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# Enumerating symmetric peaks in non-decreasing Dyck paths*

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**Abstract**

Local maxima and minima of a Dyck path are called *peaks* and *valleys*, respectively. A Dyck path is *non-decreasing* if the heights ($y$-coordinates) of its valleys increase from left to right. A peak is symmetric if it is surrounded by two valleys (or endpoints of the path) at the same height. In this paper we give multivariate generating functions, recurrence relations, and closed formulas to count the number of symmetric and asymmetric peaks in non-decreasing Dyck paths. Finally, we use Riordan arrays to study weakly symmetric peaks, namely those for which the valley preceding the peak is at least as high as the valley following it.

*Keywords: Non-decreasing Dyck path, symmetric peak, generating function, Riordan array, Fibonacci number.*

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# 1 Introduction

A *Dyck path* is a lattice path in the first quadrant of the $xy$-plane that starts at the origin, ends on the $x$-axis, and consists of (the same number of) up-steps $X = (1, 1)$ and down-steps $Y = (1, -1)$. A *peak* is a subpath of the form $XY$, and a *valley* is a subpath of the form $YX$. The height of a valley is the $y$-coordinate of its lowest point. A Dyck path is called *non-decreasing* if the heights of its valleys form a non-decreasing sequence from left to right (see Figure 1 for an example). Non-decreasing Dyck paths have been extensively studied in the literature, see [2, 5, 6, 8, 14, 16, 17, 20]. All the Dyck paths considered in this paper will be non-decreasing. Following the notation from [5, 6, 13, 14], we denote by $\mathcal{D}$ the set of all non-decreasing Dyck paths, and by $\mathcal{D}_n$ the set of all non-decreasing Dyck paths of length $2n$, where the length is defined as the number of steps. For $P \in \mathcal{D}_n$, we write $|P|= n$ to denote its semilength.

A *pyramid* of semilength $h \geq 1$ is a subpath of the form $X^hY^h$; it is *maximal* if it can not be extended to a pyramid $X^{h+1}Y^{h+1}$.

Flórez and Ramírez [15] introduced the concept of symmetric and asymmetric peaks in Dyck paths, see also recent follow-up work by Elizalde [11] and Flórez et al. [13]. This concept was motivated in part by Asakly's [1] study of symmetric and asymmetric peaks in $k$-ary words. The concept of symmetric peaks is different from the notion of *degree of symmetry*, which has been considered by Elizalde [9, 10] as a measure of how symmetric a Dyck path is.

In this paper we study symmetric peaks and asymmetric peaks in non-decreasing Dyck paths. A peak is *symmetric* if the maximal pyramid containing the peak is not preceded by an $X$ and is not followed by a $Y$. A peak is *weakly symmetric* if the maximal pyramid containing the peak is not preceded by an $X$. A peak is *asymmetric* if the maximal pyramid containing the peak is either preceded by an $X$ or followed by a $Y$. Geometrically, a peak is symmetric if the maximal pyramid containing the peak is either at ground level or bounded by two valleys at the same height, and it is asymmetric otherwise. For example, in the non-decreasing Dyck path in Figure 1, the first, third, fourth, and sixth peaks are symmetric. The weakly symmetric peaks are the symmetric ones along with the seventh peak. Finally, the second, fifth, and the seventh peaks are asymmetric.

We are also interested in the size of the maximal pyramid containing a peak. We define the *weight* of a pyramid $X^hY^h$ to be equal to $h$. In [5, 7], the authors refer to this parameter as the height, but we will use the term weight to suggest that it is not affected by the location of the pyramid. We define the *weight* of a peak to be the weight of the maximal pyramid that contains it. The *symmetric weight* of a path is the sum of the weights of its symmetric peaks. Similarly, the *asymmetric weight* of a path is the sum of the weights of its asymmetric peaks. For example, the weights of the symmetric peaks in the path depicted in Figure 1 are 4, 3, 3, 2 from left to right, and so the symmetric weight of the path is 12. The weights of asymmetric peaks are 1, 3, and 1, and the asymmetric weight of the path is 5.

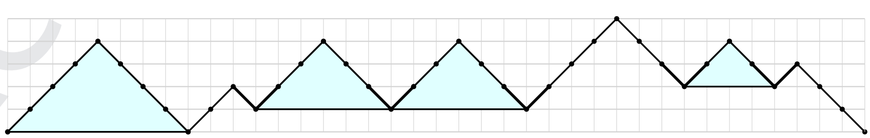

Figure 1: A non-decreasing Dyck path of length 38.

The generating functions that we present throughout the paper, are given using the symbolic method (cf. [12]). In Section 2, we give generating functions, recurrence relations, and closed formulas enumerating symmetric peaks and asymmetric peaks in non-decreasing Dyck paths. In Section 3, we focus on the enumeration of peaks with respect to their weight, and we give a connection to directed column-convex polyominoes. In Section 4, we study weakly symmetric peaks, and we synthesize the results using Riordan arrays. A summary of notation used throughout the paper appears in Tables 1 and 2 in the appendix.

## 2 Counting symmetric peaks

In this section we study the distribution of the number of symmetric peaks in $\mathcal{D}_n$. We give recurrences, generating functions and closed formulas (in terms of Fibonacci numbers) that enumerate these statistics in non-decreasing Dyck paths. Throughout the paper we will use $F_n$ and $L_n$ to denote the $n$th Fibonacci number and the $n$th Lucas number, respectively.

The set $\mathcal{D}_n$ can be partitioned into two disjoint sets $\mathcal{A}_n$ and $\mathcal{B}_n$, where $\mathcal{A}_n$ consists of the paths that have at least one valley of ground level (height 0), and $\mathcal{B}_n = \mathcal{D}_n \setminus \mathcal{A}_n$. Note that

$$\mathcal{D}_n = \mathcal{A}_n \mathbin{\dot\cup} \mathcal{B}_n \quad \text{and} \quad \mathcal{A}_n = \dot{\bigcup_{i=1}^{n-1}} \mathcal{C}_{n,i}, \tag{2.1}$$

where $\mathcal{C}_{n,i}$ consists of those paths whose first valley touches the $x$-axis at $(2i, 0)$, and $\dot\cup$ denotes disjoint union. There is a natural bijection

$$\begin{array}{ccc} \mathcal{C}_{n,i} & \to & \mathcal{D}_{n-i} \\ P & \mapsto & P \setminus \Delta_i, \end{array} \tag{2.2}$$

obtained by removing the first pyramid $\Delta_i = X^i Y^i$ of each $P \in \mathcal{C}_{n,i}$. Similarly, there is a bijection from $\mathcal{B}_n$ to $\mathcal{D}_{n-1}$ obtained by removing the first up-step and last down-step from each path.

From (2.1), a path $Q \in \mathcal{D}$ is either empty or has one of these two forms: $Q = XPY$ or $Q = X^k Y^k P$, where $k \geq 1$ and $P \in \mathcal{D}$ is non-empty. This decomposition gives rise to the following equation for the generating function $D(x) = \sum_{P \in \mathcal{D}} x^{|P|} = \sum_{n \geq 0} |\mathcal{D}_n| x^n$:

$$D(x) = 1 + xD(x) + \frac{x}{1-x}(D(x) - 1). \tag{2.3}$$

Solving this equation and removing the empty path, we obtain the generating function for non-decreasing Dyck paths with respect to their semilength:

$$D(x) = \frac{x(1-x)}{1-3x+x^2} = \sum_{n=1}^{\infty} F_{2n-1} x^n.$$

Therefore,

$$|\mathcal{D}_n| = F_{2n-1}. \tag{2.4}$$

Other derivations of this generating function appear in [2, 14].

### 2.1 A generating function for the number of symmetric and asymmetric peaks

In this section we give a multivariate generating function enumerating symmetric peaks and the number of asymmetric peaks in non-decreasing Dyck paths. We start by introducing some terminology. We define the *insertion vertices* of a path to be the lowest point of each valley $YX$, the initial point of the path, and, if the path contains no valleys at positive height, the final point of the path. For a path $P \in \mathcal{D}$, we use $\tau(P)$, $\sigma(P)$, $\overline{\sigma}(P)$, $\nu(P)$, and $\iota(P)$ to denote the number of peaks, the number of symmetric peaks, the number of asymmetric peaks, the number of valleys, and the number of insertion points of $P$, respectively. We are interested in the generating function

$$D_{\sigma,\overline{\sigma}}(t,r,x) = \sum_{P \in \mathcal{D}} t^{\sigma(P)} r^{\overline{\sigma}(P)} x^{|P|}.$$

The coefficient of $t^i r^j x^n$ in $D_{\sigma,\overline{\sigma}}(t,r,x)$ is the number of paths of length $2n$ with $i$ symmetric peaks and $j$ asymmetric peaks.

