_i = 1$.*

Proof. By way of contradiction, we may assume without loss of generality that $i = n$, and $\psi(s)_n \in \{0, 1\}$ for all $s \in V_s(\mathfrak{H})$. Let $\mathfrak{G}$ be the Hoffman graph obtained from $\mathfrak{H}$ by attaching a new fat vertex g and join it to all the slim vertices s of $\mathfrak{H}$ satisfying $\psi(s)_n = 1$. Then the composition of $\psi : V_s(\mathfrak{H}) = V_s(\mathfrak{G}) \to \mathbb{Z}^n$ and the projection $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$ gives a reduced representation of norm 3 of $\mathfrak{G}$. By Theorem 2.8, $\mathfrak{G}$ has smallest eigenvalue at least -3. This contradicts the assumption that $\mathfrak{H}$ is (-3)-saturated. $\square$

Proposition 4.3. *The graph $\mathcal{S}^-(\mathfrak{H})$ is connected.*

Proof. Before proving the proposition, we first show the following claim.

Claim 4.4. *Let s_1 and s_2 be two slim vertices such that $\phi(s_1)_i = 1$ and $\phi(s_2)_i = -1$ for some $i \in \{1, \ldots, n\}$. Then the distance between s_1 and s_2 is at most 2 in $\mathcal{S}^-(\mathfrak{H})$.*

By (4.1), we have $(\psi(s_1), \psi(s_2)) \in \{0, -1\}$. If $(\psi(s_1), \psi(s_2)) = -1$, then s_1 and s_2 are adjacent in $\mathcal{S}^-(\mathfrak{H})$ by the definition, hence the distance equals one. If $(\psi(s_1), \psi(s_2)) = 0$, then there exists a unique $j \in \{1, \ldots, n\}$ such that $\phi(s_1)_j = \phi(s_2)_j = \pm 1$. From Lemma 4.2, there exists a slim vertex s_3 such that $\phi(s_3)_j = -\phi(s_1)_j$. If $\phi(s_3)_i \neq 0$, then $(\psi(s_q), \psi(s_3)) \in \{\pm 2\}$ for some $q \in \{1, 2\}$, which is a contradiction. Hence $\phi(s_3)_i = 0$. This implies that $(\psi(s_i), \psi(s_3)) = -1$ for $i = 1, 2$, or equivalently, s_3 is a common neighbor of s_1 and s_2 in $\mathcal{S}^-(\mathfrak{H})$. This shows the claim.

Since $\mathfrak{H}$ is indecomposable, $\mathcal{S}(\mathfrak{H})$ is connected by Lemma 3.4. Thus, in order to show the proposition, we only need to show that slim vertices s_1 and s_2 with $(\psi(s_1), \psi(s_2)) = 1$ are connected by a path in $\mathcal{S}^-(\mathfrak{H})$. There exists $i \in \{1, \ldots, n\}$ such that $\phi(s_1)_i = \phi(s_2)_i = \pm 1$. From Lemma 4.2, there exists a slim vertex s_3 such that $\phi(s_3)_i = -\phi(s_1)_i$, and hence the distances between s_1 and s_3 and between s_3 and s_2 are at most 2 in $\mathcal{S}^-(\mathfrak{H})$ by Claim 4.4. Therefore, s_1 and s_2 are connected by a path of length at most 4 in $\mathcal{S}^-(\mathfrak{H})$. □

Lemma 4.5. *Let $\mathfrak{H}$ be a fat indecomposable (-3)-saturated Hoffman graph. Then the reduced representation of norm 3 of $\mathfrak{H}$ is injective unless $\mathfrak{H}$ is isomorphic to a subgraph of the graph given in Example 4.1.*

Proof. Suppose that the reduced representation ψ of norm 3 of $\mathfrak{H}$ is not injective. Then there are two distinct slim vertices x and y satisfing $\psi(x) = \psi(y)$. Then $(\psi(x), \psi(y)) = 0$ or 1.

If $(\psi(x), \psi(y)) = 0$, then $\psi(x) = \psi(y) = 0$, hence both x and y are isolated vertices, contradicting the assumption that $\mathfrak{H}$ is indecomposable.

