

The adjacency dimension of graphs*

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Abstract

It is known that the problem of computing the adjacency dimension of a graph is NP-hard. This suggests finding the adjacency dimension for special classes of graphs or obtaining good bounds on this invariant. In this work we obtain general bounds on the adjacency dimension of a graph G in terms of known parameters of G . We discuss the tightness of these bounds and, for some particular classes of graphs, we obtain closed formulae. In particular, we show the close relationships that exist between the adjacency dimension and other parameters, like the domination number, the location-domination number, the 2-domination number, the independent 2-domination number, the vertex cover number, the independence number and the super domination number.

Keywords: Adjacency dimension, metric dimension, location-domination number, independence number, super domination number.

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1 Introduction

The metric dimension of a general metric space was introduced in 1953 by Blumenthal [2, p. 95]. A *metric generator* for a metric space (X, d) is a set $S \subseteq X$ of points in the space with the property that every point of X is uniquely determined by the distances from the elements of S , i.e., $S \subseteq X$ is a metric generator for X if for any pair of distinct points $x, x' \in X$ there exists $s \in S$ such that $d(x, s) \neq d(x', s)$. A metric generator of minimum cardinality in X is called a *metric basis*, and its cardinality, which is denoted by $\dim(X)$, is called the *metric dimension* of X . The notion of metric dimension of a graph was introduced by Slater in [23], where the metric generators were called *locating sets*. Harary and Melter [11] independently introduced the same concept, where metric generators were called *resolving sets*. Given a simple and connected graph $G = (V, E)$, we consider the function $d: V \times V \rightarrow \mathbb{N} \cup \{0\}$, where $d(x, y)$ is the length of a shortest path in G between x and y and \mathbb{N} is the set of positive integers. Since (V, d) is a metric space, a metric generator for a graph $G = (V, E)$ is simply a metric generator for the metric space (V, d) and we will use the notation $\dim(G)$ instead of $\dim(V)$ for the metric dimension of G .

Several variations of metric generators have been introduced and studied, namely, resolving dominating sets [3], locating-dominating sets [24, 25], independent resolving sets [5], local metric sets [18], strong resolving sets [22], adjacency generators [15, 16], k -adjacency generators [6], k -metric generators [1, 7, 8], simultaneous metric generators [21] etc. In this article, we are interested in the study of adjacency generators.

The notion of adjacency generator was introduced by Jannesari and Omoomi in [16] as a tool to study the metric dimension of lexicographic product graphs. This concept has been studied further by Fernau and Rodríguez-Velázquez in [9, 10] where they showed that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency dimension of H . As a consequence of this strong relation they showed that the problem of computing the adjacency dimension is NP-hard. This suggests finding the adjacency dimension for special classes of graphs or obtaining good bounds on this invariant. In this work we obtain general bounds on the adjacency dimension of a graph G in terms of known parameters of G , while for some particular cases we obtain closed formulae.

In order to introduce the concept of adjacency generator for a graph $G = (V, E)$, we define the distance function $d_2: V \times V \rightarrow \mathbb{N} \cup \{0\}$, where

$$d_2(x, y) = \min\{d(x, y), 2\}.$$

An *adjacency generator* for a graph $G = (V, E)$ is a metric generator for the metric space (V, d_2) . Hence, the *adjacency dimension* of $G = (V, E)$, denoted by $\text{adim}(G)$, equals the metric dimension of (V, d_2) .

Notice that $S \subseteq V$ is an adjacency generator for $G = (V, E)$ if for every pair of vertices $x, y \in V \setminus S$ there exists $s \in S$ which is adjacent to exactly one of these two vertices x and

y . Therefore, S is an adjacency generator for G if and only if S is an adjacency generator for its complement \overline{G} . Consequently,

$$\text{adim}(G) = \text{adim}(\overline{G}). \quad (1.1)$$

From the definition of adjacency and metric bases, we deduce that S is an adjacency basis of a graph G of diameter at most two if and only if S is a metric basis of G . In these cases, $\text{adim}(G) = \text{dim}(G)$. The reader is referred to [6, 10, 15, 16, 20] for known results on the adjacency dimension.

The paper is organized as follows. Section 2 is devoted to study the variation of the adjacency dimension of a graph by removing a set of edges. In particular, we wonder how far can decrease the adjacency dimension by removing edges from a complete graph and we obtain a lower bound on the adjacency dimension of any graph in terms of the order. In Section 3 we show the close relationships that exist between the adjacency dimension and other parameters, like the domination number, the location-domination number, the 2-domination number, the independent 2-domination number, the vertex cover number, the independence number and the super domination number.

We will use the notation K_n , $K_{r,n-r}$, C_n , P_n and N_n for complete graphs, complete bipartite graphs, cycle graphs, path graphs and empty graphs of order n , respectively. We use the notation $u \sim v$ if u and v are adjacent vertices and $G \cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G , $N(v)$ will denote the set of neighbours or *open neighborhood* of v in G , i.e., $N(v) = \{u \in V(G) : u \sim v\}$. The *closed neighborhood*, denoted by $N[v]$, equals $N(v) \cup \{v\}$. We also define $\text{deg}(v) = |N(v)|$ as the degree of $v \in V(G)$, as well as, $\delta = \min_{v \in V(G)} \{\text{deg}(v)\}$ and $\Delta = \max_{v \in V(G)} \{\text{deg}(v)\}$. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 The effect of removing edges and bounds in terms of the order

The following theorem is an important tool to derive some of our results.

Theorem 2.1 ([16]). *Let G be a graph of order n . Then the following statements hold.*

- (i) $\text{adim}(G) = 1$ if and only if $n \in \{1, 2, 3\}$, $G \not\cong K_3$ and $G \not\cong N_3$.
- (ii) $\text{adim}(G) = n - 1$ if and only if $G \cong K_n$ or $G \cong N_n$.
- (iii) If $n \geq 3$ and $t \in \{1, \dots, n - 1\}$, then $\text{adim}(K_{t,n-t}) = n - 2$.
- (iv) If $n \geq 4$, then $\text{adim}(P_n) = \text{adim}(C_n) = \lfloor \frac{2n+2}{5} \rfloor$.

In this section we show the effect, on the adjacency dimension, of an operation which removes a set of edges from a graph. Given a non-empty graph $G = (V, E)$ and an edge $e \in E$ we denote by $G - e = (V, E \setminus \{e\})$ the subgraph obtained by removing the edge e from G . In general, given a set of edges $E_k = \{e_1, \dots, e_k\} \subseteq E$ we denote by $G - E_k = (V, E \setminus E_k)$ the subgraph obtained by removing the k edges in E_k from G . By analogy we define the supergraphs $G + e = (V, E \cup \{e\})$ and $G + E_k = (V, E \cup E_k)$, where $\{e\}$ and E_k are sets of edges of the complement of G .