**Theorem 2.1.** *The generating function for non-decreasing Dyck paths with respect to the number of symmetric peaks and the number of asymmetric peaks is*

$$D_{\sigma,\overline{\sigma}}(t,r,x) = \frac{1-(3+t)x+(3+2t-r)x^2-(1+t-r-r^2)x^3}{(1-(1+t)x)(1-(t+2)x+(1+t-r)x^2)}.$$

*Proof.* In order to obtain an expression for $D_{\sigma,\overline{\sigma}}(t,r,x)$, we show that non-decreasing Dyck paths where some of their symmetric peaks have been marked can be constructed by inserting marked symmetric peaks in certain positions of smaller non-decreasing Dyck paths.

First, we refine Equation (2.3) by introducing a variable $v$ that keeps track of the number of valleys in the path. Letting $D_\nu(v,x) = \sum_{P\in\mathcal{D}} v^{\nu(P)} x^{|P|}$, the same decomposition gives

$$D_\nu(v,x) = 1 + xD_\nu(v,x) + \frac{vx}{1-x}(D_\nu(v,x)-1),$$

from where

$$D_\nu(v,x) = \frac{1-(1+v)x}{1-(2+v)x+x^2}.$$

Next we introduce another refinement. Let $\mathcal{D}^\Delta \subseteq \mathcal{D}$ denote the set of paths that consist of a non-empty sequence of pyramids, that is, paths of the form $X^{k_1}Y^{k_1}\cdots X^{k_j}Y^{k_j}$, where $k_i \geq 1$ for $1 \leq i \leq j$, for some $j \geq 1$. Let $D_{\tau,\iota}(p,q,x) = \sum_{P\in\mathcal{D}} p^{\tau(P)} q^{\iota(P)} x^{|P|}$ be the generating function with respect to the number of peaks and the number of insertion vertices. Recall that insertion vertices of $P$ are the bottoms of the valleys, the initial point of $P$, and, in the case that $P \in \mathcal{D}^\Delta$, the final point of $P$. Thus, $\iota(P) = \nu(P)+2$ if $P \in \mathcal{D}^\Delta$, and $\iota(P) = \nu(P)+1$ otherwise. On the other hand, $\tau(P) = \nu(P)+1$ unless $P$ is empty, in which case $\tau(P) = 0$. Using that

$$D_\nu^\Delta(v,x) = \sum_{P\in\mathcal{D}^\Delta} v^{\nu(P)} x^{|P|} = \frac{x/(1-x)}{1-vx/(1-x)} = \frac{x}{1-x-vx},$$

it follows that

$$\begin{aligned} D_{\tau,\iota}(p,q,x) &= q + pq(D_\nu(pq,x) - D_\nu^\Delta(pq,x) - 1) + pq^2 D_\nu^\Delta(pq,x) \\ &= q + \frac{pq^2x(1-(2+pq)x+(1+p)x^2)}{(1-x+pqx)(1-(2+pq)x+x^2)}. \qquad (2.5) \end{aligned}$$

By construction, the insertion vertices of $P$ are those vertices where the insertion of a pyramid $X^kY^k$ creates a symmetric peak and results in another non-decreasing Dyck path.

Our next step is to enumerate non-decreasing Dyck paths where some of its symmetric peaks have been *marked*. Formally, we are enumerating pairs $(P, M)$ where $P \in \mathcal{D}$ and $M$ is a subset of the symmetric peaks of $P$. Let $\mathcal{D}^*$ be the set of such pairs $(P, M)$, which we refer to as *non-decreasing Dyck paths with marked symmetric peaks*, and let $D^*_\tau(p, u, x) = \sum_{(P,M)\in\mathcal{D}^*} p^{\tau(P)}u^{|M|}x^{|P|}$. The key observation is that elements of $\mathcal{D}^*$ can be uniquely obtained from paths in $\mathcal{D}$ by inserting a possibly empty sequence of marked pyramids (that is, pyramids whose symmetric peak is marked) at each insertion vertex. Since replacing each insertion vertex with a sequence of marked pyramids corresponds to the substitution

$$q = \frac{1}{1 - upx/(1-x)},$$

we get

$$D^*_\tau(p, u, x) = D_{\tau,\iota}\left(p, \frac{1}{1 - upx/(1-x)}, x\right).$$

In order to have a variable $t$ that keeps track of the total number of symmetric peaks, as opposed to marked symmetric peaks, we make the substitution $u = t - 1$. Note that, if $\Sigma(P)$ is the set of symmetric peaks of a path $P \in D$, then

$$\sum_{M\subseteq\Sigma(P)} (t-1)^{|M|} = ((t-1)+1)^{|\Sigma(P)|} = t^{\sigma(P)}. \tag{2.6}$$

It follows that

$$D_{\tau,\sigma}(p, t, x) = \sum_{P\in\mathcal{D}} p^{\tau(P)}t^{\sigma(P)}x^{|P|} = \sum_{P\in\mathcal{D}} \sum_{M\subseteq\Sigma(P)} p^{\tau(P)}(t-1)^{|M|}x^{|P|} = D^*_\tau(p, t-1, x).$$

Finally, since $\overline{\sigma}(P) = \tau(P) - \sigma(P)$, we have

$$D_{\sigma,\overline{\sigma}}(t, r, x) = D_{\tau,\sigma}(r, t/r, x) = D_{\tau,\iota}\left(r, \frac{1}{1 - (t-r)x/(1-x)}, x\right),$$

and the formula in the statement follows now from Equation (2.5). □

**Corollary 2.2.** *The generating functions for the total number of symmetric peaks and the total number of asymmetric peaks in non-decreasing Dyck paths are, respectively,*

$$S(x) := \sum_{P\in\mathcal{D}} \sigma(P)x^{|P|} = \frac{\partial}{\partial t} D_{\sigma,\overline{\sigma}}(t, 1, x)\bigg|_{t=1} = \frac{x(1 - 5x + 7x^2 - x^3 - x^4)}{(1-2x)(1-3x+x^2)^2}, \tag{2.7}$$

$$\sum_{P\in\mathcal{D}} \overline{\sigma}(P)x^{|P|} = \frac{\partial}{\partial r} D_{\sigma,\overline{\sigma}}(1, r, x)\bigg|_{r=1} = \frac{x^3(2 - 6x + 3x^2)}{(1-2x)(1-3x+x^2)^2}.$$

## 2.2 Recurrence relations and Fibonacci numbers

Let $s_n = \sum_{P\in\mathcal{D}_n} \sigma(P)$, that is, the total number of symmetric peaks in all non-decreasing Dyck paths of semilength $n$. Note that $S(x) = \sum_{n\geq 1} s_n x^n$ is the generating function in Equation (2.7). Next we give a recurrence for $s_n$ that involves the Fibonacci numbers. Define the *level* of a pyramid to be the height of the base of the pyramid.

**Theorem 2.3.** *The sequence $s_n$ satisfies the recurrence relation*

$$s_n = 3s_{n-1} - s_{n-2} + F_{2(n-2)} - 2^{n-3} \quad \text{for } n \geq 3,$$

*with initial values $s_1 = 1$ and $s_2 = 3$.*

*Proof.* Recall the decomposition given in (2.1). It is clear from the definition of non-decreasing Dyck paths that the first pyramid in every path in $\mathcal{C}_{n,i}$ has a symmetric peak. Applying the bijection $\mathcal{C}_{n,i} \to \mathcal{D}_{n-i}$ from Equation (2.2) to all paths in $\mathcal{C}_{n,i}$ removes a total of $|\mathcal{D}_{n-i}| = F_{2(n-i)-1}$ pyramids (using Equation (2.4)), each having a symmetric peak. This implies that the number of symmetric peaks in $\mathcal{C}_{n,i}$ equals $F_{2(n-i)-1}$ plus the number of symmetric peaks in $\mathcal{D}_{n-i}$. So, the total number of symmetric peaks in $\mathcal{A}_n$ is given by

$$\sum_{i=1}^{n-1} s_{n-i} + \sum_{i=1}^{n-1} F_{2(n-i)-1} = \sum_{i=1}^{n-1} s_i + F_{2(n-1)}. \tag{2.8}$$

We now count the total number of symmetric peaks in $\mathcal{B}_n$, using the fact that $\mathcal{B}_n$ maps bijectively into $\mathcal{D}_{n-1}$ by deleting the first $X$ and the last $Y$. Note, however, that the first and the last peak of paths in $\mathcal{B}_n$ are not symmetric (unless the path is a pyramid), but they may become symmetric after the first $X$ and the last $Y$ are deleted. This happens when the associated path in $\mathcal{D}_{n-1}$ starts or ends with a pyramid at ground level, without the path being itself the pyramid $\Delta_{n-1} = X^{n-1}Y^{n-1}$, resulting in more symmetric peaks in $\mathcal{D}_{n-1}$ than in $\mathcal{B}_n$. Therefore, to count the number of symmetric peaks in $\mathcal{B}_n$, we take the number of symmetric peaks in $\mathcal{D}_{n-1}$, which is $s_{n-1}$, and subtract the total number of first and last pyramids at ground level of paths in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$.