Suppose $(\psi(x), \psi(y)) = 1$. Since $\mathcal{S}^-(\mathfrak{H})$ is connected by Proposition 4.3, there exists a slim vertex z such that $(\psi(x), \psi(z)) = -1$. Then we may assume

$$\begin{aligned} \phi(x) &= \mathbf{e}_1 + \mathbf{e}_{f_1} + \mathbf{e}_{f_2}, \\ \phi(y) &= \mathbf{e}_1 + \mathbf{e}_{f_3} + \mathbf{e}_{f_4}, \\ \phi(z) &= -\mathbf{e}_1 + \mathbf{e}_{f_1} + \mathbf{e}_{f_3}. \end{aligned}$$

If $\mathfrak{H}$ has another slim vertex w, then

$$\phi(w) = -\mathbf{e}_1 + \mathbf{e}_{f_2} + \mathbf{e}_{f_4},$$

and no other possibility occurs. Therefore, $\mathfrak{H}$ is isomorphic to either the graph given in Example 4.1, or its subgraph obtained by deleting one slim vertex. □

Lemma 4.6. *Suppose that $s \in V_s(\mathfrak{H})$ has exactly two fat neighbors in $\mathfrak{H}$. Then the following statements hold.*

(i) *for each $f \in N^f_{\mathfrak{H}}(s)$, $|N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)| \leq 2$,*

(ii) *$|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq 4$, and if equality holds, then $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the graph $\tilde{\mathsf{D}}_4$,*

(iii) *if* $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 3$*, then two of the vertices in* $N_{\mathcal{S}^-(\mathfrak{H})}(s)$ *have* s *as their unique neighbor in* $\mathcal{S}^-(\mathfrak{H})$*.*

Proof. Let ψ be the reduced representation of norm 3 of $\mathfrak{H}$. Since s has exactly two fat neighbors, $(\psi(s), \psi(s)) = 1$. This means that we may assume without loss of generality $\psi(s) = \mathbf{e}_1$.

Let $f \in N^f_{\mathfrak{H}}(s)$. If t_1 and t_2 are distinct vertices of $N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)$, then

$$\begin{aligned} 1 &\geq (\phi(t_1), \phi(t_2)) \\ &\geq (\psi(t_1) + \mathbf{e}_f, \psi(t_2) + \mathbf{e}_f) \\ &= (\psi(t_1), \psi(t_2)) + 1, \end{aligned}$$

Thus we have $(\psi(t_1), \psi(t_2)) \leq 0$. Since $t_1, t_2 \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$, we have $(\psi(s), \psi(t_1)) = (\psi(s), \psi(t_2)) = -1$, and hence we may assume without loss of generality that

$$\psi(t_1) = -\mathbf{e}_1 + \mathbf{e}_2, \tag{4.2}$$

$$\psi(t_2) = -\mathbf{e}_1 - \mathbf{e}_2. \tag{4.3}$$

Now it is clear that there cannot be another $t_3 \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$. Thus $|N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)| \leq 2$. This proves (i).

As for (ii), let $N^f_{\mathfrak{H}}(s) = \{f, f'\}$. Then

$$|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq |N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f)| + |N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f')| \leq 4$$

by (i). To prove (iii) and the second part of (ii), we may assume that $\{t_1, t_2\} = N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N_{\mathfrak{H}}(f)$. We claim that neither t_1 nor t_2 has a neighbor in $\mathcal{S}^-(\mathfrak{H})$ other than s. Suppose by contradiction, that $t_3 \neq s$ is a neighbor in $\mathcal{S}^-(\mathfrak{H})$ of t_1. By (4.2) (resp. (4.3)), f is the unique fat neighbor of t_1 (resp. t_2). In particular, f is also a neighbor of t_3. Observe

$$\begin{aligned} &1 \geq (\phi(s), \phi(t_3)) \geq (\psi(s), \psi(t_3)) + 1, \\ &1 \geq (\phi(t_i), \phi(t_3)) = (\psi(t_i), \psi(t_3)) + 1 \quad (i = 1, 2). \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{e}_1, \psi(t_3)) &\leq 0, \\ (-\mathbf{e}_1 \pm \mathbf{e}_2, \psi(t_3)) &\leq 0. \end{aligned}$$

These imply $(\mathbf{e}_1, \psi(t_3)) = (\mathbf{e}_2, \psi(t_3)) = 0$. But then $-1 = (\psi(t_1), \psi(t_3)) = (-\mathbf{e}_1 + \mathbf{e}_2, \psi(t_3)) = 0$. This is a contradiction, and we have proved our claim. Now (iii) is an immediate consequence of this claim.