Theorem 2.2. *Let $G = (V, E)$ be a non-empty graph. For any set $E_k = \{e_1, \dots, e_k\} \subseteq E$,*

$$\text{adim}(G) - k \leq \text{adim}(G - E_k) \leq \text{adim}(G) + k.$$

Proof. Since $(G - E_{k-1}) - e_k = G - E_k$, it is enough to prove that, for any $e \in E$,

$$\text{adim}(G) - 1 \leq \text{adim}(G - e) \leq \text{adim}(G) + 1.$$

Let S be an adjacency basis of $G - e$, where $e = xy$. Since $S \cup \{y\}$ is an adjacency generator for G , we have that $\text{adim}(G) \leq |S \cup \{y\}| \leq |S| + 1 = \text{adim}(G - e) + 1$. Hence, $\text{adim}(G) - 1 \leq \text{adim}(G - e)$.

Finally, let us observe that $\text{adim}(G - e) = \text{adim}(\overline{G - e}) = \text{adim}(\overline{G} + e) \leq \text{adim}((\overline{G} + e) - e) + 1 = \text{adim}(\overline{G}) + 1 = \text{adim}(G) + 1$. Therefore, the result follows. \square

Since $\text{adim}(G - E_k) = \text{adim}(\overline{G - E_k}) = \text{adim}(\overline{G} + E_k)$, we conclude that $\text{adim}(G - E_k) = \text{adim}(G) - k$ if and only if the graph $H = \overline{G} + E_k$ satisfies $\text{adim}(H - E_k) = \text{adim}(H) + k$. Therefore, in order to show that the inequalities above are tight, we only need to consider one of them. For instance, $\text{adim}(K_n - e) = n - 2 = \text{adim}(K_n) - 1$. With the aim of showing a more general example, let us consider s stars $H_i \cong K_{1,r}$, $r \geq 4$, such that v_i is the center and u_{i1}, \dots, u_{ir} are the leaves of H_i , for $i \in \{1, \dots, s\}$. Let $e_i = u_{i1}u_{i2}$, $G_i = H_i + e_i$ and $M = \{v_i v_{i+1} : 1 \leq i < s\}$, and define $G_{r,s} = (V, E)$, where $V = \bigcup_{i=1}^s V(G_i)$ and $E = (\bigcup_{i=1}^s E(G_i)) \cup M$. It is readily seen that $\text{adim}(G_{r,s}) = s(r - 1) - 1$, while for any $k \leq s$ and $E_k = \{e_1, \dots, e_k\}$, $\text{adim}(G_{r,s} - E_k) = s(r - 1) - 1 + k = \text{adim}(G_{r,s}) + k$.

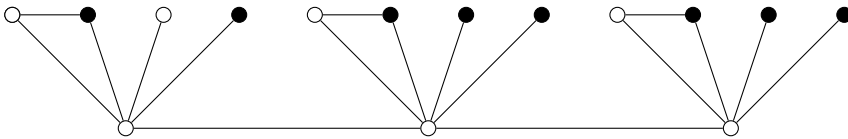


Figure 1: The set of black-colored vertices is an adjacency basis of $G_{4,3}$.

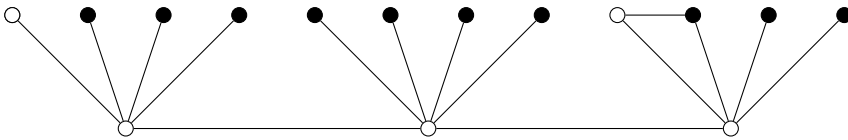


Figure 2: The set of black-colored vertices is an adjacency basis of $G_{4,3} - E_2$.

Figure 1 shows the graph $G_{4,3}$, while Figure 2 shows the graph $G_{4,3} - E_2$. In this case, $\text{adim}(G_{4,3} - E_2) = 10 = \text{adim}(G_{4,3}) + 2$.

All graphs of order n are obtained by successive elimination of edges from a complete graph (or by successive addition of edges to an empty graph). We know from Theorem 2.1 that for any graph G of order n , $\text{adim}(G) \leq n - 1$ and the equality holds if and only if $G \cong K_n$ or $G \cong N_n$. Hence, by Theorem 2.2 we conclude that $\text{adim}(K_n - e) = n - 2$ for every $e \in E(K_n)$. Now we wonder how far can decrease the adjacency dimension by removing edges from K_n , i.e., which is the lower bound for the adjacency dimension in terms of the order of the graph. This problem is addressed in Propositions 2.3 and 2.4. Before stating it we need to introduce the following notation.

Given a positive integer s , let \mathcal{G}' be the family of all graphs of order s and \mathcal{G}'' the family of all graphs of order 2^s . We can assume that the graphs in \mathcal{G}' are defined on

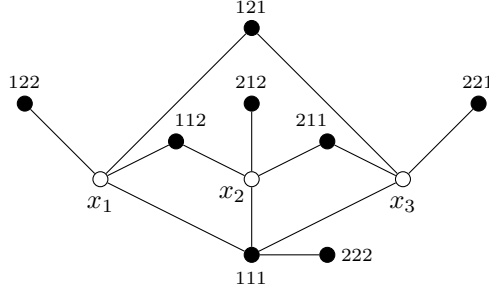


Figure 3: A graph $G \in \mathcal{G}_3$ constructed from $G' \cong N_3 \in \mathcal{G}'$ and $G'' \cong (K_2 \cup N_6) \in \mathcal{G}''$.

$S = \{x_1, \dots, x_s\}$ and the graphs in \mathcal{G}'' are defined on the set $\{1, 2\}^s$ of binary words of length s . Let \mathcal{G}_s be the family of graphs constructed from \mathcal{G}' and \mathcal{G}'' as follows. We say that $G \in \mathcal{G}_s$ if and only if there exist $G' \in \mathcal{G}'$ and $G'' \in \mathcal{G}''$ such that $V(G) = V(G') \cup V(G'')$ and $E(G) = E(G') \cup E(G'') \cup E^*$, where E^* is the set of edges connecting vertices of G' with vertices of G'' in such a way that x_i is adjacent to $y \in \{1, 2\}^s$ if and only if the i -th letter of y is 1. Notice that S is an adjacency generator for every $G \in \mathcal{G}_s$. Figure 3 shows a graph $G \in \mathcal{G}_3$ constructed from $G' \cong N_3 \in \mathcal{G}'$ and $G'' \cong (K_2 \cup N_6) \in \mathcal{G}''$.