First of all, we want to know the total number of pyramids at ground level that occur at the end of the paths in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$. Note that if the last pyramid of a non-decreasing Dyck path is at ground level, then the path consists of a sequence of pyramids at ground level. From [14, Corollary 6.3], we deduce that the number of paths in $\mathcal{D}_{n-1}$ ending with a pyramid $\Delta_i = X^iY^i$ at ground level, for $1 \leq i \leq n-2$, is $2^{(n-1-i)-1}$. This implies that the total number of last pyramids at ground level in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$ is $\sum_{i=0}^{n-3} 2^i = 2^{n-2} - 1$. From a similar analysis as in the first paragraph of this proof, we have that the total number of first pyramids at ground level in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$ is $\sum_{i=1}^{n-2} F_{2i-1} = F_{2(n-2)}$. So, the total number of symmetric peaks in $\mathcal{B}_n$ is given by $s_{n-1} - F_{2(n-2)} - 2^{n-2} + 1$. Adding this to (2.8), we get

$$s_n = \left( \sum_{i=1}^{n-1} s_i + F_{2(n-1)} \right) + \left( s_{n-1} - F_{2(n-2)} - 2^{n-2} + 1 \right),$$

with $s_1 = 1$, and $s_2 = 3$. We can simplify the recurrence by computing $s_{n+1} - s_n = 2s_n - s_{n-1} + F_{2(n-1)} - 2^{n-2}$. Therefore,

$$s_{n+1} = 3s_{n-1} - s_{n-2} + F_{2(n-2)} - 2^{n-3}. \qquad \square$$

The first few values of the sequence $s_n$ for $n \geq 1$ are

$$1, \quad 3, \quad 8, \quad 22, \quad 62, \quad 177, \quad 508, \quad 1459, \quad 4182, \quad 11946, \quad \ldots.$$

For example, Figure 2 shows the non-decreasing Dyck paths of length 6, where the total number of symmetric peaks is $s_3 = 8$.

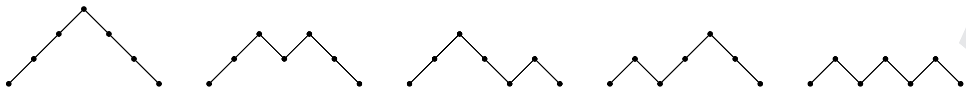

Figure 2: Non-decreasing Dyck paths of length 6.

Next we give another expression for $s_n$ in terms of the Fibonacci and the Lucas numbers.

**Theorem 2.4.** *The sequence $s_n$ satisfies*

$$s_n = F_{2n} + \sum_{\ell=3}^{n}(F_{2\ell-2} - 2^{\ell-2})F_{2(n-\ell)} = \frac{2F_{2n-2} + (n-1)L_{2n-2}}{5} + 2^{n-1}.$$

*Proof.* We first consider the generating function of the bisection of the Fibonacci sequence

$$F(x) = \sum_{n\geq 0} F_{2n}x^n = \frac{x}{1-3x+x^2}.$$

By Equation (2.7), the generating function $S(x)$ can be decomposed as

$$\begin{aligned} S(x) &= F(x)\,\frac{1-5x+7x^2-x^3-x^4}{(1-2x)(1-3x+x^2)} = F(x)\left(1 + \frac{x^2}{1-3x+x^2} - \frac{x^2}{1-2x}\right) \\ &= F(x)\left(1 + xF(x) - \frac{x^2}{1-2x}\right). \end{aligned}$$

Using the Cauchy product of series we obtain the desired result. The second equality follows from the recurrence relation given in Theorem 2.3. □

In [6, Theorem 2], the authors prove that the total number of peaks in $\mathcal{D}_n$ is

$$t_n = \frac{(2n-1)F_{2n} - (n-5)F_{2n-1}}{5}. \tag{2.9}$$

The next corollary is a direct consequence of Theorem 2.4 and Equation (2.9).

**Corollary 2.5.** *Let $\overline{s}_n$ be the total number of asymmetric peaks in $\mathcal{D}_n$. Then, for $n \geq 2$,*

$$\overline{s}_n = \frac{2F_{2n+1} + (n-2)L_{2n-3}}{5} - 2^{n-1}.$$

The first few values of the sequence $\overline{s}_n$ for $n \geq 1$ are

$$0, \quad 0, \quad 2, \quad 10, \quad 37, \quad 122, \quad 379, \quad 1136, \quad 3326, \quad 9580, \quad \dots.$$

From the identities in Theorem 2.4 and Corollary 2.5, we obtain some asymptotic results about the proportion of peaks in non-decreasing Dyck paths that are symmetric.

**Theorem 2.6.** *Among all peaks of non-decreasing Dyck paths, the proportion of those that are symmetric is asymptotically*

$$\lim_{n\to\infty}\frac{s_n}{t_n} = \frac{-1+\sqrt{5}}{2} \approx 0.618034.$$

*Proof.* From the well-known limits

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{L_n}{F_n} = \sqrt{5},$$

we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{s_n}{t_n} &= \lim_{n\to\infty} \frac{(2F_{2n-2} + (n-1)L_{2n-2})/5 + 2^{n-1}}{((2n-1)F_{2n} - (n-5)F_{2n-1})/5} \\
&= \lim_{n\to\infty} \frac{2 + (n-1)L_{2n-2}/F_{2n-2} + 5\cdot 2^{n-1}/F_{2n-2}}{(2n-1)F_{2n}/F_{2n-2} - (n-5)F_{2n-1}/F_{2n-2}} \\
&= \frac{\sqrt{5}}{2\phi^2 - \phi} = \frac{-1+\sqrt{5}}{2}.
\end{aligned}$$

□

**Corollary 2.7.** *Among all peaks of non-decreasing Dyck paths, the proportion of those that are asymmetric is asymptotically*

$$\lim_{n\to\infty} \frac{\overline{s}_n}{t_n} = \frac{3-\sqrt{5}}{2} \approx 0.381966.$$

We say that a symmetric peak is *low* if the $y$-coordinate of its top vertex is one, and that it is *high* if this coordinate is greater than 1. Note that every low peak is symmetric. By [6, Corollary 6], the total number of high peaks in $\mathcal{D}_n$ is $((2n-1)F_{2n} - nF_{2n-1})/5$. Together with Corollary 2.5, this implies the following.

**Corollary 2.8.** *The total number of high symmetric peaks in $\mathcal{D}_n$ is*

$$\frac{1}{5}\left(F_{2n-3} + (n-4)L_{2n-2}\right) + 2^{n-1}.$$

# 3 Symmetric weight and symmetric height

Recall that the weight of a pyramid $X^hY^h$ is equal to $h$ and that the weight of a peak is the weight of the maximal pyramid that contains it. In this section we give a multivariate generating function for non-decreasing Dyck paths with respect to the weight of their symmetric peaks, as well a recurrence relation for the total symmetric weight over $\mathcal{D}_n$. We also give a recurrence relation for the total sum of the heights of symmetric peaks over $\mathcal{D}_n$. At the end of the section we describe a connection with polyominoes.