Continuing the proof of the second part of (ii), if $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 4$, then we may assume

$$\begin{aligned} \psi(t'_1) &= -\mathbf{e}_1 + \mathbf{e}_3, \\ \psi(t'_2) &= -\mathbf{e}_1 - \mathbf{e}_3, \end{aligned}$$

where $\{t'_1, t'_2\} = N_{\mathcal{S}^-(\mathfrak{H})}(s) \cap N^s_{\mathfrak{H}}(f')$. It follows that t_1, t_2, t'_1, t'_2 are pairwise non-adjacent in $\mathcal{S}^-(\mathfrak{H})$. By our claim, none of t_1, t_2, t'_1, t'_2 is adjacent to any vertex other than s in $\mathcal{S}^-(\mathfrak{H})$. Since $\mathcal{S}^-(\mathfrak{H})$ is connected by Proposition 4.3, $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the graph $\tilde{\mathrm{D}}_4$. □

Lemma 4.7. *Suppose that slim vertices s, t^+, t^- share a common fat neighbor and that they are represented as*

$$\psi(s) = \mathbf{e}_1 + \mathbf{e}_2,$$
$$\psi(t^\pm) = -\mathbf{e}_1 \pm \mathbf{e}_3.$$

If there exists a slim vertex t with

$$\psi(t) = -\mathbf{e}_2 + \mathbf{e}_j \quad \text{for some } j \notin \{1, 2, 3\},$$

then the vertices $t^\pm$ have s as their unique neighbor in $\mathcal{S}^-(\mathfrak{H})$.

Proof. Note that each of the vertices $s, t^\pm, t$ has a unique fat neighbor. Since $(\psi(s), \psi(t^\pm)) = (\psi(s), \psi(t)) = -1$, these vertices share a common fat neighbor f. Suppose that there exists a slim vertex t' adjacent to t^- in $\mathcal{S}^-(\mathfrak{H})$. This means $(\psi(t'), \psi(t^-)) = -1$. Since f is the unique fat neighbor of t^-, t' is adjacent to f, and hence $(\psi(t'), \psi(u)) \leq 0$ for $u \in \{t, t^+, s\}$. This is impossible. Similarly, there exists no slim vertex adjacent to t^+ in $\mathcal{S}^-(\mathfrak{H})$. □

Lemma 4.8. *Suppose that $s \in V_s(\mathfrak{H})$ has exactly one fat neighbor in $\mathfrak{H}$. Then the following statements hold:*

(i) $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq 4$, *and if equality holds, then $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the graph $\tilde{\mathsf{D}}_4$,*

(ii) *if $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 3$, then two of the vertices in $N_{\mathcal{S}^-(\mathfrak{H})}(s)$ have s as their unique neighbor in $\mathcal{S}^-(\mathfrak{H})$.*

Proof. Let ψ be the reduced representation of norm 3 of $\mathfrak{H}$. Since s has exactly one fat neighbor, $(\psi(s), \psi(s)) = 2$. This means that we may assume without loss of generality $\psi(s) = \mathbf{e}_1 + \mathbf{e}_2$. Let f be the unique fat neighbor of s. If $t \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$, then t is adjacent to f, hence

$$\psi(t) \in \{-\mathbf{e}_1, -\mathbf{e}_2\} \cup \{-\mathbf{e}_i \pm \mathbf{e}_j \mid 1 \leq i \leq 2 < j \leq n\}. \tag{4.4}$$

If $t, t' \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$ are distinct, then

$$\begin{aligned} 1 &\geq (\phi(t), \phi(t')) \\ &\geq (\psi(t) + \mathbf{e}_f, \psi(t') + \mathbf{e}_f) \\ &= (\psi(t), \psi(t')) + 1, \end{aligned}$$

Thus we have $(\psi(t), \psi(t')) \leq 0$. If $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \geq 3$, then by (4.4), we may assume without loss of generality that there exists $t \in N_{\mathcal{S}^-(\mathfrak{H})}(s)$ with $\psi(t) = -\mathbf{e}_1 + \mathbf{e}_3$. Then $\psi(N_{\mathcal{S}^-(\mathfrak{H})}(s) \setminus \{t\})$ is contained in

$$\{-\mathbf{e}_2 - \mathbf{e}_3\}, \{-\mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_3\}, \text{ or } \{-\mathbf{e}_1 - \mathbf{e}_3\} \cup \{-\mathbf{e}_2 \pm \mathbf{e}_j\}$$

for some j with $3 \leq j \leq n$. Thus $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| \leq 4$, and equality holds only if

$$\psi(N_{\mathcal{S}^-(\mathfrak{H})}(s)) = \{-\mathbf{e}_1 \pm \mathbf{e}_3, -\mathbf{e}_2 \pm \mathbf{e}_j\}$$

for some j with $3 \leq j \leq n$. In this case, Lemma 4.7 implies that each of the vertices in $N_{\mathcal{S}^-(\mathfrak{H})}(s)$ has s as a unique neighbor. This means that $\mathcal{S}^-(\mathfrak{H})$ contains a subgraph isomorphic to the graph $\tilde{\mathsf{D}}_4$ as a connected component. Since $\mathcal{S}^-(\mathfrak{H})$ is connected by Proposition 4.3, we have the desired result.