The following inequality appeared recently in [15], but we characterize here all graphs satisfying the equality.

Proposition 2.3. *For any graph G of order n ,*

$$2^{\text{adim}(G)} + \text{adim}(G) \geq n. \tag{2.1}$$

Furthermore, a graph G of order $n = 2^s + s$ satisfies $\text{adim}(G) = s$ if and only if $G \in \mathcal{G}_s$.

Proof. As we mentioned before, the inequality was proved in [15]. By definition of \mathcal{G}_s , if $G \in \mathcal{G}_s$, then $\{x_1, \dots, x_s\}$ is an adjacency generator. Now, if $\text{adim}(G) = r < s$, then Equation (2.1) leads to $n = s + 2^s > r + 2^r \geq n$, which is a contradiction. Therefore, $G \in \mathcal{G}_s$ leads to $\text{adim}(G) = s$.

Conversely, suppose that G has order $n = 2^s + s$ and $\text{adim}(G) = s$. In this case, for any adjacency basis $S = \{x_1, \dots, x_s\}$ of G , the function $\psi: V(G) \setminus S \rightarrow \{1, 2\}^s$ defined by

$$\psi(x) = (d_2(x, x_1), \dots, d_2(x, x_s)),$$

is bijective, as it is injective and $|V(G) \setminus S| = 2^s$. Hence, taking $G' \in \mathcal{G}'$ as the subgraph of G induced by S , $G'' \in \mathcal{G}''$ as the subgraph of G induced by $V(G) \setminus S$ and E^* as the set of edges connecting vertices in S with vertices in $V(G) \setminus S$, we can conclude that $G \in \mathcal{G}_s$. \square

Proposition 2.4. *For any graph G of order n , $\text{adim}(G) \geq \left\lceil \frac{\ln(\frac{2n}{3})}{\ln(2)} \right\rceil$.*

Proof. If G is a graph with order n and $\text{adim}(G) = k$, since

$$n \leq 2^k + k \leq 2^k + 2^{k-1} = 2^k \left(1 + \frac{1}{2}\right) = 2^k \left(\frac{3}{2}\right),$$

we conclude that $k \geq \frac{\ln(\frac{2n}{3})}{\ln(2)}$. \square

The bound above is tight. It is achieved, for instance, for the family \mathcal{G}_s of graphs constructed prior to Proposition 2.3. These graphs have order $n = s + 2^s$ and metric dimension s . To check the tightness of the bound we only need to observe that $\frac{2(s+2^s)}{3} > 2^{s-1}$, for every positive integer s . Examples of graphs of small order for which the bound is achieved are the path P_3 , the cycles C_r ($4 \leq r \leq 6$), and the cube $Q_3 = K_2 \square K_2 \square K_2$, as $\text{adim}(P_3) = 1$, $\text{adim}(C_4) = \text{adim}(C_5) = \text{adim}(C_6) = 2$ and $\text{adim}(Q_3) = 3$.

By Theorem 2.1, for any non-complete and non-empty graph of order n , $\text{adim}(G) \leq n - 2$. The characterization for graphs G such that $\text{adim}(G) = n - 2$ appeared recently in [15].

Theorem 2.5. *Let G be a connected graph of order $n \geq 5$. Then $\text{adim}(G) = n - 2$ if and only if one of the following conditions holds.*

- (i) $G \cong K_{t,n-t}$, for some $t \in \{1, \dots, n - 1\}$.
- (ii) $G \cong K_{n-t} + N_t$, for some $t \in \{2, \dots, n - 2\}$.
- (iii) $G \cong (K_1 \cup K_t) + K_{n-t-1}$, for some $t \in \{2, \dots, n - 2\}$.

We conclude this section with a characterization of all graphs G satisfying that $\text{adim}(G) = 2$, which also appeared in [15].

Theorem 2.6. *Let G be a connected graph of order n . Then $\text{adim}(G) = 2$ if and only if one of the following conditions holds for G (or \overline{G}).*

- (a) $G \cong K_3$.
- (b) $n = 4$ and $G \not\cong K_4$.
- (c) $n = 5$ and $G \not\cong K_5$, $G \not\cong K_{t,5-t}$ for $t \in \{1, \dots, 4\}$, $G \not\cong K_{5-t} + N_t$ and $G \not\cong (K_1 \cup K_t) + K_{4-t}$ for $t \in \{2, 3\}$.
- (d) $n = 6$ and $G \in \mathcal{G}_2$.

3 Relationship between the adjacency dimension and other parameters

A set $D \subseteq V(G)$ is a *dominating set* of G if $N(x) \cap D \neq \emptyset$ for every vertex $x \in V(G) \setminus D$. The *domination number*, $\gamma(G)$, is the minimum cardinality among all dominating sets of G . A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$ -set. The reader is referred to the books [12, 13] on the domination theory.

The following result is immediate from Equation (1.1) and the fact that at most one vertex of G is not dominated by the vertices in an adjacency generator of G .

Remark 3.1 ([10]). For any graph G ,

$$\text{adim}(G) \geq \max\{\gamma(G), \gamma(\overline{G})\} - 1.$$

The bound above is tight. For instance, it is attained by the corona graphs $K_r \odot K_1$, $r \geq 3$, as $\text{adim}(K_r \odot K_1) = r - 1$ and $\gamma(K_r \odot K_1) = r$. Another example is any graph $G \in \mathcal{G}$ with $\gamma(G) = s + 1$. A particular case is shown in Figure 3.

A *locating-dominating* set is a dominating set D that locates/distinguishes all the vertices in the sense that every vertex not in D is uniquely determined by its neighbourhood in D , i.e., $N(u) \cap D \neq N(v) \cap D$ for every pair of vertices $u, v \in V(G) \setminus D$. The *location-domination number* of G , denoted $\lambda(G)$, is the minimum cardinality among all locating-dominating sets in G . A locating-dominating set of cardinality $\lambda(G)$ is called a $\lambda(G)$ -set. The concept of a locating-dominating set was introduced and first studied by Slater [24, 25] and studied, for instance, in [4, 14, 19] and elsewhere.

Since every locating-dominating set is an adjacency generator and any adjacency basis S dominates at least all but one vertex in $V(G) \setminus S$, we deduce the following remark.

Remark 3.2. For any graph G ,

$$\text{adim}(G) \leq \lambda(G) \leq \text{adim}(G) + 1.$$

Furthermore, $\lambda(G) = \text{adim}(G) + 1$ if and only if no adjacency basis of G is a dominating set.

In general, for non-connected graphs we can state the following remark.