## 3.1 A generating function for symmetric weight

We introduce an infinite family of variables $\mathbf{t} = (t_1, t_2, \dots)$ in order to keep track of symmetric peaks of a given weight. For $P \in \mathcal{D}$ and $i \geq 1$, let $\omega_i(P)$ be the number of symmetric peaks of weight $i$ in $P$. Let $\boldsymbol{\omega}(P) = (\omega_1(P), \omega_2(P), \dots)$, and let $\mathbf{t}^{\boldsymbol{\omega}(P)} = \prod_{i\geq 1} t_i^{\omega_i(P)}$. We are interested in the generating function

$$D_{\boldsymbol{\omega}}(\mathbf{t}, x) = \sum_{P\in\mathcal{D}} \mathbf{t}^{\boldsymbol{\omega}(P)} x^{|P|}.$$

**Theorem 3.1.** *Let* $P(\mathbf{t}, x) = \sum_{i\geq 1} t_i x^i$. *The generating function for non-decreasing Dyck paths with respect to the weights of their symmetric peaks is*

$$D_{\boldsymbol{\omega}}(\mathbf{t}, x) = \frac{1 - 3x + 2x^2 + x^3 - (1-x)^3 P(\mathbf{t}, x)}{(1-x)(1 - P(\mathbf{t}, x))(1 - 2x - (1-x)^2 P(\mathbf{t}, x))}.$$

*Proof.* We modify the proof of Theorem 2.1 in order to keep track of the weight of the inserted marked symmetric peaks. Replacing insertion vertices in non-decreasing Dyck paths with sequences of marked pyramids, with variable $u_i$ keeping track of marked pyramids of the form $X^i Y^i$ for each $i \geq 1$, corresponds to the substitution

$$q = \frac{1}{1 - \sum_{i\geq 1} u_i x^i}$$

in $D_{\tau,\iota}(1, q, x)$. A variant of Equation (2.6), where we replace $\Sigma(P)$ with the set of symmetric peaks of weight $i$, shows that the substitutions $u_i = t_i - 1$ yield the generating function where $t_i$ keeps track of the total number of symmetric peaks of weight $i$ in non-decreasing Dyck paths. It follows that

$$D_{\boldsymbol{\omega}}(\mathbf{t}, x) = D_{\tau,\iota}\left(1, \frac{1}{1 - \sum_{i\geq 1}(t_i - 1)x^i}, x\right) = D_{\tau,\iota}\left(1, \frac{1}{\frac{1}{1-x} - P(\mathbf{t}, x)}, x\right),$$

and the formula is now obtained from Equation (2.5). □

The symmetric weight of a path $P \in \mathcal{D}$ is defined as the sum of the weights of its symmetric peaks, and it is denoted by $\omega(P) = \sum_{i\geq 1} \omega_i(P)$. From Theorem 3.1, one can easily obtain a generating function for this statistic. Let

$$D_{\sigma,\omega}(t, w, x) = \sum_{P\in\mathcal{D}} t^{\sigma(P)} w^{\omega(P)} x^{|P|}$$

be the generating function for non-decreasing Dyck paths with respect to the number of symmetric peaks and the symmetric weight of the path.

**Corollary 3.2.** *The generating function* $D_{\sigma,\omega}(t, w, x)$ *is equal to*

$$\frac{(1 - wx)\left(1 - (3 + w + tw)x + (2 + 3w + 3tw)x^2 + (1 - 2w - 3tw)x^3 - (1-t)wx^4\right)}{(1-x)\left(1 - (t+1)wx\right)\left(1 - (2 + w + tw)x + 2(t+1)wx^2 - twx^3\right)}.$$

*Proof.* By definition, $D_{\sigma,\omega}(t, w, x)$ is obtained from $D_{\boldsymbol{\omega}}(\mathbf{t}, x)$ by making the substitution $t_i = tw^i$ for all $i \geq 1$. When applied to $P(\mathbf{t}, x)$, this substitution yields $\sum_{i\geq 1} tw^i x^i = twx/(1 - wx)$, and so the formula follows immediately from Theorem 3.1. □

**Corollary 3.3.** *The generating function for the total symmetric weight in non-decreasing Dyck paths is*

$$W(x) := \sum_{P\in\mathcal{D}} \omega(P) x^{|P|} = \frac{\partial}{\partial w} D_{\sigma,\omega}(1, w, x)\bigg|_{w=1} = \frac{x(1 - 5x + 7x^2 - x^3 - x^4)}{(1-x)(1-2x)(1-3x+x^2)^2}.$$

Comparing this formula with Equation (2.7), we see that

$$W(x) = \frac{S(x)}{1-x}.$$

Taking the coefficients of $x^n$ on both sides, and letting $w_n = \sum_{P\in\mathcal{D}_n} \omega(P)$ denote the total symmetric weight of $\mathcal{D}_n$, we get

$$w_n = \sum_{k=1}^{n} s_k, \tag{3.1}$$

that is, the total symmetric weight of paths in $\mathcal{D}_n$ equals the total number of symmetric peaks of paths in $\bigcup_{k=1}^n \mathcal{D}_k$. Next we give a bijective proof of this equality.

The right-hand side of (3.1) can be interpreted as counting paths in $\bigcup_{k=1}^n \mathcal{D}_k$ with a distinguished symmetric peak. Indeed, for each $k$, the number of ways to choose path in $\mathcal{D}_k$ and select a symmetric peak of such path equals the total number of symmetric peaks of paths in $\mathcal{D}_k$, namely $s_k$. Similarly, the left-hand side of (3.1) can be interpreted as counting pairs $(\hat{P}, i)$, where $\hat{P}$ is a path in $\mathcal{D}_n$ with a distinguished symmetric peak, and $i$ is an integer between 1 and the weight of the distinguished peak of $\hat{P}$. This is because, for a given path $P \in \mathcal{D}_n$, the number of ways to choose a symmetric peak of $P$ and then an integer $i$ between 1 and the weight of that peak equals the sum of the weights of the symmetric peaks of $P$, which is $\omega(P)$.

Let us describe a bijection between the sets counted by both sides of (3.1). Given a path in $\mathcal{D}_k$ (for some $k \le n$) with a distinguished symmetric peak, insert a pyramid $X^{n-k}Y^{n-k}$ at the top of the distinguished peak to obtain a pair $(\hat{P}, i)$, where $\hat{P}$ is a path in $\mathcal{D}_n$ with a distinguished symmetric peak (the same distinguished peak where the pyramid was inserted), and $i = n-k$. Conversely, given such a pair $(\hat{P}, i)$, delete the pyramid $X^iY^i$ around the distinguished peak, to obtain a path in $\mathcal{D}_{n-i}$ with a distinguished symmetric peak (the same distinguished peak from where the pyramid was removed).

### 3.2 Recurrence relations and Fibonacci numbers

Recall that $w_n$ denotes the sum of the symmetric weights of all paths in $\mathcal{D}_n$. Similarly, let $\overline{w}_n$ denote the sum of the asymmetric weights of all paths in $\mathcal{D}_n$. For example, the paths in Figure 2 give $w_3 = 3+0+3+3+3 = 12$ and $\overline{w}_3 = 2$. The next theorem follows immediately by applying Equation (3.1) to Theorem 2.3.

**Theorem 3.4.** *The sequence $w_n$ satisfies the recurrence relation*

$$w_n = 3w_{n-1} - w_{n-2} + F_{2n-3} - 2^{n-2} + 1 \quad \text{for } n \ge 3,$$

*with initial values $w_1 = 1$ and $w_2 = 4$.*

The first few values of the sequence $w_n$ for $n \ge 1$ are

$$1, \quad 4, \quad 12, \quad 34, \quad 96, \quad 273, \quad 781, \quad 2240, \quad 6422, \quad 18368, \quad \ldots.$$

From the expression for $W(x)$ in Corollary 3.3, we obtain the following corollary.

**Corollary 3.5.** *We have*

$$w_n = F_{2n} + \sum_{\ell=1}^{n} (F_{2\ell-1} - 2^{\ell-1} + 1) F_{2(n-\ell)}$$

*and*

$$w_n = \frac{1}{5}(nL_{2n-1} - F_{2n}) + 2^n - 1.$$

In [5, Theorem 8] the authors prove that the sum of the weights of all peaks in $\mathcal{D}_n$ is

$$\frac{2nF_{2n+1} + (2-n)F_{2n}}{5}.$$

As a direct application of Corollary 3.5, we obtain the following formula for the sum of the asymmetric weights of all paths in $\mathcal{D}_n$.

**Corollary 3.6.** *We have*

$$\overline{w}_n = \frac{1}{5}(3F_{2n} + nL_{2n-2}) - 2^n + 1.$$

### 3.3 Symmetric height

The height of a peak is the $y$-coordinate of the vertex at the top of the peak. Denote by $h_n$ the total sum of the heights of all symmetric peaks of paths in $\mathcal{D}_n$. For example, from the paths in Figure 2, we see that $h_3 = 12$.