To prove (ii), suppose $|N_{\mathcal{S}^-(\mathfrak{H})}(s)| = 3$. Then we may assume without loss of generality that $N_{\mathcal{S}^-(\mathfrak{H})}(s) = \{t, t^+, t^-\}$, where $\psi(t^\pm) = -\mathbf{e}_2 \pm \mathbf{e}_4$. Then by Lemma 4.7, the vertices $t^\pm$ have s as their unique neighbor in $\mathcal{S}^-(\mathfrak{H})$. □

Theorem 4.9. *Let $\mathfrak{H}$ be a fat indecomposable (-3)-saturated Hoffman graph such that $\Lambda^{\mathrm{red}}(\mathfrak{H}, 3)$ is isomorphic to a sublattice of the standard lattice $\mathbb{Z}^n$. Then $\mathcal{S}^-(\mathfrak{H})$ is a connected graph which is isomorphic to $\mathsf{A}_m, \mathsf{D}_m, \tilde{\mathsf{A}}_m$ or $\tilde{\mathsf{D}}_m$ for some positive integer m.*

Proof. From Proposition 4.3, $\mathcal{S}^-(\mathfrak{H})$ is connected. First we suppose that the maximum degree of $\mathcal{S}^-(\mathfrak{H})$ is at most 2. Then $\mathcal{S}^-(\mathfrak{H}) \cong \tilde{\mathsf{A}}_m$ or $\mathcal{S}^-(\mathfrak{H}) \cong \mathsf{A}_m$ for some positive integer m.

Next we suppose that the degree of some vertex s in $\mathcal{S}^-(\mathfrak{H})$ is at least 3. From Lemma 4.6(ii) and Lemma 4.8(i), $\deg_{\mathcal{S}^-(\mathfrak{H})}(s) \le 4$, and $\mathcal{S}^-(\mathfrak{H}) \cong \tilde{\mathsf{D}}_4$ if $\deg_{\mathcal{S}^-(\mathfrak{H})}(s) = 4$. Thus, for the remainder of this proof, we suppose that the maximum degree of $\mathcal{S}^-(\mathfrak{H})$ is 3 and $\deg_{\mathcal{S}^-(\mathfrak{H})}(s) = 3$.

It follows from Lemma 3.5 that if $\mathfrak{H}$ has a subgraph isomorphic to $\mathfrak{H}^{(3)}$, then $\mathfrak{H} \cong \mathfrak{H}^{(3)}$, in which case $\mathcal{S}^-(\mathfrak{H})$ consists of a single vertex, and the assertion holds. Hence it remains to consider two cases: s is adjacent to exactly two fat vertices, and s is adjacent to exactly one fat vertex. In either cases, by Lemma 4.6(iii) or Lemma 4.8(ii), s has at most one neighbor t with degree greater than 1. Thus, the only way to extend this graph is by adding a slim neighbor adjacent to t. We can continue this process, but once we encounter a vertex of degree 3, then the process stops by Lemma 4.6(iii) or Lemma 4.8(ii). Thus, $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to one of the graphs D_m or $\tilde{\mathsf{D}}_m$. □

Example 4.10. Let $n_1, \dots, n_k$ be positive integers satisfying $n_i \ge 2$ for $1 < i < k$. Set $m_j = \sum_{i=1}^{j} n_i$ and $\ell_j = m_j - j$ for $j = 0, 1, \dots, k$. Let $\mathfrak{H}$ be the Hoffman graph with $V_s(\mathfrak{H}) = \{v_i \mid i = 0, 1, \dots, m_k\}$, $V_f(\mathfrak{H}) = \{f_j \mid j = 0, 1, \dots, k+1\}$, and

$$\begin{aligned} E(\mathfrak{H}) =& \{\{v_i, v_{i'}\} \mid 1 \le j \le k,\ m_{j-1} < i+1 < i' \le m_j\} \\ & \cup \{\{v_{m_j-1}, v_{m_j+1}\} \mid 1 < j < k\} \\ & \cup \{\{f_j, v_i\} \mid 1 \le j < k,\ m_{j-1} \le i \le m_j\} \cup \{\{f_0, v_0\}, \{f_{k+1}, v_{m_k}\}\}. \end{aligned}$$