Remark 3.3. Let $\{G_1, \dots, G_k\}$ be the set of components of a graph G . If there exists at least one component where no adjacency basis is a dominating set, then

$$\text{adim}(G) = -1 + \sum_{i=1}^k \lambda(G_i).$$

Otherwise,

$$\text{adim}(G) = \sum_{i=1}^k \lambda(G_i) = \sum_{i=1}^k \text{adim}(G_i).$$

Furthermore, if there are exactly $t \geq 1$ components where no adjacency basis is a dominating set, then

$$\text{adim}(G) = t - 1 + \sum_{i=1}^k \text{adim}(G_i).$$

According to the two remarks above, tight bounds on $\text{adim}(G)$ impose good bounds on $\lambda(G)$. In any case, the problem of obtaining the location-domination number of a graph G from the adjacency dimension of G , forces us to know whether G has dominating basis or not. Therefore, we can state the following open problem.

Problem 3.4. Characterize the graphs where no adjacency basis is a dominating set.

In order to show some families of graphs where every adjacency basis is a dominating set, we proceed to state the following lemma obtained previously in [20].

Lemma 3.5 ([20]). *Let G be a connected graph. If has diameter $\mathcal{D} \geq 6$, or $G \cong C_n$ with $n \geq 7$, or G is a graph of girth $g \geq 5$ and minimum degree $\delta \geq 3$, then for every adjacency generator B for G and every $v \in V(G)$, $B \not\subseteq N(v)$.*

Theorem 3.6. *Let G be a connected graph. If G has diameter $\mathcal{D} \geq 6$, or $G \cong C_n$ with $n \geq 7$, or G is a graph of girth $g \geq 5$ and minimum degree $\delta \geq 3$, then*

$$\text{adim}(G) = \lambda(\overline{G}).$$

Proof. Let G be a graph satisfying the hypothesis and let S be an adjacency basis of G . By Lemma 3.5 we deduce that S is a dominating set of \overline{G} and, since S is an adjacency basis of \overline{G} , we can conclude that S is a locating-dominating set of \overline{G} . Therefore, $\text{adim}(G) = \lambda(\overline{G})$, as required. \square

Theorem 3.7. *Let G be a graph of order n and maximum degree Δ . If $\Delta \ln(2) < \ln\left(\frac{2n}{3}\right)$, then $\text{adim}(G) = \lambda(\overline{G})$.*

Proof. Let S be an adjacency basis of G . If $\Delta \ln(2) < \ln\left(\frac{2n}{3}\right)$, then Proposition 2.4 leads to $\deg(u) \leq \Delta < \frac{\ln\left(\frac{2n}{3}\right)}{\ln(2)} \leq |S|$ for every $u \in V(G) \setminus S$, concluding that S is a locating-dominating set of \overline{G} . Therefore, $\text{adim}(G) = \lambda(\overline{G})$. \square

The following result is a direct consequence of the theorem above.

Corollary 3.8. *Let G be a graph of order n and minimum degree δ . If $\delta > n - \left\lceil \frac{\ln\left(\frac{2n}{3}\right)}{\ln(2)} \right\rceil - 1$, then $\text{adim}(G) = \lambda(G)$.*

Theorem 3.9. *Given a graph G of order n , the following assertions hold.*

- (i) *If G has at most one isolated vertex, then $\text{adim}(G) \leq n - \gamma(G)$.*
- (ii) *If G has at most one vertex of degree $n - 1$, then $\text{adim}(G) \leq n - \gamma(\overline{G})$.*
- (iii) *If G has no isolated vertices, then $\lambda(G) \leq n - \gamma(G)$.*

Proof. In this proof, the number of edges of a graph H will be denoted by $m(H)$. Let G be a graph having at most one isolated vertex and let S be a $\gamma(G)$ -set such that for any $\gamma(G)$ -set S' it is satisfied $m(\langle S \rangle) \geq m(\langle S' \rangle)$. We shall show that $V(G) \setminus S$ is an adjacency generator. Suppose to the contrary that $V(G) \setminus S$ is not an adjacency generator. In such a case, there exist $x, y \in S$ such that for every $z \in V(G) \setminus S$, either $x, y \in N(z)$ or $x, y \notin N(z)$. As a result, neither x nor y has any private neighbour (with respect to S) in $V(G) \setminus S$. We can assume that x is not an isolated vertex. Now, if $N(x) \cap S \neq \emptyset$, then $S \setminus \{x\}$ is a dominating set, which is a contradiction. If $N(x) \cap S = \emptyset$, then taking any $z \in N(x)$ we have that $S' = (S \setminus \{x\}) \cup \{z\}$ is a $\gamma(G)$ -set such that $m(\langle S' \rangle) > m(\langle S \rangle)$, which is a contradiction. Therefore, $V(G) \setminus S$ is an adjacency generator, and so (i) and (ii) follow.

Furthermore, if G has no isolated vertices, then the complement of every $\gamma(G)$ -set is a dominating set, which implies that $V(G) \setminus S$ is a locating-dominating set. Therefore, (iii) follows. \square

The bounds above are tight. For instance, bounds (i) and (iii) are achieved by $G \cong K_n$, $G \cong P_4$ and $K_{p,q}$ ($2 \leq p \leq q$). Bound (i) is also achieved by $G \cong K_1 \cup K_r$ ($r \geq 2$), as $\text{adim}(K_1 \cup K_r) = r - 1$ and $\gamma(K_1 \cup K_r) = 2$, and (iii) is also achieved by any corona graph $G \cong H \odot K_1$, as in this case $\lambda(G) = |V(H)| = \gamma(G) = \frac{n}{2}$. Obviously, bound (i) is achieved by a graph G if and only if bound (ii) is achieved by \overline{G} .

We now emphasize two well-known bounds on the domination number.

Theorem 3.10 ([26]). *For any graph G of order n and maximum degree $\Delta \geq 1$,*

$$\gamma(G) \geq \left\lceil \frac{n}{\Delta + 1} \right\rceil.$$

A graph invariant closely related to the domination number is the 2-packing number. A set $S \subseteq V(G)$ is a 2-packing if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The 2-packing number $\rho(G)$ is the cardinality of a maximum 2-packing.

Theorem 3.11 ([13]). *For any graph G ,*

$$\gamma(G) \geq \rho(G).$$

The following result is a direct consequence of combining Remark 3.1 and Theorems 3.9, 3.10 and 3.11.