**Theorem 3.7.** *The sequence $h_n$ satisfies the recurrence relation*

$$h_n = 3h_{n-1} - h_{n-2} + \frac{nL_{2n-5} + 7F_{2n-5}}{5} - 2^{n-2} + 1 \quad \text{for } n \geq 3,$$

*with initial values $h_1 = 1$ and $h_2 = 4$.*

*Proof.* We will find the total sum of the heights of all symmetric peaks of paths in $\mathcal{D}_n = \mathcal{A}_n \uplus \mathcal{B}_n$ by adding the total sum of the heights of all symmetric peaks in $\mathcal{A}_n$ and the total sum of the heights of all symmetric peaks in $\mathcal{B}_n$. Recall that $\mathcal{A}_n = \uplus_{i=1}^{n-1} \mathcal{C}_{n,i}$, and that the first peak of every path in $\mathcal{C}_{n,i}$ is symmetric. From (2.2) we know that every path $P \in \mathcal{C}_{n,i}$ is a concatenation of the pyramid $\Delta_i = X^i Y^i$ with a path $Q \in \mathcal{D}_{n-i}$. So, the total sum of the heights of all symmetric peaks in $P$ is given by the hight of $\Delta_i$ (which is equal to $i$) plus the total sum of the heights of all symmetric peaks in $Q$. Summing over all paths $P \in \mathcal{C}_{n,i}$, we deduce that the total sum of the heights of all symmetric peaks of $\mathcal{C}_{n,i}$ is $i|\mathcal{D}_{n-i}| + h_{n-i} = iF_{2(n-i)-1} + h_{n-i}$ (using that $|\mathcal{D}_{n-i}| = F_{2(n-i)-1}$, see (2.4)). Therefore, the total sum of the heights of all symmetric peaks in $\mathcal{A}_n$ is given by

$$\sum_{i=1}^{n-1} h_{n-i} + \sum_{i=1}^{n-1} iF_{2(n-i)-1} = \sum_{i=1}^{n-1} h_i + F_{2n-1} - 1. \tag{3.2}$$

We now count the sum of the heights of all symmetric peaks in $\mathcal{B}_n$, using the fact that $\mathcal{B}_n$ is in bijection with $\mathcal{D}_{n-1}$, for which the sum of the heights of all symmetric peaks is $h_{n-1}$. The bijection is given by removing the first and the last step of the path. Let us carefully analyze how the sum of the heights of the symmetric peaks is changed by this bijection. On the one hand, removing the first and last step of the path decreases the heights of the peaks by one. On the other hand, for paths in $\mathcal{D}_{n-1}$ that begin or end with a pyramid at ground level, those pyramids contain a symmetric peak that does not give a symmetric peak in the corresponding path in $\mathcal{B}_n$. To account for these cases, we subtract, from the total sum of heights of symmetric peaks in $\mathcal{D}_{n-1}$, the heights of the first and last peaks belonging to pyramids at ground level, and then we add one for each symmetric peak whose height has increased.

We recall that the paths in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$, whose first pyramid is at ground level have the form $\Delta_i P_{n-1-i}$, where $P_{n-1-i} \in \mathcal{D}_{n-1-i}$ and $1 \leq i \leq n-2$. For fixed $i$, the height of all first pyramids in all such paths is given by $i\,|\mathcal{D}_{n-1-i}| = i\,F_{2(n-1-i)-1}$. So, the total height of all first pyramids at ground level of paths in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$ is given by

$$\sum_{i=1}^{n-2} (n-1-i) F_{2i-1} = F_{2n-3} - 1. \tag{3.3}$$

We count the total height of pyramids at ground level that occur at the end of the paths in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$. If the last pyramid of a non-decreasing Dyck path is at ground level, then the whole path consists of a sequence of pyramids at ground level. From [14, Corollary 6.3], we deduce that the number of paths in $\mathcal{D}_{n-1}$ ending with a pyramid $\Delta_i$ at ground level, for $1 \leq i \leq n-2$, is $2^{(n-1-i)-1}$. So, the total height of all last pyramids at ground level of paths in $\mathcal{D}_{n-1} \setminus \{\Delta_{n-1}\}$ is given by

$$\sum_{i=1}^{n-2} i\, 2^{n-i-2} = 2^{n-1} - n. \tag{3.4}$$

Now, —to account for the increase by one of peak heights caused by the addition of the initial $X$ and the final $Y$ to paths in $\mathcal{D}_{n-1}$— we add the total number of symmetric peaks in $\mathcal{D}_{n-1}$, which equals $s_{n-1}$ (see Theorem 2.4). But this results in some over-counting due to the first and last pyramids at ground level of the paths in $\mathcal{D}_{n-1}$, so we have to subtract $F_{2n-4}$ and $2^{n-2} - 1$. All in all, the term that needs to be added to account for the increase in peak heights is

$$\left( \frac{2F_{2n-4} + (n-2)L_{2n-4}}{5} + 2^{n-2} \right) - F_{2n-4} - 2^{n-2} + 1. \tag{3.5}$$

Adding (3.2), $h_{n-1}$, and (3.5), and subtracting (3.3) and (3.4), we get the recurrence relation

$$h_n = \sum_{i=1}^{n-1} h_i + F_{2n-1} - 1 + h_{n-1} +$$
$$\left( \frac{2F_{2n-4} + (n-2)L_{2n-4}}{5} + 2^{n-2} - F_{2n-4} - 2^{n-2} + 1 \right) - \left( F_{2n-3} - 1 + 2^{n-1} - n \right).$$

Simplifying, we have that

$$h_n = \sum_{i=1}^{n-1} h_i + h_{n-1} + \frac{F_{2n-1} + nL_{2n-4} + L_{2n-5}}{5} - 2^{n-1} + n + 1,$$

where $h_1 = 1$, and $h_2 = 4$. Now it is easy to see that

$$h_{n+1} - h_n = 2h_n - h_{n-1} + \frac{F_{2n} + nL_{2n-3} + 5F_{2n-3}}{5} - 2^{n-1} + 1.$$

Therefore,

$$h_n = 3h_{n-1} - h_{n-2} + \frac{nL_{2n-5} + 7F_{2n-5}}{5} - 2^{n-2} + 1. \qquad \square$$

The first few values of the sequence $h_n$ for $n \geq 1$ are

$$1, \quad 4, \quad 12, \quad 35, \quad 104, \quad 315, \quad 964, \quad 2957, \quad 9044, \quad 27502, \quad \ldots.$$

### 3.4 Connections with dccp-polyominoes

Non-decreasing Dyck paths are in bijection with a family of polyominoes called *directed column-convex polyominoes* (dccp). A polyomino is *directed* if each of its cells can be reached from its bottom left-hand corner by a path which is contained in the polyomino and uses only north and east steps. A dccp polyomino is a directed polyomino such that every column consists of contiguous cells [3]. Deutsch and Prodinger [8] give a bijection between the set of non-decreasing Dyck paths of length $2n$ and the set of dccp of area $n$, where the *area* of a polyomino is defined as its number of cells. Figure 3 shows a dccp of area 19. The numbers in the first (second) row represent the final (initial) altitude of each column.

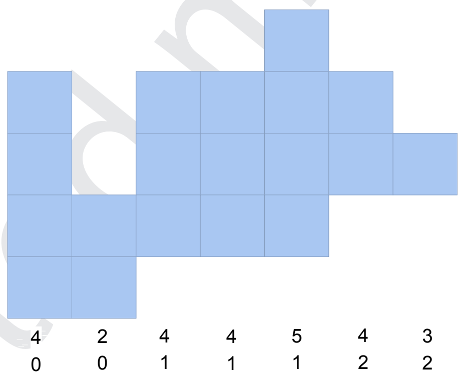


Figure 3: A direct column-convex polyomino (dccp).

The bijection from [8] can be described as follows. Given a dccp whose columns have initial altitudes $A = (0, a_2, \ldots, a_k)$ and final altitudes $B = (b_1, b_2, \ldots, b_k)$, from left to right, its corresponding non-decreasing Dyck path has valleys at heights $(a_2, \ldots, a_k)$, and peaks at heights $(b_1, b_2, \ldots, b_k)$, from left to right. For example, the dccp in Figure 3 is mapped to the path in Figure 1.

We say that two consecutive columns in a dccp polyomino are at the *same level* if their initial altitudes are the same. For example, the polyomino in Figure 3 has 4 pairs of consecutive columns at the same level; columns 1 and 2, columns 3 and 4, columns 4 and 5, and columns 6 and 7. Thus, the sequence $s_n$ that we introduced in Section 2.2 also counts the total number of pairs of consecutive columns at the same level in all dccp polyominoes of area $n$. Moreover, if we define the weight of a pair of consecutive columns at the same level as the number of cells in the first of these two columns, then the total weight over all dccp polyominoes of area $n$ is given by $w_n$.