Then $\mathfrak{H}$ is a fat Hoffman graph with smallest eigenvalue at least -3, and $\mathcal{S}^-(\mathfrak{H})$ is isomorphic to the Dynkin graph A_{m_k+1}. Indeed, $\mathfrak{H}$ has a reduced representation ψ of norm 3 defined by

$$\psi(v_i) = \begin{cases} (-1)^j \mathbf{e}_{\ell_j} & \text{if } i = m_j, 0 \le j \le k, \\ (-1)^j (\mathbf{e}_{i-j} - \mathbf{e}_{i-j-1}) & \text{if } m_j < i < m_{j+1}, 0 \le j < k. \end{cases}$$

Moreover, $\mathfrak{H}$ is (-3)-saturated. Indeed, suppose not, and let $\tilde{\mathfrak{H}}$ be a Hoffman graph obtained by attaching a new fat vertex f to $\mathfrak{H}$, and let $\tilde{\psi}$ be a reduced representation of norm 3 of $\tilde{\mathfrak{H}}$. If f is adjacent to v_{m_j} for some $j \in \{0, 1, \dots, k\}$, then v_{m_j} has three fat neighbors in $\tilde{\mathfrak{H}}$, hence $\tilde{\psi}(v_{m_j}) = 0$. This is absurd, since $(\tilde{\psi}(v_{m_j}), \tilde{\psi}(v_{m_j \pm 1})) = -1$. If f is adjacent to v_i with $m_{j-1} < i < m_j$, then $\|\tilde{\psi}(v_i)\| = 1$. Since $(\tilde{\psi}(v_{i-1}), \tilde{\psi}(v_i)) = (\tilde{\psi}(v_{i+1}), \tilde{\psi}(v_i)) =$

-1 while $(\tilde{\psi}(v_{i-1}), \tilde{\psi}(v_{i+1})) = 0$, we may assume $\tilde{\psi}(v_{i\pm1}) = \mathbf{e}_1 \pm \mathbf{e}_2$, $\tilde{\psi}(v_i) = -\mathbf{e}_1$. Then $i+1 < m_j$, and

$$\begin{aligned} 0 &= (\tilde{\psi}(v_{i+2}), \tilde{\psi}(v_{i-1})) \\ &= (\tilde{\psi}(v_{i+2}), \mathbf{e}_1 - \mathbf{e}_2) \\ &= (\tilde{\psi}(v_{i+2}), 2\mathbf{e}_1 - (\mathbf{e}_1 + \mathbf{e}_2)) \\ &= -2(\tilde{\psi}(v_{i+2}), \tilde{\psi}(v_i)) - (\tilde{\psi}(v_{i+2}), \tilde{\psi}(v_{i+1})) \\ &= 1, \end{aligned}$$

which is absurd.

We note that the graph $\mathcal{S}^+(\mathfrak{H})$ has the following edges:

$$\{\{v_{m_j-1}, v_{m_j+1}\} \mid 1 < j < k\}.$$

Example 4.11. Let $\mathfrak{H}$ be the Hoffman graph constructed in Example 4.10 by setting $n_1 = 1$, $n_2 = 2$, and $n_3 = 1$. Let $\mathfrak{H}_0$ (resp. $\mathfrak{H}_1$) be the Hoffman graph obtained from $\mathfrak{H}$ by identifying the fat vertices f_4 and f_0 (resp. f_4 and f_1), and adding edges $\{v_0, v_2\}, \{v_2, v_4\}$. Then $\mathfrak{H}_0$ and $\mathfrak{H}_1$ are fat (-3)-saturated Hoffman graphs and $\mathcal{S}^-(\mathfrak{H}_i)$ is isomorphic to the Dynkin graph A_5 for $i = 0, 1$. We note that $\mathcal{S}^+(\mathfrak{H}_i)$ has two edges $\{v_0, v_2\}, \{v_2, v_4\}$.

Examples 4.10 and 4.11 indicate that $\mathfrak{H}$ is not determined by $\mathcal{S}^\pm(\mathfrak{H})$. We plan to discuss the classification of fat indecomposable (-3)-saturated Hoffman graphs with prescribed special graph in subsequent papers.

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