Theorem 3.12. *Let G be a non-empty graph of order n , maximum degree Δ and minimum degree δ . The following assertions hold.*

- (a) $\text{adim}(G) \geq \max \left\{ \left\lceil \frac{\delta}{n-\delta} \right\rceil, \left\lceil \frac{n-\Delta-1}{\Delta+1} \right\rceil \right\}$.
- (b) $\text{adim}(G) \geq \max\{\rho(G), \rho(\overline{G})\} - 1$.
- (c) *If $\delta \geq 1$, then $\lambda(G) \leq n - \max \left\{ \rho(G), \left\lceil \frac{n}{\Delta+1} \right\rceil \right\}$.*
- (d) *If G has at most one isolated vertex, then $\text{adim}(G) \leq n - \max \left\{ \rho(G), \left\lceil \frac{n}{\Delta+1} \right\rceil \right\}$.*
- (e) *If G has at most one vertex of degree $n - 1$, then*

$$\text{adim}(G) \leq n - \max \left\{ \rho(\overline{G}), \left\lceil \frac{n}{n-\delta} \right\rceil \right\}.$$

Bound (a) is achieved by complete graphs, while bounds (b) and (c) are achieved by the corona graphs $K_r \odot K_1$, $r \geq 3$, as in this case $\text{adim}(K_r \odot K_1) = r - 1$ and $\rho(K_r \odot K_1) = r = \lambda(K_r \odot K_1)$. Bounds (c) and (d) are achieved by $G = K_n$. Obviously, bound (e) is achieved by a graph G if and only if bound (d) is achieved by \overline{G} .

A set $S \subseteq V(G)$ is a k -dominating set if $|N(v) \cap S| \geq k$ for every $v \in V(G) \setminus S$. The minimum cardinality among all k -dominating sets is called the k -domination number of G and it is denoted by $\gamma_k(G)$. A set $S \subseteq V(G)$ is an independent k -dominating set if it is both an independent set and a k -dominating set. The minimum cardinality among all independent k -dominating sets is called the independent k -domination number of G and it is denoted by $i_k(G)$.

Theorem 3.13. *If G is a non-trivial graph which does not have cycles of length four, then $\lambda(G) \leq \gamma_2(G)$.*

Proof. Let S be a 2-dominating set. If S is not an adjacency basis, then there exist $u, v \in V \setminus S$ such that $N(u) \cap S = N(v) \cap S$. Since $|N(v) \cap S| \geq 2$, there exists a cycle with four vertices, which is a contradiction. \square

The inequality above is tight. For instance, for the graph shown in Figure 4 we have that $\text{adim}(G) = \lambda(G) = \gamma_2(G) = 4$.

Theorem 3.14. *Let G be a graph which does not have cycles of length four, and let S be a $\gamma_2(G)$ -set. If there exists $s \in S$ such that $N[s] \cap S \neq N(x) \cap S$ for every $x \in N(s) \setminus S$, then $\text{adim}(G) \leq \gamma_2(G) - 1$.*

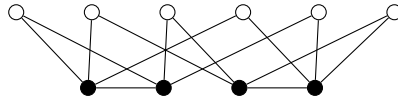


Figure 4: The set of black-colored vertices is an adjacency basis and a 2-dominating set. Hence, $\text{adim}(G) = \lambda(G) = \gamma_2(G) = 4$.

Proof. Let $s \in S$ such that $N[s] \cap S \neq N(x) \cap S$ for every $x \in N(s) \setminus S$. Let us see that $S' = S \setminus \{s\}$ is an adjacency generator. Suppose to the contrary, that S' is not an adjacency generator. In such a case, there exist $u, v \in V(G) \setminus S'$ such that $N(u) \cap S' = N(v) \cap S'$. We differentiate three cases for these two vertices.

Case 1: $u = s$. In this case $v \notin N(s)$ and so $|N(v) \cap S'| \geq 2$. Hence, there exists a cycle with four vertices, which is a contradiction.

Case 2: $u \notin N[s]$. Since $|N(u) \cap S'| \geq 2$, there exists a cycle with four vertices, which is a contradiction.

Case 3: $u, v \in N(s)$. Since $|N(u) \cap S| \geq 2$, there exists a cycle with four vertices, which is a contradiction.

Therefore, the result follows. □

In the next result we are assuming that any acyclic graph has girth $g = +\infty$.

Corollary 3.15. *Let G be a graph of minimum degree $\delta \geq 1$. Then the following assertions hold.*

- (i) *If G has girth $g \geq 5$, then $\text{adim}(G) \leq \gamma_2(G) - 1$.*
- (ii) *If G has an independent 2-dominating set and does not have cycles of length four, then $\text{adim}(G) \leq i_2(G) - 1$.*

The bounds above are tight. For instance, for $3 \leq k \leq 7$ we have that $i_2(C_{2k}) = \gamma_2(C_{2k}) = k$ and $\text{adim}(C_{2k}) = k - 1$.

Recall that a set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S . The *vertex cover number* of G , denoted by $\beta(G)$, is the smallest cardinality of a vertex cover of G . We refer to a $\beta(G)$ -set in a graph G as a vertex cover of cardinality $\beta(G)$. The largest cardinality of a set of vertices of G , no two of

which are adjacent, is called the *independence number* of G and it is denoted by $\alpha(G)$.

The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph.

Theorem 3.16. (Gallai’s theorem) *For any graph G of order n ,*

$$\alpha(G) + \beta(G) = n.$$

A *leaf* is a vertex of degree one and a *strong support vertex* is a vertex which is adjacent to more than one leaf.

Theorem 3.17. *Let G be a graph of order n without isolated vertices. If G does not have neither cycles of four vertices nor strong support vertices, then*

$$\lambda(G) \leq \beta(G) = n - \alpha(G).$$

Proof. Let S be a $\beta(G)$ -set. Since $V(G) \setminus S$ is an independent set and G does not have isolated vertices, S is a dominating set. Suppose to the contrary that S is not an adjacency generator. In such a case, there exist $u, v \in V(G) \setminus S$ such that $N(u) \cap S = N(v) \cap S$. If $|N(v) \cap S| \geq 2$, then there exists a cycle with four vertices, which is a contradiction. Now, if $|N(v) \cap S| = \{w\}$, then w is a strong support vertex, which is a contradiction again. Therefore, the results follows. \square

To see that the above inequality is tight, we can consider the graph shown in Figure 4. In this case, the set of black-colored vertices is a $\beta(G)$ -set and $\text{adim}(G) = \lambda(G) = \beta(G) = n - \alpha(G) = 4$.

A set $S \subseteq V(G)$ is called a *super dominating set* of G if for every vertex $u \in V(G) \setminus S$, there exists $u' \in S$ such that $N(u') \setminus S = \{u\}$. The *super domination number* of G , denoted by $\gamma_{\text{sp}}(G)$, is the minimum cardinality among all super dominating sets in G . A super dominating set of cardinality $\gamma_{\text{sp}}(G)$ is called a $\gamma_{\text{sp}}(G)$ -set. The study of super domination in graphs was introduced in [17].