## 4 Weakly symmetric peaks

In this section we consider a variation of symmetric peaks. We recall from Section 1 that a peak is *weakly symmetric* if the maximal pyramid containing the peak is not preceded by an $X$. Figure 4 shows different possibilities for the steps preceding and following the maximal pyramid of a weakly symmetric peak. Note that the last configuration in Figure 4 can only occur in the last peak of a path.

In Section 2, we gave generating functions to count symmetric and asymmetric peaks in non-decreasing Dyck paths, in this section we also give generating functions to count the number of weakly symmetric peaks. Surprisingly, the generating functions in this section have a simpler construction.

We will find formulas, involving Fibonacci numbers, for the total number of weakly symmetric peaks, as well as the sum of their weights, using generating functions and recurrence relations. The results in this section are synthesized using Riordan arrays.

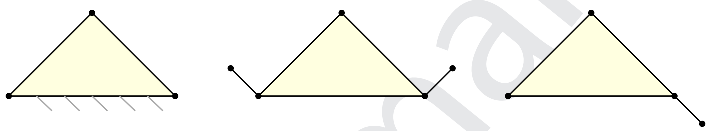

Figure 4: Weakly symmetric peaks.

### 4.1 A generating function for the number of weakly symmetric peaks

Let $\tilde{s}_n$ be the total number of weakly symmetric peaks in $\mathcal{D}_n$. For example, we see from the paths in Figure 2 that $\tilde{s}_3 = 9$. The first few values of $\tilde{s}_n$ for $n \geq 1$ are

$$1, \quad 3, \quad 9, \quad 27, \quad 80, \quad 234, \quad 677, \quad 1941, \quad 5523, \quad 15615, \quad \dots,$$

which correspond to sequence A059502 in [21].

Given a non-decreasing Dyck path $P$, we denote by $\tilde{\sigma}(P)$ the number of weakly symmetric peaks of $P$, and recall that $|P|$ denotes the semilength of $P$. We introduce the generating function

$$D_{\tilde{\sigma}}(x, y) = \sum_{P \in \mathcal{D}} x^{|P|} y^{\tilde{\sigma}(P)}.$$

**Theorem 4.1.** *The generating function $D_{\tilde{\sigma}}(x, y)$ is given by*

$$D_{\tilde{\sigma}}(x, y) = \frac{(1-x)xy}{1-(2+y)x+yx^2}.$$

*Proof.* Recall the decomposition in (2.1). Non-empty paths in $\mathcal{B}_n$ can be written as $XY$ or $XT'Y$, where $T'$ is a non-decreasing Dyck paths. Paths in $\mathcal{A}_n$ are of the form $X\Delta YT''$, where $\Delta$ is a pyramid and $T''$ is a non-decreasing Dyck paths. Figure 5 illustrates the three cases.

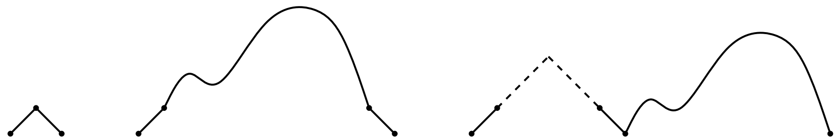

Figure 5: Decomposition of a non-decreasing Dyck path.

Using the symbolic method, we obtain the relation

$$D_{\tilde{\sigma}}(x,y) = xy + x(\underbrace{D_{\tilde{\sigma}}(x,y) - \frac{xy}{1-x}D_{\tilde{\sigma}}(x,y) + \frac{x}{1-x}D_{\tilde{\sigma}}(x,y)}_{(a)}) + \frac{xy}{1-x}D_{\tilde{\sigma}}(x,y).$$

The term $(a)$ corresponds to the case where $T'$ starts with a pyramid, which was symmetric in $T'$ but is no longer weakly symmetric in the big path. This completes the proof. □

**Corollary 4.2.** *The total number of weakly symmetric peaks in $\mathcal{D}_n$ satisfies these*

*(i) The generating function for $\tilde{s}_n$ is given by*

$$\sum_{n=1}^{\infty} \tilde{s}_n x^n = \frac{(1-x)(1-2x)x}{(1-3x+x^2)^2}.$$

*(ii) The sequence $\tilde{s}_n$ satisfies the recurrence relation*

$$\tilde{s}_n = 6\tilde{s}_{n-1} - 11\tilde{s}_{n-2} + 6\tilde{s}_{n-3} - \tilde{s}_{n-4} \quad \text{for } n \geq 5,$$

*with initial values $\tilde{s}_1 = 1$, $\tilde{s}_2 = 3$, $\tilde{s}_3 = 9$ and $\tilde{s}_4 = 27$.*

*(iii) The sequence $\tilde{s}_n$ satisfies the recurrence relation*

$$\tilde{s}_n = 3\tilde{s}_{n-1} - \tilde{s}_{n-2} + F_{2(n-2)} \quad \text{for } n \geq 3,$$

*with initial values $\tilde{s}_1 = 1$ and $\tilde{s}_2 = 3$.*

*(iv) For $n \geq 1$, we have the convolution*

$$\tilde{s}_n = \sum_{\ell=0}^{n-1} F_{2\ell-1} F_{2(n-\ell)-1}.$$

*(v) The sequence $\tilde{s}_n$ satisfies that $\tilde{s}_n = (3F_{2n} + nL_{2n-2})/5$.*

*Proof.* By Theorem 4.1,

$$\sum_{n=0}^{\infty} \tilde{s}_n x^n = \left.\frac{\partial D_{\tilde{\sigma}}(x,y)}{\partial y}\right|_{y=1} = \frac{(1-x)(1-2x)x}{(1-3x+x^2)^2}.$$

This proves part (i). The recurrence in part (ii) is obtained from this rational generating function. The proof of (iii) is similar to the proof of Theorem 2.3, but in this case we do not subtract the last pyramid at ground level of paths in $\mathcal{B}_n$.

To prove part (iv), note that

$$\sum_{n=1}^{\infty} \tilde{s}_n x^n = \left(\frac{(1-x)x}{1-3x+x^2}\right)\left(\frac{1-2x}{1-3x+x^2}\right)$$
$$= \left(\sum_{n=1}^{\infty} F_{2n-1}x^n\right)\left(\sum_{n=0}^{\infty} F_{2n-1}x^n\right)$$
$$= \sum_{n=1}^{\infty}\left(\sum_{\ell=0}^{n-1} F_{2\ell-1}F_{2(n-\ell)-1}\right)x^n.$$

Comparing coefficients of $x^n$ yields the identity.

Finally, it is easy to verify that the right side of part (v) satisfies the same recurrence relation as $\tilde{s}_n$ given in part (2), or alternatively in (3). □

From Part (v) of Corollary 4.2 and Equation (2.9), we conclude the following.

**Theorem 4.3.** *Among all peaks of all non-decreasing Dyck paths, the proportion of those that are weakly symmetric is asymptotically*

$$\lim_{n\to\infty} \frac{\tilde{s}_n}{t_n} = \frac{-1+\sqrt{5}}{2} \approx 0.618034.$$

Notice that this coincides with the asymptotic proportion of symmetric peaks given in Theorem 2.6.

### 4.2 A connection with Riordan arrays

In this section we use Riordan arrays to describe the distribution of the number of weakly symmetric peaks in non-decreasing Dyck paths. We start by giving some background on Riordan arrays [23]. We will say that an infinite column vector $(a_0, a_1, \dots)^T$ has generating function $f(x)$ if $f(x) = \sum_{n\geq 0} a_n x^n$, and we index rows and columns starting at 0. A *Riordan array* is an infinite lower triangular matrix whose $k$th column has generating function $g(x)f(x)^k$ for all $k \geq 0$, for some formal power series $g(x)$ and $f(x)$ with $g(0) \neq 0$, $f(0) = 0$, and $f'(0) \neq 0$. Such a Riordan array is denoted by $(g(x), f(x))$. If we multiply this matrix by a column vector $(c_0, c_1, \dots)^T$ having generating function $h(x)$, then the resulting column vector has generating function $g(x)h(f(x))$. This property is known as the fundamental theorem of Riordan arrays, or as the summation property.