Theorem 3.18. *For any graph G ,*

$$\lambda(G) \leq \gamma_{\text{sp}}(G).$$

Furthermore, if G has minimum degree $\delta \geq 3$ and does not have cycles of length four, then

$$\lambda(G) \leq \gamma_{\text{sp}}(G) - 1.$$

Proof. Let S be a $\gamma_{\text{sp}}(G)$ -set, $C = V(G) \setminus S$ and the function $f: C \rightarrow S$ where $f(u)$ is one of the vertices in S satisfying that $N(f(u)) \setminus S = \{u\}$. Since, $f(u)$ distinguishes $u \in C$ from any $v \in C \setminus \{u\}$, we conclude that S is a locating-dominating set of G . Hence, $\lambda(G) \leq |S| = \gamma_{\text{sp}}(G)$.

Assume that G has minimum degree $\delta \geq 3$ and does not have cycles of length four. Let $A = f(C)$ be the image of f and $B = S \setminus A$. We differentiate the following two cases.

Case 1: There exists $u \in C$ such that $N(u) \cap B \neq \emptyset$. We claim that $S' = S \setminus \{f(u)\}$ is a locating-dominating set. Since $N(f(u)) \cap C = \{u\}$ and $\deg(f(u)) \geq 3$, we have that $|N(f(u)) \cap S'| \geq 2$. Hence, S' is a dominating set. Now, every $v \in C \setminus \{u\}$

is distinguished from u by $f(v) \in S'$. Finally, if $f(u)$ and $v \in C$ are not distinguished by some vertex in S' , then $v, f(u)$ and two vertices belonging to $N(f(u)) \cap S'$ form a cycle of length four, which is a contradiction. Therefore, S' is a locating-dominating set, and so $\lambda(G) \leq |S'| = \gamma_{\text{sp}}(G) - 1$.

Case 2: $N(u) \cap B = \emptyset$ for every $u \in C$. Notice that $|N(f(u)) \cap S| \geq 2$ for every $u \in C$. Let $u, v \in C$ be two adjacent vertices. We claim that $S' = (S \setminus \{f(u), f(v)\}) \cup \{v\}$ is a locating-dominating set. Obviously, S' is a dominating set. Now, u is distinguished from any $u' \in C \setminus \{u, v\}$ by $f(u') \in S'$, and v distinguishes $f(u)$ from $f(v)$. Notice also that if $x \in C \setminus \{u, v\}$, then $|N(x) \cap (S' \setminus \{v\})| = 1$ and, since $u \sim v$, we have that $f(u) \not\sim f(v)$, which implies that $|N(y) \cap (S' \setminus \{v\})| \geq 2$ for every $y \in \{f(u), f(v)\}$. Thus, if $x \in C \setminus \{v\}$ and $y \in \{f(u), f(v)\}$, then $N(x) \cap S' \neq N(y) \cap S'$. In summary, S' is a locating-dominating set and, as a result, $\lambda(G) \leq |S'| = \gamma_{\text{sp}}(G) - 1$. \square

To show that the inequality $\lambda(G) \leq \gamma_{\text{sp}}(G)$ is tight we consider the following cases: $\lambda(K_n) = \gamma_{\text{sp}}(K_n) = n - 1$, $\lambda(K_{1,n-1}) = \gamma_{\text{sp}}(K_{1,n-1}) = n - 1$, $\lambda(K_{r,n-r}) = \gamma_{\text{sp}}(K_{r,n-r}) = n - 2$ for $2 \leq r \leq n - 2$ and $\lambda(H \odot N_t) = \gamma_{\text{sp}}(H \odot N_t) = |V(H)|t$. For the Petersen graph, shown in Figure 5, we have that $\lambda(G) = \gamma_{\text{sp}}(G) - 1 = 3$.

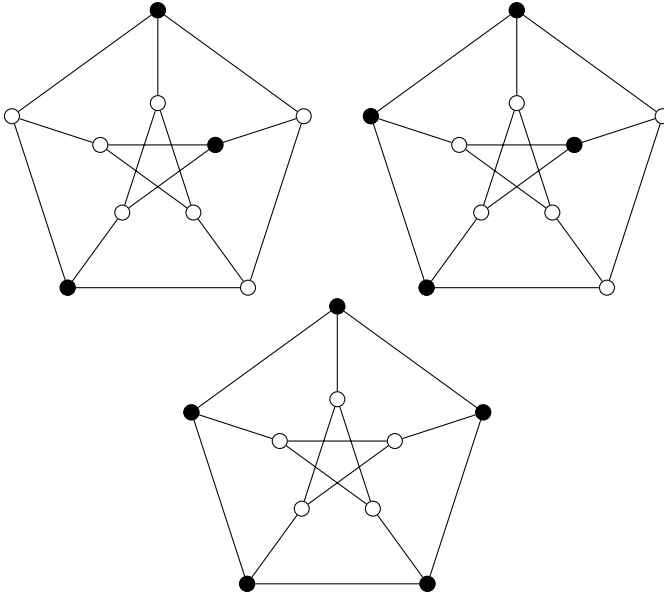


Figure 5: This figure shows three copies of the Petersen graph. The set of black-coloured vertices, on the left forms an adjacency basis, on the center forms a $\lambda(G)$ -set, while on the right forms a $\gamma_{sp}(G)$ -set.

Lemma 3.19. *Let G be a graph with two adjacent vertices $x, y \in V(G)$ such that $\deg(x) = 1$ and $\deg(y) = 2$. If $G' = G - \{x, y\}$, then $\text{adim}(G) \leq \text{adim}(G') + 1$ and $\gamma_{sp}(G) = \gamma_{sp}(G') + 1$.*

Proof. If S is an adjacency basis of G' , then $S \cup \{x\}$ is an adjacency generator of G , which implies that $\text{adim}(G) \leq \text{adim}(G') + 1$.

Assume that D' is a $\gamma_{sp}(G')$ -set and $u \in V(G')$ is adjacent to y in G . If $u \in D'$, then $D' \cup \{y\}$ is a super dominating set of G , while if $u \notin D'$, then $D' \cup \{x\}$ is a super dominating set of G . Therefore, $\gamma_{sp}(G) \leq \gamma_{sp}(G') + 1$. Now, let D be a $\gamma_{sp}(G)$ -set and $v \in N(y) \setminus \{x\}$. If $x, y \in D$, then $v \notin D$ and $(D \cup \{v\}) \setminus \{x, y\}$ is a super dominating set of G' , which implies that $\gamma_{sp}(G') \leq \gamma_{sp}(G) - 1$. Now, if $|D \cap \{x, y\}| = 1$, $D \setminus \{x, y\}$ is a super dominating set of G' and so $\gamma_{sp}(G') \leq \gamma_{sp}(G) - 1$. \square

We know that $\text{adim}(P_4) = \lambda(P_4) = \gamma_{sp}(P_4) = 2$, $\text{adim}(K_n) = \lambda(K_n) = \gamma_{sp}(K_n) = n - 1$, $\text{adim}(K_{p,q}) = \lambda(K_{p,q}) = \gamma_{sp}(K_{p,q}) = p + q - 2$ ($2 \leq p \leq q$). We proceed to show that for the remaining graphs, $\text{adim}(G) \leq \gamma_{sp}(G) - 1$.

Theorem 3.20. *For any connected graph $G \notin \{P_4, K_n, K_{p,q}\}$, with $2 \leq p \leq q$,*

$$\text{adim}(G) \leq \gamma_{sp}(G) - 1.$$

Proof of Theorem 3.20. Let G be a connected graph such that $G \notin \{P_4, K_n, K_{p,q}\}$ for $2 \leq p \leq q$. If $G' = G - \{x, y\}$, where $\deg(x) = 1$ and $\deg(y) = 2$, then we have the following:

- If $G' \cong P_4$, then $\text{adim}(G) = 2 < 3 = \gamma_{\text{sp}}(G)$.
- If $G' \cong K_1$, then $\text{adim}(G) = 1 < 2 = \gamma_{\text{sp}}(G)$.
- If $G' \cong K_{n-2}$ ($n \geq 5$), then $\text{adim}(G) = n - 3 < n - 2 = \gamma_{\text{sp}}(G)$.
- If $G' \cong K_{p,q}$ ($2 \leq p \leq q$), then $\text{adim}(G) = p + q - 2 < p + q - 1 = \gamma_{\text{sp}}(G)$.

Hence, by Lemma 3.19 we only need to consider the case where G does not have vertices of degree one which are adjacent to vertices of degree two.

Let D be a $\gamma_{\text{sp}}(G)$ -set, $C = V(G) \setminus D$ and $f: C \rightarrow D$ a function such that, for every $u \in C$, $f(u)$ is one of the vertices in S satisfying that $N(f(u)) \setminus S = \{u\}$. Let $A = f(C)$ be the image of f and $B = S \setminus A$. Notice that $D_c = C \cup B$ is also a $\gamma_{\text{sp}}(G)$ -set, so any condition given on A could be also considered on C .

Suppose to the contrary that $\text{adim}(G) \geq \gamma_{\text{sp}}(G)$. With the assumptions above in mind, we proceed to prove the following eight claims.

Claim 1. For any vertex $x \in C$, $|N(x) \cap C| \leq 1$ and $|N(f(x)) \cap A| \leq 1$.

Proof of Claim 1. If there exist $y, z \in C$ such that $f(y), f(z) \in N(f(x)) \cap A$, then x and $f(x)$ are distinguished by $f(y)$; x and any $u \in C \setminus \{x\}$ are distinguished by $f(u)$; while $f(x)$ and any $u \in C \setminus \{x\}$ are distinguished by $f(y)$ or by $f(z)$. Hence, $D \setminus \{f(x)\}$ is an adjacency generator, which is a contradiction.

If $|N(x) \cap C| \leq 1$, then we proceed by analogy to the proof above using D_c instead of D . \square

Claim 2. For any vertex $x \in C$, $\deg(x) \geq 2$ and $\deg(f(x)) \geq 2$.

Proof of Claim 2. Suppose that there exists $x \in C$ such that $\deg(x) = 1$. If $N(f(x)) \cap B = \emptyset$, then (by the connectivity of G) Claim 1 leads to $\deg(f(x)) = 2$, which is a contradiction with our assumptions. Now, if there exists $v \in N(f(x)) \cap B$, then $f(x)$ and x are distinguished by v ; for any $y \in C \setminus \{x\}$, $f(y)$ and $f(x)$ are distinguished by y ; while $f(y)$ and x are distinguished by y . Thus, $D_c \setminus \{x\}$ is an adjacency generator, which is a contradiction.

If $\deg(f(x)) = 1$, then we proceed by analogy to the proof above using D instead of D_c . \square

Claim 3. Let $x \in C$. If $N(x) \cap C = \emptyset$ or $N(f(x)) \cap A = \emptyset$, then $N(x) \cap B = N(f(x)) \cap B$.

Proof of Claim 3. If $N(f(x)) \cap A = \emptyset$, then for any $z \in C \setminus \{x\}$, $f(x)$ and z are distinguished by $f(z)$. Since $D \setminus \{f(x)\}$ is not an adjacency generator, $N(f(x)) \cap B = N(x) \cap B$. A similar argument works for the case $N(x) \cap C = \emptyset$. \square

Claim 4. Let $x, y \in C$. If $N(f(x)) \cap A = \{f(y)\}$, then $N(f(x)) \cap B = N(y) \cap B$ and $N(f(y)) \cap B = N(x) \cap B$.

Proof of Claim 4. Since $D \setminus \{f(x)\}$ is not an adjacency generator, if $N(f(x)) \cap A = \{f(y)\}$, then $N(f(x)) \cap B = N(y) \cap B$. Furthermore, by Claim 1, $N(f(x)) \cap A = \{f(y)\}$ leads to $N(f(y)) \cap A = \{f(x)\}$, and since $D \setminus \{f(y)\}$ is not an adjacency generator, we have that $N(f(y)) \cap B = N(x) \cap B$. \square

Claim 5. If $v \in B$, then $|N(v) \cap A| = 1$ and $|N(v) \cap C| = 1$.

Proof of Claim 5. If $v \in B$ and $N(v) \cap A = \emptyset$, then v and any $x \in C$ are distinguished by $f(x)$. Now, if $v \in B$ and there exist $y, z \in C$ such that $f(y), f(z) \in N(v) \cap A$, then v and any $x \in C$ are distinguished by $f(y)$ or by $f(z)$. In both cases, $D \setminus \{v\}$ is an adjacency generator, which is a contradiction. Therefore, $|N(v) \cap A| = 1$. By analogy we deduce that $|N(v) \cap C| = 1$. \square

Claim 6. If $v_1, v_2 \in B$ are adjacent vertices, $N(v_1) \cap A = \{f(x)\}$ and $N(v_1) \cap C = \{y\}$, then $N(v_2) \cap A = \{f(y)\}$ and $N(v_2) \cap C = \{x\}$.