The product of two Riordan arrays $(g(x), f(x))$ and $(h(x), l(x))$ is defined by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x))). \tag{4.1}$$

Under this operation, the set of all Riordan arrays is a group [23]. The identity element is $I = (1, x)$, and the inverse of $(g(x), f(x))$ is

$$(g(x), f(x))^{-1} = \left(1/\left(g \circ f^{<-1>}\right)(x), f^{<-1>}(x)\right), \tag{4.2}$$

where $f^{<-1>}(x)$ denotes the compositional inverse of $f(x)$.

Let $r_{n,k}$ be the number of paths in $\mathcal{D}_n$ with exactly $k$ weakly symmetric peaks, that is,

$$D_{\tilde{\sigma}}(x, y) = \sum_{n,k\geq 1} r_{n,k} x^n y^k.$$

By definition, $\sum_{k=1}^{n} k\, r_{n,k} = \tilde{s}_n$.

Consider the matrix $\mathcal{R} = [r_{n,k}]_{n,k\geq 1}$. The first few rows of $\mathcal{R}$ are

$$\mathcal{R} = [r_{n,k}]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 4 & 5 & 3 & 1 & 0 & 0 & 0 & 0 & \\ 8 & 12 & 9 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 16 & 28 & 25 & 14 & 5 & 1 & 0 & 0 & \\ 32 & 64 & 66 & 44 & 20 & 6 & 1 & 0 & \\ 64 & 144 & 168 & 129 & 70 & 27 & 7 & 1 & \\ \vdots & & & \vdots & & & & & \ddots \end{pmatrix},$$

which correspond to array A105306 in [21]. Even though rows and columns of Riordan arrays are indexed starting at 0, the elements of $\mathcal{R}$ are shifted so that the entry in row 0 and column 0 is in fact $r_{1,1}$. The goal of this shift is to simplify some of our formulas.

**Theorem 4.4.** *The matrix $\mathcal{R}$ is a Riordan array given by*

$$\mathcal{R} = \left(\frac{1-x}{1-2x}, \frac{x(1-x)}{1-2x}\right).$$

*Proof.* Multiplying the right-hand side of the equality by the vector $(1, y, y^2, \dots)^T$, which has generating function $\frac{1}{1-xy}$, and using the summation property, the resulting vector has bivariate generating function

$$\left(\frac{1-x}{1-2x}, \frac{x(1-x)}{1-2x}\right)\frac{1}{1-xy} = \frac{1-x}{1-2x}\,\frac{1}{1-\dfrac{x(1-x)}{1-2x}y} = \frac{1-x}{1-(2+y)x+yx^2} = \frac{D_{\tilde{\sigma}}(x,y)}{xy},$$

by Theorem 4.1. □

**Theorem 4.5.** *For $n, k \geq 0$,*

$$r_{n+1,k+1} = \sum_{\ell=0}^{n} \binom{k+1}{\ell}\binom{n-\ell}{k}(-1)^{\ell} 2^{n-k-\ell}.$$

*Proof.* From the definition of the Riordan array $\mathcal{R}$, we have

$$\begin{aligned} r_{n+1,k+1} &= [x^n]\,\frac{1-x}{1-2x}\left(\frac{x(1-x)}{1-2x}\right)^{k} \\ &= [x^{n-k}]\left(\frac{1-x}{1-2x}\right)^{k+1} \\ &= [x^{n-k}]\sum_{n\geq 0}\sum_{\ell=0}^{n}\binom{k+1}{\ell}\binom{k+n-\ell}{n-\ell}(-1)^{\ell}2^{n-\ell}x^{n}. \end{aligned}$$

□

Let $\mathcal{P} = \left[\binom{n}{k}\right]_{n,k\geq 0}$, often called Pascal's matrix, and let $\overline{\mathcal{P}} = [\overline{p_{i,j}}]$ be the matrix defined by

$$\overline{p_{i,j}} = \begin{cases} \binom{(i+j)/2}{j}, & \text{if } i+j \equiv 0 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that $\mathcal{P}$ and $\overline{\mathcal{P}}$ are Riordan arrays given by

$$\mathcal{P} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right) \quad \text{and} \quad \overline{\mathcal{P}} = \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right).$$

**Theorem 4.6.** *The matrix $\mathcal{R}$ factors as $\mathcal{R} = \mathcal{P}\overline{\mathcal{P}}$.*

*Proof.* By Equation (4.1),

$$\begin{aligned} \mathcal{P}\overline{\mathcal{P}} &= \left(\frac{1}{1-x}, \frac{x}{1-x}\right)\left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) \\ &= \left(\frac{1}{1-x}\left(\frac{1}{1-\left(\frac{x}{1-x}\right)^2}\right), \frac{\frac{x}{1-x}}{1-\left(\frac{x}{1-x}\right)^2}\right). \end{aligned}$$

Simplifying,

$$\mathcal{P}\overline{\mathcal{P}} = \left(\frac{1-x}{1-2x}, \frac{x(1-x)}{1-2x}\right) = \mathcal{R}.$$ □

From above theorem and the product of matrices we obtain the following combinatorial identities.

**Theorem 4.7.** *For $n, k \geq 0$,*

$$\begin{aligned} r_{n+1,2k+1} &= \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell}\binom{\ell+k}{2k}, \\ r_{n+1,2k+2} &= \sum_{\ell=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2\ell+1}\binom{\ell+k+1}{2k+1}. \end{aligned}$$

Rogers [22], observed that every element not belonging to row 0 or column 0 in a Riordan array can be expressed as a fixed linear combination of the elements in the preceding row. The $A$-sequence is defined to be the sequence coefficients of this linear combination. Similarly, Merlini et al. [19] introduced the $Z$-sequence, that characterizes the elements in column 0, except for the top one. Therefore, the $A$-sequence, the $Z$-sequence and the upper-left element completely characterize a Riordan array. We summarize this characterization in the following two theorems.

**Theorem 4.8** ([19]). *An infinite lower triangular array $\mathcal{F} = [d_{n,k}]_{n,k\geq 0}$ is a Riordan array if and only if $d_{0,0} \neq 0$ and there exist two sequences $(a_0, a_1, a_2, \ldots)$, with $a_0 \neq 0$, and $(z_0, z_1, z_2, \ldots)$ (called the $A$-sequence and the $Z$-sequence, respectively), such that*

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots && \text{for } n,k \geq 0, \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots && \text{for } n \geq 0. \end{aligned}$$

**Theorem 4.9** ([19, 18]). *Let $\mathcal{F} = (g(x), f(x))$ be a Riordan array with inverse $\mathcal{F}^{-1} = (d(x), h(x))$. Then the $A$-sequence and the $Z$-sequence of $\mathcal{F}$ have generating functions*

$$A(x) = \frac{x}{h(x)}, \qquad Z(x) = \frac{1}{h(x)}\left(1 - d_{0,0}d(x)\right),$$

*respectively.*

Next we describe the $A$-sequence and $Z$-sequence for the Riordan array $\mathcal{R}$.

**Theorem 4.10.** *If $C_n$ denotes the $n$-th Catalan number, then for $n, k \geq 2$,*

$$r_{n,k} = \sum_{\ell=0}^{n} r_{n-1,k-1+\ell}\, c_\ell,$$

*where*

$$c_n = \begin{cases} 1, & \text{if } n = 0, 1; \\ (-1)^{\frac{n+2}{2}} C_{\frac{n-2}{2}}, & \text{if } n \geq 2 \text{ is even}; \\ 0, & \text{otherwise.} \end{cases}$$

*Moreover, for $n \geq 2$*

$$r_{n,1} = \sum_{\ell=0}^{n} r_{n-1,k-1+\ell}\, c_{\ell+1},$$

*with initial value $r_{1,1} = 1$.*

*Proof.* By Equation (4.2), the inverse of the matrix $\mathcal{R}$ is given by

$$\mathcal{R}^{-1} = \left(\frac{1 + 2x - \sqrt{1+4x^2}}{2x}, \frac{1 + 2x - \sqrt{1+4x^2}}{2}\right).$$

Therefore, by Theorem 4.9, the $A$-sequence and $Z$-sequence of the Riordan array $\mathcal{R}$ have generating functions given by

$$A(x) = \frac{1 + 2x + \sqrt{1+4x^2}}{2} \quad \text{and} \quad Z(x) = \frac{-1 + 2x + \sqrt{1+4x^2}}{2x}.$$