Proof of Claim 6. Assume that $v_1, v_2 \in B$ are adjacent vertices, $N(v_1) \cap A = \{f(x)\}$ and $N(v_1) \cap C = \{y\}$. Since $D \setminus \{v_1\}$ is not an adjacency generator and $f(x)$ distinguishes v_1 and z for every $z \in C \setminus \{x\}$, we have that $x \in N(v_2)$. Thus, by Claim 5, $N(v_2) \cap C = \{x\}$. Furthermore, since $D \setminus \{v_2\}$ is not an adjacency generator and v_1 distinguishes v_2 and z for every $z \in C \setminus \{y\}$, we have that $f(y) \in N(v_2)$. Hence, by Claim 5 we conclude that $N(v_2) \cap A = \{f(y)\}$. \square

Claim 7. If there exists $x \in C$ such that $N(x) \cap C = \emptyset$ and $N(f(x)) \cap A = \emptyset$, then $|C| = 1$ and G is a complete graph.

Proof of Claim 7. Assume that there exists a vertex $x \in C$ such that $N(x) \cap C = \emptyset$ and $N(f(x)) \cap A = \emptyset$. By Claim 3, $N(x) \cap B = N(f(x)) \cap B$. Let $B_x = N(x) \cap B$, which is nonempty, as G is connected and $G \not\cong K_2$. If there exist two nonadjacent vertices $v_r, v_s \in B_x$, then $D \setminus \{v_s\}$ is an adjacency generator because x and v_s are distinguished by v_r , and any $u \in C \setminus \{x\}$ is distinguished from v_s by $f(u)$. Therefore, $X = \{x, f(x)\} \cup B_x$ induces a complete graph. Now, by the connectivity of G , if $V(G) \neq X$, then there exist two adjacent vertices $b, b' \in B$ such that $b \in B_x$ and $b' \in B \setminus B_x$. In such a case, applying Claim 6 for $x = y$, we conclude that $b' \in B_x$, which is a contradiction. Therefore, $V(G) = X$, $|C| = 1$ and G is a complete graph. \square

Claim 8. If there exist $x, y \in C$ such that $N(f(x)) \cap A = \{f(y)\}$, then $G \cong K_{p,q}$, where $2 \leq p \leq q$.

Proof of Claim 8. We differentiate two cases.

Case 1: $N(x) \cap C = \{y\}$. Since the subgraph induced by $U = \{x, f(x), y, f(y)\}$ is isomorphic to $K_{2,2}$, $V(G) \setminus U \neq \emptyset$. By Claims 1, 4 and 5, every vertex in $V(G) \setminus U$ which is adjacent to some vertex in U has to belong to $B_1 = B \cap N(f(x)) \cap N(y)$ or to $B_2 = B \cap N(x) \cap N(f(y))$. Notice that $B_1 \cap B_2 = \emptyset$. Let $X_1 = \{x, f(y)\} \cup B_1$ and $X_2 = \{f(x), y\} \cup B_2$. Let us see that G is a complete bipartite graph. Firstly, if there exist two adjacent vertices $u \in V(G) \setminus (X_1 \cup X_2)$ and $v \in B_1 \cup B_2$, by the definition of B_1 and B_2 , we know that $u \notin A \cup C$. Hence, if v belongs, for instance, to B_1 , by Claim 6, $u \in B_2$, which is a contradiction. Consequently, $V(G) = X_1 \cup X_2$. Secondly, if there exist two adjacent vertices $u, v \in B$, by Claim 6, either $u \in B_1$ and $v \in B_2$ or $u \in B_2$ and $v \in B_1$. Finally, if there exist two nonadjacent vertices $u \in B_1$ and $v \in B_2$, since u and x are distinguished by v , while u and any $z \in C \setminus \{x\}$ are distinguished by $f(z)$, we have that $D \setminus \{u\}$ is an adjacency generator, which is a contradiction. Therefore, $G = (X_1 \cup X_2, E)$ is a complete bipartite graph with $|X_1| \geq 2$ and $|X_2| \geq 2$.

Case 2: For any $x, y \in C$ such that $N(f(x)) \cap A = \{f(y)\}$, the subgraph induced by $\{x, f(x), y, f(y)\}$ is not isomorphic to $K_{2,2}$. By Claim 2, for every $x \in C$, $\deg(x) \geq 2$

and $\deg(f(x)) \geq 2$. Since $\text{adim}(C_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 4$ and $\gamma_{\text{sp}}(C_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 3$, we have that $\text{adim}(C_n) \leq \gamma_{\text{sp}}(C_n) - 1$ for any $n \geq 5$. Hence, $G \not\cong C_n$ and so $B \neq \emptyset$. By Claim 7, for every $x \in C$ either $N(x) \cap C \neq \emptyset$ or $N(f(x)) \cap A \neq \emptyset$. Suppose that there exist $x, y \in C$ such that $y \notin N(x)$ and $N(f(x)) \cap A = \{f(y)\}$. If there exists $b \in B \cap N(x) \cap N(f(y))$, then $D' = (D \cup \{y\}) \setminus \{b\}$, is also a $\gamma_{\text{sp}}(G)$ -set. In such a case, we define a new function $f' : (C \cup \{b\}) \setminus \{y\} \rightarrow D'$ where $f'(b) = f(y)$ and $f'(w) = f(w)$ for every $w \in C \setminus \{y\}$. Since the subgraph induced by $\{x, f'(x), b, f'(b)\}$ is isomorphic to $K_{2,2}$, we can conclude the proof using again Case 1. Analogously, suppose that there exist $x, y \in C$ such that $f(y) \notin N(f(x))$ and $N(x) \cap C = \{y\}$. If there exists $b \in B \cap N(x) \cap N(f(y))$, then we can take the $\gamma_{\text{sp}}(G)$ -set $D' = (D_c \cup \{f(x)\}) \setminus \{b\}$ and $f' : (A \cup \{b\}) \setminus \{f(x)\} \rightarrow D'$ where $f'(b) = x$ and $f'(f(z)) = z$ for every $z \in C \setminus \{x\}$ to obtain a subgraph isomorphic to $K_{2,2}$. Applying again Case 1 we get the result. \square

End of *Proof of Theorem 3.20*. \square

The bound above is tight. For instance, for any graph H , $\text{adim}(H \odot N_t) = \gamma_{\text{sp}}(H \odot N_t) - 1 = |V(H)|t - 1$ and for the Petersen graph shown in Figure 5 we have $\text{adim}(G) = \gamma_{\text{sp}}(G) - 1 = 3$.

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