We recall that the generating function of the Catalan numbers is given by

$$C(x) = \sum_{n\geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

Therefore, $A(x) = 1 + x + x^2C(-x^2) = \sum_{n\geq 0} c_n x^n$, where $c_n$ is as in the statement of the theorem. Similarly, $Z(x) = 1 + xC(-x^2)$. The recurrences from Theorem 4.8 now give the desired result. □

The first few values of the sequence $c_n$ for $n \geq 0$ are

$$1, \quad 1, \quad 1, \quad 0, \quad -1, \quad 0, \quad 2, \quad 0, \quad -5, \quad 0, \quad 14, \quad 0, \quad -42, \quad 0, \quad 132, \quad \ldots.$$

So, the recurrence for $r_{n,k}$ starts as

$$r_{n-1,k-1} + r_{n-1,k} + r_{n-1,k+1} - r_{n-1,k+3} + 2r_{n-1,k+5} - 5r_{n-1,k+7} + \cdots.$$

Next we analyze the central diagonal of the matrix $\mathcal{R}$, that is, the sequence $u_n = r_{2n+1,n+1}$ for $n \geq 0$ (recall that the entry in row $i$ and column $j$ of $\mathcal{R}$ is $r_{i+1,j+1}$). The first few values of $u_n$ are

$$1, \quad 2, \quad 9, \quad 44, \quad 225, \quad 1182, \quad 6321, \quad 34232, \quad 187137, \quad 1030490, \quad 5707449, \quad \ldots,$$

which correspond to the sequence A176479 in [21].

Barry [4] proved that for any Riordan array $(g(x), f(x)) = [d_{n,k}]_{n,k\geq 0}$ the generating function of its central diagonal is given by

$$\sum_{n\geq 0} d_{2n,n}x^n = \frac{v(x)g(v(x))}{f(v(x))}v'(x),$$

where

$$v(x) = \left(\frac{x^2}{f(x)}\right)^{<-1>}.$$

Therefore, by Theorem 4.4,

$$\sum_{n\geq 0} u_n x^n = \frac{3 - x + \sqrt{1-6x+x^2}}{4\sqrt{1-6x+x^2}}.$$

Other combinatorial interpretations of the sequence $u_n$ are given in [21]. For example, it counts the number of Dyck paths having exactly $n$ peaks at height 1, $n$ peaks at height 2, and no other peaks. It is also equal to $n+1$ times the $n$th little Schröder number. The little Schröder numbers have several combinatorial interpretations in terms of leaves in plane trees, parenthesizations, and dissections of convex polygons [24].

### 4.3 A generating function for total weight

Let $\tilde{\omega}(P)$ be the sum of the weights of the weakly symmetric peaks of a path $P$. Define the generating function

$$D_{\tilde{\omega}}(x,y) = \sum_{P\in\mathcal{D}} x^{|P|}y^{\tilde{\omega}(P)}.$$

**Theorem 4.11.** *The generating function $D_{\tilde{\omega}}(x,y)$ is given by*

$$D_{\tilde{\omega}}(x,y) = \frac{(1-x)^2xy}{1-2(1+y)x+4yx^2-yx^3}.$$

*Proof.* We again use the refinement of the decomposition (2.1) illustrated in Figure 5: every non-empty non-decreasing Dyck path can be written as either $XY$, $XT'Y$, or $X\Delta YT''$, where $T'$ and $T''$ are non-decreasing Dyck paths and $\Delta$ is a pyramid. It follows that

$$D_{\tilde{\omega}}(x,y) = xy + x(\underbrace{D_{\tilde{\omega}}(x,y) - \frac{xy}{1-xy}D_{\tilde{\omega}}(x,y) + \frac{x}{1-x}D_{\tilde{\omega}}(x,y)}_{(a)} - \underbrace{\frac{xy}{1-xy} + \frac{xy^2}{1-xy}}_{(b)}))$$
$$+ \frac{xy}{1-xy}D_{\tilde{\omega}}(x,y).$$

The correction term $(a)$ corresponds to the case where $T'$ consists of a pyramid followed by a non-empty path, whereas the term $(b)$ corresponds to the case where $T'$ is a pyramid. □

From Theorem 4.11 we obtain the following corollary, whose proof is similar to that of Corollary 4.2. Let $\tilde{w}_n$ be the sum of the weights of all weakly symmetric peaks of paths in $\mathcal{D}_n$.

**Corollary 4.12.** *The sum of the weights of all weakly symmetric peaks in $\mathcal{D}_n$ satisfies the following:*

*(i) The generating function for $\tilde{w}_n$ is given by*

$$\sum_{n=1}^{\infty} \tilde{w}_n x^n = \frac{(1-2x)x}{(1-3x+x^2)^2}.$$

*(ii) The sequence $\tilde{w}_n$ satisfies the recurrence relation*

$$\tilde{w}_n = 6\tilde{w}_{n-1} - 11\tilde{w}_{n-2} + 6\tilde{w}_{n-3} - \tilde{w}_{n-4} \quad \text{for } n \geq 5,$$

*with initial values $\tilde{w}_1 = 1, \tilde{w}_2 = 4, \tilde{w}_3 = 13$ and $\tilde{w}_4 = 40$.*

*(iii) For $n \geq 1$, we have the convolution*

$$\tilde{w}_n = \sum_{\ell=0}^{n} F_{2\ell-1} F_{2(n-\ell)} = \frac{4F_{2n} + nL_{2n-1}}{5}.$$

The first few values of $\tilde{w}_n$ for $n \geq 1$ are

$$1, \quad 4, \quad 13, \quad 40, \quad 120, \quad 354, \quad 1031, \quad 2972, \quad 8495, \quad 24110, \quad \dots,$$

which correspond to the sequence A238846 in [21].

Let $q_{n,k}$ be the number of paths in $\mathcal{D}_n$ which have weakly symmetric weight $k$, that is,

$$D_{\tilde{\omega}}(x,y) = \sum_{n,k\geq 1} q_{n,k} x^n y^k.$$

Notice that $\sum_{k=1}^{n} k\, q_{n,k} = \tilde{w}_n$. Consider the matrix defined by $\mathcal{Q} = [q_{n,k}]_{n,k\geq 1}$. The first few rows of $\mathcal{Q}$ are

$$\mathcal{Q} = [q_{n,k}]_{n,k\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & \\ 2 & 3 & 0 & 8 & 0 & 0 & 0 & 0 & \\ 4 & 6 & 8 & 0 & 16 & 0 & 0 & 0 & \cdots \\ 8 & 13 & 16 & 20 & 0 & 32 & 0 & 0 & \\ 16 & 28 & 37 & 40 & 48 & 0 & 64 & 0 & \\ 32 & 60 & 84 & 98 & 96 & 112 & 0 & 128 & \\ \vdots & & & \vdots & & & & & \ddots \end{pmatrix}.$$

Again, as in the matrix $\mathcal{R}$, the elements of $\mathcal{Q}$ are shifted so that the entry in row 0 and column 0 is $q_{1,1}$. The proof of our last result is similar to that of Theorem 4.4.

**Theorem 4.13.** *The matrix $\mathcal{Q}$ is a Riordan array given by*

$$\mathcal{Q} = \left( \frac{1-2x+x^2}{1-2x}, \frac{2x-4x^2+x^3}{1-2x} \right).$$

# 5 Appendix. Notation tables

| | type of peaks | | | |
|---|---|---|---|---|
| | symmetric | asymmetric | weakly symmetric | all |
| number of such peaks in $P$ | $\sigma(P)$ | $\overline{\sigma}(P)$ | $\tilde{\sigma}(P)$ | $\tau(P)$ |
| total number over $\mathcal{D}_n$ | $s_n$ | $\overline{s}_n$ | $\tilde{s}_n$ | $t_n$ |
| vector of peak weights of $P$ | $\boldsymbol{\omega}(P) = (\omega_1(P), \dots)$ | | | |
| sum of peak weights of $P$ | $\omega(P)$ | | $\tilde{\omega}(P)$ | |
| total sum of weights over $\mathcal{D}_n$ | $w_n$ | | $\tilde{w}_n$ | |
| total sum of heights over $\mathcal{D}_n$ | $h_n$ | | | |

Table 1: Summary of notation for peak statistics.

| Notation | Page | Notation | Page | Notation | Page |
|---|---|---|---|---|---|
| $\mathcal{D}_n, \mathcal{D}$ | 2 | $\iota(P), \nu(P)$ | 4 | $r_{n,k}$ | 16 |
| $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_{n,i}$ | 3 | $S(x)$ | 5 | $q_{n,k}$ | 21 |

Table 2: Other notation, along with the page where it is first introduced.

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